The Cotype of Operators from $C(K)$

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Dedicated to the Glory of God, by whom this was inspired.

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Statement of Originality

All work contained in this thesis is the result of my own work carried out between October 1985 and August 1988. No part of this thesis contains work done in collaboration with any other person. No part of this thesis has been submitted for a degree or diploma at any other University.

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Contents

Introduction

In 1987, Jameson [J1] studied the relationship between the $(2,1)$-summing norm and the 2-summing norm for operators from l_∞^N. He showed that, in general, these norms are not equivalent. At the end of his paper, he observed that the Rademacher cotype 2 constant of operators from l_∞^N lay between these two summing norms, and he asked whether it was indeed equivalent to one of them.

Answering this question proved to be very hard. By delicate averaging arguments, I managed to prove that the Rademacher cotype 2 constant for an operator from l_∞^N is very close to its $(2,1)$-summing norm; they are within about $\log\log N$ of each other, and hence, in general, the cotype 2 constant and the 2-summing norms are inequivalent. The techniques used also enabled me to compare the Rademacher and Gaussian cotype p constants for many operators from l_∞^N, deducing that these are not the same.

Studying this problem also led me to consider quite a different subject. I defined new spaces which are a common generalization of the Lorentz $L_{p,q}$ and the Orlicz L_Φ spaces. As well as rederiving results of Bennett and Rudnick, I sought to calculate the Boyd indices of these new spaces.

The Structure of the Thesis

The thesis is split up into three parts, and each part is split up into three or four chapters. The first two parts are about the cotype of operators from $C(K)$, and the third is on the generalized Lorentz spaces.

The purpose of Part 1 is to introduce the problems that are solved in Part 2. Part 1 starts in Chapter 1A, which introduces the notation and preliminary results to be used. Chapter 1B studies (p,q)-summing operators from $C(K)$, and includes the result that reduces the study of $(p,1)$-summing operators to consideration of the formal identity maps $l_\infty^N \to L_{p,1}^N$. Chapter 1C describes random walks and cotype, and in doing so, motivates the main problems of this thesis. It culminates in Section 5, where the main results to be proved are stated. Last, and least, Chapter 1D discusses other miscellaneous results.

Part 2 is devoted to proving the results of Section 1C:5. Chapter 2A proves the most difficult result of this thesis, that is, showing that the Rademacher cotype 2 constant of an operator from l_∞^N is no more than $\log\log N$ times its $(2,1)$-summing norm. The techniques used are discussed in Chapter 2B, and then Chapter 2C proves the second major result, showing that the Gaussian cotype 2 of an operator from l_∞^N is no more than $\sqrt{\log\log N}$ times its $(2,1)$-summing norm. Finally, Chapter 2D shows that the Gaussian and Rademacher cotype p constants of many operators from l_∞^N differ by $\sqrt{\log N}$.

Part 3 is in many ways an appendix, and it proves some of the results about Orlicz and Lorentz spaces needed in Parts 1 and 2. It gives new proofs and extensions of known results. It also discusses the problem of calculating Boyd indices of the generalized Lorentz spaces.

Part 1 — Introducing the Problems

Contents

Introduction to Part 1

The first part of my thesis consists of four chapters. The purpose of this part is to set the scene for Part 2, where I prove the main results.

Chapter 1A contains most of the definitions and many of the preliminary results required for Parts 1 and 2. Some of the material is standard, and is presented only to avoid any possible misunderstanding, but I also present some new ideas.

Chapter 1B is a discussion about summing operators. It contains results due to Pietsch, Maurey and Pisier. It also gives a finite dimensional version of Pisier's theorem, which is of importance later on.

After this comes Chapter 1C, which gives an introduction to random walks and cotype, and culminates in Section 5, where the main results of the thesis are stated.

Finally, Chapter 1D gives some results which are of some interest, but of no great importance to the main arguments in this thesis. First it discusses the constants that appear in Pisier's theorem on $(p,1)$-summing operators. Then it shows that $L_{2,1}$ is not a $(2,1)$-summing space. After this, it presents a proof of Grothendieck's Theorem. Finally it gives a result like Pisier's theorem for $(\Phi,1)$-summing operators.

Throughout Part 1, the familiar and the unfamiliar are presented side by side, to aid the flow of the arguments.

Chapter 1A — Preliminaries

In this chapter we give definitions and preliminary results. Section 1 merely gives the most elementary notation. Section 2 gives the notation about Banach spaces. It includes an elementary introduction to Orlicz functions and spaces, and also gives a definition of Lorentz spaces that extends the usual definition (so that, for example, it includes the Zygmund–Lorentz spaces (see [B–R])). It also contains results about the so called J and K interpolation functors. Section 3 gives the notation for ideals, and is standard with the single exception that we define the (Φ, Ψ)-summing norm.

1) Basic Notation

1.1) Constants and the Letter 'c'

Much of the time, particularly in Part 2, we will be using numerical constants in whose precise values we are not interested. For this reason we will reserve the lower case letter c to mean 'a numerical constant'. This numerical constant will not depend on any parameters, that is, it will be a universal constant, but different occurrences of the letter c may represent different constants. The constants so represented are to be thought of as large; a small numerical constant will be represented by c^{-1}.

Given a number $C < \infty$, and functions X and Y, we say that X *is approximately equal to* Y, *with constant of approximation* C (in symbols $X \overset{C}{\approx} Y$), if $X \le CY$ and $Y \le CX$. If the constant of approximation is a universal constant, then we simply say that X is *approximately equal to* Y (in symbols, $X \approx Y$). If we have $X \le cY$, then we say that X is *approximately less than* Y, or that Y is *approximately greater than* X.

1.2) Notation involving Integers

We write $\{\, j \in \mathbf{Z} : m \le j \le n\}$ as $[m, n]$, and $[1, n]$ as $[n]$. It should be clear from the context as to whether $[m, n]$ means $\{\, j \in \mathbf{Z} : m \le j \le n\}$ or $\{\, t \in \mathbf{R} : m \le t \le n\}$. We write $\mathbf{N}$ for the natural numbers, $\{\, j \in \mathbf{Z} : j \ge 1\}$.

If x is a real number, we write $\lfloor x \rfloor$ for the *integer part* of x, that is, the greatest integer not exceeding x. We write $\lceil x \rceil$ for $-\lfloor -x \rfloor$. Often we use a real number where an integer is to be expected, for example, in the limits of a sum; this should be understood as meaning the integer part of the number. Occasionally we make implicit use of the inequality

$$\lfloor x \rfloor \ge c^{-1} x \quad \text{for } x \ge 1.$$

If m, m' and n are integers with $n > 0$, then we write

$$m \equiv m' \pmod n$$

if $m - m'$ is divisible by n. If m and n are integers with $n > 0$, then we write $m \bmod n$ for the integer $m' \in [1, n]$ such that $m \equiv m' \pmod n$, that is,

$$m \bmod n = m - n \lfloor \tfrac{m-1}{n} \rfloor.$$

1.3) Notation involving Real and Complex Numbers

If x and y are real numbers, we write $x \vee y$ for $\max\{x, y\}$ and $x \wedge y$ for $\min\{x, y\}$. If $x = re^{i\theta}$ is a complex number (with $r \geq 0$ and $\theta \in [0, 2\pi)$), then we set $|x| = r$, and $\operatorname{sign}(x) = e^{i\theta}$ if $r \neq 0$, and $\operatorname{sign}(0) = 0$. We write $\mathbf{R}_+$ for $\{ t \in \mathbf{R} : t \geq 0\}$. If $t \in \mathbf{R}_+$, then we write $\log^+ t = 0 \vee \log t$.

We define the *modified logarithm* and *modified exponential* functions for $t > 0$:

$$\operatorname{lm}(t) = \begin{cases} 1 + \log t & \text{for } t \geq 1 \\ \dfrac{1}{1 + \log \frac{1}{t}} & \text{for } t \leq 1; \end{cases}$$

$$\operatorname{em}(t) = \operatorname{lm}^{-1}(t) = \begin{cases} e^{t-1} & \text{for } t \geq 1 \\ e^{-\frac{1}{t}+1} & \text{for } t \leq 1. \end{cases}$$

The function lm is designed to behave very much like the function log for large values, but modified so that it behaves well around 1, and so that $\operatorname{lm} \frac{1}{t} = \frac{1}{\operatorname{lm} t}$. Similarly, em is just a modified version of exp.

We have the following elementary result about lm.

Proposition 1.1. *If $0 < p < \infty$, and $\alpha \in \mathbf{R}$, then there is a number $C < \infty$, depending on p and α only, such that if*

$$s = t^p (\operatorname{lm} t)^\alpha,$$

then

$$t \overset{C}{\approx} \left(s / (\operatorname{lm} s)^\alpha \right)^{\frac{1}{p}}.$$

Proof: It is easy to see that for some $C < \infty$, we have $\operatorname{lm} s \overset{C}{\approx} \operatorname{lm} t$, and the result follows. □

We also have the following elementary results about summing various series. These results will be heavily used throughout the thesis, especially in Part 2.

Proposition 1.2. *Let $0 < p < \infty$, and $\alpha \in \mathbf{R}$.*

i) If $p < 1$, then there is a number $C < \infty$, depending on α only, such that

$$\begin{aligned} C^{-1} N^{1-p} (\operatorname{lm} N)^\alpha &\leq \sum_{n=1}^{N} n^{-p} (\operatorname{lm} n)^\alpha \\ &\leq C \frac{1}{1-p} N^{1-p} (\operatorname{lm} N)^\alpha, \end{aligned}$$

$$\begin{aligned} C^{-1} N^{1-p} (\operatorname{lm}(N/N_1))^\alpha &\leq \sum_{n=1}^{N_1} n^{-p} (\operatorname{lm}(N/n))^\alpha \\ &\leq C \frac{1}{1-p} N^{1-p} (\operatorname{lm}(N/N_1))^\alpha. \end{aligned}$$

ii) If $p > 1$, then there is a number $C < \infty$, depending on α only, such that

$$\sum_{n=1}^{N} n^{-p} (\operatorname{lm} n)^\alpha \leq C \left(1 \vee \frac{1}{p-1} \right),$$

$$\sum_{n=1}^{N} n^{-p} (\operatorname{lm}(N/n))^\alpha \leq C \left(1 \vee \frac{1}{p-1} \right) (\operatorname{lm} N)^\alpha.$$

iii) If $\alpha > -1$, then

$$c^{-1}\left(1 \wedge \frac{1}{1+\alpha}\right)(\operatorname{lm} N)^{1+\alpha} \le \sum_{n=1}^{N} n^{-1}(\operatorname{lm} n)^{\alpha}$$
$$\le c\left(1 \vee \frac{1}{1+\alpha}\right)(\operatorname{lm} N)^{1+\alpha},$$

$$c^{-1}\left(1 \wedge \frac{1}{1+\alpha}\right)(\operatorname{lm} N)^{1+\alpha} \le \sum_{n=1}^{N} n^{-1}(\operatorname{lm}(N/n))^{\alpha}$$
$$\le c\left(1 \vee \frac{1}{1+\alpha}\right)(\operatorname{lm} N)^{1+\alpha}.$$

iv) If $\alpha < -1$, then

$$\sum_{n=1}^{N} n^{-1}(\operatorname{lm} n)^{\alpha} \le c\left(1 \vee \frac{-1}{1+\alpha}\right),$$

$$\sum_{n=1}^{N} n^{-1}(\operatorname{lm}(N/n))^{\alpha} \le c\left(1 \vee \frac{-1}{1+\alpha}\right)(\operatorname{lm} N)^{\alpha}.$$

v)

$$\sum_{n=1}^{N} n^{-1}(\operatorname{lm} n)^{-1} \approx \operatorname{lm}\operatorname{lm} N,$$

$$\sum_{n=1}^{N} n^{-1}(\operatorname{lm}(N/n))^{-1} \approx \operatorname{lm}\operatorname{lm} N.$$

Proof: All these results come from approximating sums by integrals. For example, with (v), we have

$$\sum_{n=1}^{N} n^{-1}(\operatorname{lm} n)^{-1} \le 1 + \int_1^N \tfrac{1}{t(1+\log t)}\,dt = 1 + \log(1+\log n)$$
$$= \operatorname{lm}\operatorname{lm} N,$$

and

$$\sum_{n=1}^{N} n^{-1}(\operatorname{lm} n)^{-1} \ge \int_1^N \tfrac{1}{t(1+\log t)}\,dt = -1 + \operatorname{lm}\operatorname{lm} N.$$

$\square$

1.4) Notation involving Sets and Functions

Let A be a set. We write $|A|$ for the number of elements in A, and $A^{(k)}$ for the collection of k-subsets (that is, subsets of size k) of A.

If $a\colon X \to \mathbf{C}$ is a function and $A \subseteq X$, then the symbol $a\big|_A$ denotes the function $X \to \mathbf{C}$ defined by

$$a\big|_A(x) = \begin{cases} a(x) & \text{if } x \in A \\ 0 & \text{if } x \in X \setminus A. \end{cases}$$

If $A \subseteq X$, then the characteristic function of A in X, $\chi_A^X = \chi_A$, is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in X \setminus A. \end{cases}$$

If X is any set, then the identity function on X is denoted by Id_X. If $a\colon X \to Y$ and $b\colon Y \to Z$ are functions, then the composite, $b \circ a\colon X \to Z$, is the function $b \circ a(x) = b(a(x))$.

If $N \in \mathbf{N}$, then we write S_N for the symmetry group on $[N]$, that is, the set of permutations $\pi\colon [N] \to [N]$, with function composition as the group operation.

1.5) Notation involving Probability

With very few exceptions, we desire to treat random variables in a naive fashion. For this reason, we make the following conventions throughout the thesis. We assume the existence of an underlying probability space with measure Pr. As usual, measurable functions from the probability space are called random variables, and for any integrable random variable X, we write $\mathbf{E}X$ for $\int X \, d\,\mathrm{Pr}$. We suppose that there are independent random variables $\gamma_1, \gamma_2, \ldots, \varepsilon_1, \varepsilon_2, \ldots$ such that for each $n \in \mathbf{N}$, γ_n is normalised Gaussian, that is,

$$\Pr(\gamma_n > t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}s^2} \, ds,$$

and ε_n is Bernoulli, that is,

$$\Pr(\varepsilon_n = 1) = \Pr(\varepsilon_n = -1) = \frac{1}{2}.$$

2) Notation involving Quasi-Banach Spaces

The notation for Banach spaces and quasi-Banach spaces is fairly standard, and is loosely based on that used in [L–T1] and [L–T2]. We never worry as to whether the spaces are real or complex. If X is any Banach space, than its dual is denoted X^*. The action of $\xi \in X^*$ on $x \in X$ is denoted by $\langle \xi, x \rangle$. Note, we do *not* use x^* to represent a typical element of X^*. If $T\colon X \to Y$ is a bounded linear operator between Banach spaces, then we denote its dual map by $T^*\colon Y^* \to X^*$. We denote the unit ball of a quasi-Banach space by B_X.

The notation for Banach lattices is also standard, and we follow [L–T2]. In particular, we have the functional calculus (see [L–T2 Thm.1.d.1]), so that if f, g, f_1, $f_2, \ldots,$ f_S are elements of a lattice X, then so are $f \vee g$, $f \wedge g$, $|f|$ and $\left(\sum_{s=1}^S |f_s|^q\right)^{\frac{1}{q}}$ (where $0 < q < \infty$).

Definition. *If X is a Banach lattice, then we say that X is q-concave if there is a number $C < \infty$ such that for all f_1, $f_2, \ldots,$ $f_S \in X$ we have*

$$\left(\sum_{s=1}^S \|f_s\|_X^q\right)^{\frac{1}{q}} \le C \left\| \left(\sum_{s=1}^S |f_s|^q\right)^{\frac{1}{q}} \right\|_X .$$

Practically all the quasi-Banach spaces we refer to are rearrangement invariant function spaces or 1-symmetric sequence spaces, which we introduce here.

2.1) The $C(K)$, L_∞ and l_∞ Spaces

I shall assume that the reader is familiar with the elementary properties of the L_∞ spaces, which we define here.

Symbol	Vectors	Norm
$C(K)$	Continuous functions on a compact Hausdorff space K	$\|f\|_\infty = \sup_{x\in K} |f(x)|$
$L_\infty(\Omega, \mathcal{F}, \mu)$, $L_\infty(\Omega, \mu)$, $L_\infty(\Omega)$, $L_\infty(\mu)$, L_∞	Essentially bounded functions on measure space $(\Omega, \mathcal{F}, \mu)$	$\|f\|_\infty = \operatorname{ess\,sup}_{w\in\Omega} |f(w)|$
l_∞	Bounded functions from $\mathbf{N}$	$\|f\|_\infty = \sup_{n\in\mathbf{N}} |f(n)|$
l_∞^N	Functions from $[N]$	$\|f\|_\infty = \sup_{n\in[N]} |f(n)|$

2.2) The L_p and l_p Spaces

Here we define the L_p spaces for $0 < p < \infty$. The reader will recall that L_p is a normed space for $p \geq 1$, otherwise it is only a quasi-normed space.

Symbol	Vectors	Quasi-norm
$L_p(\Omega, \mathcal{F}, \mu)$, $L_p(\Omega, \mu)$, $L_p(\Omega)$, $L_p(\mu)$, L_p	Measurable functions on measure space $(\Omega, \mathcal{F}, \mu)$ such that $\int \lvert f\rvert^p \, d\mu < \infty$	$\lVert f\rVert_{L_p} = \lVert f\rVert_p = \left(\int \lvert f\rvert^p \, d\mu\right)^{\frac{1}{p}}$
l_p	Functions from $\mathbf{N}$ such that $\sum_{n=1}^{\infty} \lvert f(n)\rvert^p < \infty$	$\lVert f\rVert_p = \left(\sum_{n=1}^{\infty} \lvert f(n)\rvert^p\right)^{\frac{1}{p}}$
l_p^N	Functions from $[N]$	$\lVert f\rVert_{l_p^N} = \lVert f\rVert_p = \left(\sum_{n=1}^{N} \lvert f(n)\rvert^p\right)^{\frac{1}{p}}$
L_p^N	Functions from $[N]$	$\lVert f\rVert_{L_p^N} = \left(\frac{1}{N}\sum_{n=1}^{N} \lvert f(n)\rvert^p\right)^{\frac{1}{p}}$

Note the difference between l_p^N and L_p^N. Both spaces consist of functions from $[N]$, but in the first case, l_p^N, the measure on $[N]$ is $|\cdot|$, whereas with L_p^N, the measure is the probability measure $\frac{1}{N}|\cdot|$.

2.3) Orlicz Spaces

These spaces will also be discussed in Part 3, where we will prove many of the results quoted here. The definitions here are slight extensions of those usually given (see, for example, [B–S Ch.4], [K–R], [L–T1 Ch.4], [L–T2 p120] or [Mu]).

Definition. *A normal quasi-Orlicz function is a function* $\Phi\colon [0,\infty) \to [0,\infty)$ *satisfying the following.*

i) $\Phi(0) = 0$, *and* $\Phi(t) \to \infty$ *as* $t \to \infty$.

ii) Φ *is continuous and strictly increasing.*

iii) The $\frac{1}{2}$*-dilatory constant of* Φ, $\sup_{x\in\mathbf{R}_+} \Phi^{-1}(2x)/\Phi^{-1}(x)$, *is finite.*

If Φ *is a normal quasi-Orlicz function, we write* $\tilde{\Phi}$ *for the normal quasi-Orlicz function*

$$\tilde{\Phi}(x) = 1/\Phi(\tfrac{1}{x}).$$

We often label a function by its effect on T, for example the function $\Phi\colon x \mapsto x^2$ is called T^2. Typical normal quasi-Orlicz functions include $T^p(\operatorname{lm} T)^\alpha$ and $\operatorname{em}(T^p)$, where $0 < p < \infty$ and $\alpha \in \mathbf{R}$. Where no confusion can arise, we write p for the function T^p. Thus $L_p = L_{T^p}$.

Practically all the normal quasi-Orlicz functions that we will consider will satisfy $\tilde{\Phi} = \Phi$.

Definition. *A normal quasi-Orlicz function* Φ *is said to satisfy the* Δ_2*-condition if* $\sup_{x\in\mathbf{R}_+} \Phi(2x)/\Phi(x)$ *is finite.*

Proposition 2.1. *Let* Φ *be a convex normal quasi-Orlicz function that satisfies the* Δ_2*-condition. Then there are numbers* $0 < q < \infty$, $C < \infty$, *and a normal quasi-Orlicz function* Ψ *such that*

i) $\Psi^{-1} \overset{C}{\approx} \Phi^{-1}$;

ii) Ψ *is convex;*

iii) $\Psi \circ T^{\frac{1}{q}}$ *is concave.*

Proof: See Proposition 3A:2.9. $\square$

Definition. *Let Φ be a normal quasi-Orlicz function, and $(\Omega, \mathcal{F}, \mu)$ be any measure space. Then for any measurable $f : \Omega \to \mathbf{C}$, we define its Φ-quasi-Orlicz norm to be*

$$\|f\|_\Phi = \inf \left\{ \lambda : \int_\Omega \Phi\left(\tfrac{|f(w)|}{\lambda}\right) d\mu(w) \le 1 \right\}.$$

Proposition 2.2. *Let Φ be a normal quasi-Orlicz function. Then $\|\cdot\|_\Phi$ is a quasi-norm. If Φ convex, then $\|\cdot\|_\Phi$ is a norm.*

Proof: See Proposition 3A:3.5, or any of the above references. □

If Φ is a normal quasi-Orlicz function, then we form the *Orlicz spaces.*

Symbol	Vectors	Quasi-norm
$L_\Phi(\Omega, \mathcal{F}, \mu)$, $L_\Phi(\Omega, \mu)$, $L_\Phi(\Omega)$, $L_\Phi(\mu)$, L_Φ	Measurable functions on measure space $(\Omega, \mathcal{F}, \mu)$ such that $\|f\|_\Phi < \infty$	$\|f\|_{L_\Phi} = \|f\|_\Phi$
l_Φ	$L_\Phi(\mathbf{N})$	$\|f\|_\Phi$
l_Φ^N	$L_\Phi([N], \lvert\cdot\rvert)$	$\|f\|_\Phi$
L_Φ^N	$L_\Phi([N], \frac{1}{N}\lvert\cdot\rvert)$	$\|f\|_{L_\Phi^N}$

For comparing quasi-Orlicz norms, we have the following result.

Proposition 2.3. *Let Φ and Ψ be normal quasi-Orlicz functions, and let $0 < C < \infty$.*
i) If $\tilde\Phi^{-1}(t) \le C\tilde\Psi^{-1}(t)$ for all $t \in [0, \infty)$, then $\|\cdot\|_\Phi \le C\|\cdot\|_\Psi$.
ii) There is a constant C', depending on the $\frac{1}{2}$-dilatory constant of Φ only, such that the following holds. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and set $\mu_{\min} = \inf\{\mu(A) : \mu(A) > 0\}$. If $\tilde\Phi^{-1}(t) \le C\tilde\Psi^{-1}(t)$ for $\mu_{\min} \le t \le \mu(\Omega)$, then $\|\cdot\|_{L_\Phi(\mu)} \le CC'\|\cdot\|_{L_\Psi(\mu)}$.

Proof: See Propositions 3A:3.6 and 3B:3.1 (see also [K–R Thm.8.1] or [L–T1 4.a.5]). □

2.4) Lorentz Spaces

These spaces are discussed in greater detail in Part 3.

Definition. *Let $(\Omega, \mathcal{F}, \mu)$ be any measure space, and $f : \Omega \to \mathbf{C}$ be a measurable function. Then the decreasing rearrangement of $|f|$ is*

$$f^*(t) = \sup\{ s : \mu\{|f| \ge s\} \ge t\},$$

that is, $f^* : [0, \infty) \to [0, \infty)$ is the left continuous inverse of the function $s \mapsto \mu\{|f| \ge s\}$. Thus, for example, if we have $f : [N] \to \mathbf{C}$ (where $[N]$ is given the counting measure), then $(f^*(n) : n \in [N])$ is the sequence of numbers $(|f(n)| : n \in [N])$ rearranged in decreasing order.

Definition. *Let Φ be a normal quasi-Orlicz function, $(\Omega, \mathcal{F}, \mu)$ be any measure space, and $0 < q \le \infty$. Then for any measurable $f : \Omega \to \mathbf{C}$, we define its $L_{\Phi,q}$ quasi-norm to be*

$$\begin{aligned}
\|f\|_{\Phi,q} &= \left(\int_0^\infty \left(f^*(\tilde\Phi(t^{\frac{1}{q}})) \right)^q dt \right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \left(f^*(t) \right) d\left(\tilde\Phi^{-1}(t) \right)^q \right)^{\frac{1}{q}} \qquad \text{if } q < \infty, \\
\|f\|_{\Phi,\infty} &= \sup_{t \in \mathbf{R}_+} \tilde\Phi^{-1}(t) f^*(t).
\end{aligned}$$

Proposition 2.4. *Let Φ be a normal quasi-Orlicz function, and $0 < q \le \infty$. Then $\|\cdot\|_{\Phi,q}$ is a quasi-norm.*

Proof: See Proposition 3A:5.3 and 3A:5.5. □

If Φ is a normal quasi-Orlicz function, and $0 < q \le \infty$, then we form the *Lorentz spaces.*

Symbol	Vectors	Quasi-norm
$\left.\begin{array}{l} L_{\Phi,q}(\Omega,\mathcal{F},\mu) \\ L_{\Phi,q}(\Omega,\mu) \\ L_{\Phi,q}(\Omega) \\ L_{\Phi,q}(\mu) \\ L_{\Phi,q} \end{array}\right\}$	Measurable functions on measure space $(\Omega,\mathcal{F},\mu)$ such that $\Vert f\Vert_{\Phi,q} < \infty$	$\Vert f\Vert_{L_{\Phi,q}} = \Vert f\Vert_{\Phi,q}$
$l_{\Phi,q}$	$L_{\Phi,q}(\mathbf{N})$	$\Vert f\Vert_{\Phi,q}$
$l^N_{\Phi,q}$	$L_{\Phi,q}([N],\lvert\cdot\rvert)$	$\Vert f\Vert_{\Phi,q}$
$L^N_{\Phi,q}$	$L_{\Phi,q}([N],\frac{1}{N}\lvert\cdot\rvert)$	$\Vert f\Vert_{L^N_{\Phi,q}}$

If $\Phi = T^p$, then $L_{\Phi,q} = L_{p,q}$, the familiar Lorentz quasi-norm. If $\Phi = T^p(\mathrm{lm}\, T)^\alpha$, then the $L_{\Phi,q}$ spaces are the same as the Lorentz–Zygmund spaces described in [B–R] or [B–S Ch.4 §6].

The properties of the $L_{p,q}$ norm are well documented (see for example [B–L], [B–S], [H], [L–T2] or [Lo]). If $p = q$ then $\Vert\cdot\Vert_{p,q} = \Vert\cdot\Vert_p$, and if $1 \le q \le p < \infty$ then $\Vert\cdot\Vert_{p,q}$ is a norm. We also have the following comparison result.

Proposition 2.5. *Let $0 < p < \infty$ and $0 < q_1 \le q_2 \le \infty$.*
i) $\Vert\cdot\Vert_{p,q_1} \ge \Vert\cdot\Vert_{p,q_2}$.
ii) $\Vert\cdot\Vert_{l^N_{p,q_1}} \le \left(1 + c\left(\frac{1}{p}\log N\right)^{\frac{1}{q_1}-\frac{1}{q_2}}\right) \Vert\cdot\Vert_{l^N_{p,q_2}}$.

Proof of i): This is well known. See the above references (see also Theorem 3B:2.1). □

Proof of ii): If $f \in \mathbf{C}^N$, then

$$\begin{aligned}
\Vert f\Vert_{p,q_1} &= \left(\frac{q_1}{p}\int_0^N t^{\frac{q_1}{p}-1}(f^*(t))^{q_1}\,dt\right)^{\frac{1}{q_1}} \\
&\le (2^{\frac{1}{q_1}} \vee 1)\left(f^*(1) + \left(\frac{q_1}{p}\int_1^N t^{\frac{q_1}{p}-1}(f^*(t))^{q_1}\,dt\right)^{\frac{1}{q_1}}\right) \\
&\le \Vert f\Vert_{p,q_2} + \left(\frac{q_1}{p}\right)^{\frac{1}{q_1}}\left(\frac{p}{q_2}\right)^{\frac{1}{q_2}}\left(\int_1^N t^{-1}\,dt\right)^{\frac{1}{q_1}-\frac{1}{q_2}}\left(\frac{q_2}{p}\int_1^N t^{\frac{q_2}{p}-1}(f^*(t))^{q_2}\,dt\right)^{\frac{1}{q_2}} \\
&\le \left(1 + c\left(\frac{1}{p}\log N\right)^{\frac{1}{q_1}-\frac{1}{q_2}}\right)\Vert f\Vert_{p,q_2}.
\end{aligned}$$

□

Now, we will give some elementary results on calculating $L_{\Phi,q}$ norms for spaces of functions from $[N]$.

Definition. *Let $N \in \mathbf{N}$, and $f\colon [N] \to \mathbf{C}$ be a function. A bijection $\pi\colon [N] \to [N]$ is called an ordering permutation for f if for all $m, n \in [N]$ we have*

$$|f(m)| \ge |f(n)| \Leftrightarrow \pi(m) \le \pi(n).$$

Thus $f^*(\pi(n)) = f(n)$.

Proposition 2.6. *Let $f \in \mathbf{C}^N$, $\pi\colon [N] \to [N]$ be an ordering permutation for f, $0 < p, q < \infty$, and $\alpha \in \mathbf{R}$.*
i) If $q \le p$ then

$$\left(\frac{q}{p}\sum_{n=1}^N \big(\pi(n)\big)^{\frac{q}{p}-1}|f(n)|^q\right)^{\frac{1}{q}} \le \Vert f\Vert_{p,q} \le \left(\sum_{n=1}^N \big(\pi(n)\big)^{\frac{q}{p}-1}|f(n)|^q\right)^{\frac{1}{q}},$$

and if $p \leq q$ then

$$\left(\sum_{n=1}^{N} \big(\pi(n)\big)^{\frac{q}{p}-1} |f(n)|^q\right)^{\frac{1}{q}} \leq \|f\|_{p,q} \leq \left(\frac{q}{p} \sum_{n=1}^{N} \big(\pi(n)\big)^{\frac{q}{p}-1} |f(n)|^q\right)^{\frac{1}{q}}.$$

In both cases, $\|f\|_{L^N_{p,q}} = N^{\frac{1}{p}} \|f\|_{p,q}$.

ii) There is a number $C < \infty$, depending on p, q and α only, such that

$$\|f\|_{T^p(\operatorname{lm} T)^\alpha,q} \overset{C}{\approx} \left(\sum_{n=1}^{N} \big(\pi(n)\big)^{\frac{q}{p}-1} \big(\operatorname{lm} \pi(n)\big)^{-\alpha\frac{q}{p}} |f(n)|^q\right)^{\frac{1}{q}},$$

$$\|f\|_{L^N_{T^p(\operatorname{lm} T)^\alpha,q}} \overset{C}{\approx} N^{-\frac{1}{p}} \left(\sum_{n=1}^{N} \big(\pi(n)\big)^{\frac{q}{p}-1} \big(\operatorname{lm}(N/\pi(n))\big)^{\alpha\frac{q}{p}} |f(n)|^q\right)^{\frac{1}{q}}.$$

Proof of i): Clearly

$$N^{\frac{q}{p}} \|f\|^q_{L^N_{p,q}} = \|f\|^q_{p,q} = \sum_{n=1}^{N} \left(\big(\pi(n)\big)^{\frac{q}{p}} - \big(\pi(n)-1\big)^{\frac{q}{p}}\right) |f(n)|^q,$$

and the result follows easily. □

Proof of ii): Let $\Phi = \tilde{\Phi} = T^p(\operatorname{lm} T)^\alpha$. By Proposition 1.1, there is a number $C < \infty$, such that $\Phi^{-1} = \tilde{\Phi}^{-1} \overset{C^{\frac{\alpha}{p}}}{\approx} (T/(\operatorname{lm} T)^\alpha)^{\frac{1}{p}}$. So

$$\begin{aligned}
\|f\|^q_{\Phi,q} &= \int_0^\infty (f^*(t))^q \, d(\tilde{\Phi}^{-1}(t))^q \\
&\overset{C^{\alpha\frac{q}{p}}}{\approx} \sum_{n=1}^{N} \left(\left(\frac{\pi(n)}{\big(\operatorname{lm}\pi(n)\big)^\alpha}\right)^{\frac{q}{p}} - \left(\frac{\pi(n)-1}{\big(\operatorname{lm}(\pi(n)-1)\big)^\alpha}\right)^{\frac{q}{p}}\right) |f(n)|^q \\
&\overset{C_1}{\approx} \sum_{n=1}^{N} \frac{\big(\pi(n)\big)^{\frac{q}{p}-1}}{\big(\operatorname{lm}\pi(n)\big)^{\alpha\frac{q}{p}}} |f(n)|^q.
\end{aligned}$$

for some number $C_1 < \infty$. The argument for $\|f\|_{L^N_{\Phi,q}}$ is similar. □

We also have the following result, that provides some comparison between the Orlicz and Lorentz quasi-norms.

Theorem 2.7. *(Bennett and Rudnick) Let $0 < p < \infty$, and $\alpha \in \mathbf{R}$.*

i) There is a number $C < \infty$, depending on p and α, such that

$$\|\cdot\|_{T^p(\operatorname{lm} T)^\alpha} \overset{C}{\approx} \|\cdot\|_{T^p(\operatorname{lm} T)^\alpha,p}.$$

ii) There is a number $C < \infty$, depending on α only, such that

$$\|\cdot\|_{\operatorname{em}(T^p)} \overset{C}{\approx} \|\cdot\|_{\operatorname{em}(T^p),\infty}.$$

Proof: See Theorem 3B:2.7. (See also [B–R] and [B–S Ch.4 §6].) □

Finally, we introduce notation for the natural embedding map.

Definition. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and L_A, L_B be any of the spaces described in Sections 2.1 to 2.4 such that for some number $C < \infty$, we have $\|\cdot\|_{L_A} \geq C^{-1} \|\cdot\|_{L_B}$. Then we denote the natural embedding $f \mapsto f$, from L_A to L_B, by $L_A \hookrightarrow L_B$.*

Examples are $C(K) \hookrightarrow L_{\Phi,q}(K,\mu)$ and $L^N_{\Phi_1,q_1} \hookrightarrow L^N_{\Phi_2,q_2}$.

2.5) Interpolation Norms

These norms are well known, see for example, [B–L] or [B–S].

Definition. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be two norms on $\mathbf{C}^N$. If $0 \leq t < \infty$ then*

i) the K-interpolation norm on $\mathbf{C}^N$ is defined by

$$K(a, t, \|\cdot\|_A, \|\cdot\|_B) = \inf\{\, \|a'\|_A + t\,\|a''\|_B : a' + a'' = a\};$$

ii) the J-interpolation norm on $\mathbf{C}^N$ is defined by

$$J(a, t, \|\cdot\|_A, \|\cdot\|_B) = \max\{\|a\|_A, t\,\|a\|_B\}.$$

Proposition 2.8. *Let $\|\cdot\|_A$ and $\|\cdot\|_B$ be two norms on $\mathbf{C}^N$, and let $\|\cdot\|^*_A$ and $\|\cdot\|^*_B$ be their dual norms respectively, with respect to the standard duality, $\langle \xi, x\rangle = \sum_{n=1}^N \xi(n)x(n)$. If $0 < t < \infty$, then the norms $K(\cdot, t, \|\cdot\|_A, \|\cdot\|_B)$ and $J(\cdot, \frac{1}{t}, \|\cdot\|_A \|\cdot\|_B)$ are also dual to one another.*

Proof: This is elementary. See [B–L Thm.2.71]. □

We also define a norm which we will use once in Chapter 2A. As far as I know, this norm, and the proposition following, are new.

Definition. *Let $M, N \in \mathbf{N}$. An M-partition of $[N]$ is a collection of M disjoint subsets of $[N]$, $\{\beta_1, \beta_2, \ldots, \beta_M\}$, such that $\bigcup_{m=1}^M \beta_m = [N]$. If $0 < p < q < \infty$, then the (p,q)-M-partition norm on $\mathbf{C}^N$ is defined by*

$$\|a\|_{P_{p,q}(M)} = \sup\left\{ \left(\sum_{m=1}^M \left\| a|_{\beta_m} \right\|_q^p \right)^{\frac{1}{p}} \right\},$$

where the supremum is over all M-partitions of $[N]$.

Proposition 2.9. *If $M \in \mathbf{N}$, and $0 < p < q < \infty$, then there is a number $C < \infty$ depending on p and q only, such that*

$$K(\cdot, M^{\frac{1}{p}-\frac{1}{q}}, \|\cdot\|_p, \|\cdot\|_q) \overset{C}{\approx} \|\cdot\|_{P_{p,q}(M)}.$$

Proof: First we show that $\|\cdot\|_{P_{p,q}(M)} \leq CK(\cdot, M^{\frac{1}{p}-\frac{1}{q}}, \|\cdot\|_p, \|\cdot\|_q)$. For it is easy to see that $\|\cdot\|_{P_{p,q}(M)} \leq \|\cdot\|_p$ and that $\|\cdot\|_{P_{p,q}(M)} \leq M^{\frac{1}{p}-\frac{1}{q}} \|\cdot\|_q$. Hence, if $a \in \mathbf{C}^N$, then

$$\begin{aligned} K(a, M^{\frac{1}{p}-\frac{1}{q}}, \|\cdot\|_p, \|\cdot\|_q) &= \inf\{\, \|a'\|_p + M^{\frac{1}{p}-\frac{1}{q}} \|a''\|_q : a = a' + a''\} \\ &\geq \inf\{\, \|a'\|_{P_{p,q}(M)} + \|a''\|_{P_{p,q}(M)} : a = a' + a''\} \\ &\geq C^{-1} \|a\|_{P_{p,q}(M)}, \end{aligned}$$

where the last inequality follows as $\|\cdot\|_{P_{p,q}(M)}$ is a quasi-norm.

For the converse inequality, suppose that $K(a, M^{\frac{1}{p}-\frac{1}{q}}, \|\cdot\|_p, \|\cdot\|_q) \geq 1$. We will let $b = |a|^p$, so that $K(b, M^{\frac{1}{r}}, \|\cdot\|_1, \|\cdot\|_{\frac{q}{p}}) \geq 1$, where $\frac{1}{r} + \frac{p}{q} = 1$. By the duality given in Proposition 2.8, there is a $\zeta \in \mathbf{C}^N$ such that $J(\zeta, M^{-\frac{1}{r}}, \|\cdot\|_\infty, \|\cdot\|_r) = 1$ and $\sum_{n=1}^N \zeta(n)b(n) \geq 1$.

Since $\sup_{n\in[N]} |\zeta(n)| \leq 1$ and $\sum_{n=1}^N |\zeta(n)|^r \leq M$, we can choose a M-partition of $[N]$, $\{\beta_1, \beta_2, \ldots, \beta_M\}$, such that $\sum_{n\in\beta_m} |\zeta(n)|^r \leq 2$ for each $m \in [M]$. Hence

$$\|b\|_{P_{1,\frac{q}{p}}(M)} \geq \sum_{m=1}^M \left\| b|_{\beta_m} \right\|_{\frac{q}{p}} \geq 2^{-\frac{1}{r}} \sum_{m=1}^M \sum_{n\in\beta_m} \zeta(n)b(n) \geq 2^{-\frac{1}{r}}.$$

Hence $\|a\|_{P_{p,q}(M)} \geq \|b\|^{\frac{1}{p}}_{P_{1,\frac{q}{p}}(M)} \geq 2^{\frac{1}{q}-\frac{1}{p}}$. □

2.6) Spanning Points of $C(K, l_1)$

As the last result of this section, we give a result due to Maurey.

Definition. *Let K be a compact Hausdorff space and $1 \le p < \infty$. Then the Banach space $C(K, l_p)$ is the space of sequences $a = \big((a_s)_{s=1}^{\infty} : a_s \in C(K)\big)$ such that*

$$\|a\|_{C(K,l_p)} = \left\| \left(\sum_{s=1}^{\infty} |a_s|^p \right)^{\frac{1}{p}} \right\|_{\infty} < \infty.$$

Proposition 2.10. *Let $T\colon C(K, l_1) \to Y$ be a linear operator to a normed space Y. Then $\|T\|$ is equal to the supremum of $\big\|T\big((a_1, a_2, \ldots, a_S, 0, 0, \ldots)\big)\big\|$, over all continuous functions a_1, $a_2, \ldots,$ a_S from K with disjoint support and $\|a_s\|_{\infty} \le 1$.*

Proof: See [Ma2 Lem.4] or [J2 Prop.14.4]. $\square$

3) Notation involving Ideals

The notion of an operator ideal is introduced in [Pt]. Our definition is essentially the same, but for ease we define the norm first.

Definition. *A quasi-ideal norm is a function, α, from the class of all bounded linear operators between Banach spaces, to $[0,\infty]$, such that the following hold.*

i) *If T is a finite rank operator, then $\alpha(T)<\infty$.*

ii) *There is a number $C<\infty$ such that for bounded linear operators $S, T\colon X\to Y$, we have $\alpha(S+T)\le C\big(\alpha(S)+\alpha(T)\big)$.*

iii) *If S is a bounded linear operator, and $\lambda\in\mathbf{C}$, then $\alpha(\lambda S)=|\lambda|\,\alpha(S)$.*

iv) *If $R\colon X\to Y$, $S\colon Y\to Z$, $T\colon Z\to W$ are bounded linear operators, then $\alpha(T\circ S\circ R)\le\|T\|\,\alpha(S)\,\|R\|$.*

The ideal *defined by α is the class of operators T such that $\alpha(T)<\infty$. If in (ii) we have $C=1$, then we say that α is an* ideal norm. *If X is any Banach space, we write $\alpha(X)$ for $\alpha(\mathrm{Id}_X)$.*

All the ideals we will use we introduce here. They are all standard (and can be found in [Pt]) except for the (Φ,Ψ)-summing norm.

3.1) Operators that Factor through Hilbert Space

This is one of the most elementary ideals. All the work in the first two parts of this thesis can be thought of as having stemmed from studying this particular ideal. Its definition is very simple.

Definition. *The Hilbert space factorization norm is defined by*

$$\gamma(T\colon X\to Y)=\inf\{\,\|R\|\,\|S\| : T=S\circ R,\ R\colon X\to H,\ S\colon H\to Y\},$$

where H is a Hilbert space.

We note that if X is a Banach space, then $\gamma(X)$ is the Banach–Mazur distance of X from Hilbert space.

3.2) The (p,q)-Summing Operators

Definition. *Suppose $0<q\le p<\infty$. The (p,q)-summing norm of a bounded linear operator $T\colon X\to Y$, denoted by $\pi_{p,q}(T)$, is the least number C such that for all $x_1, x_2,\dots, x_S\in X$ we have*

$$\left(\sum_{s=1}^S\|T(x_s)\|^p\right)^{\frac1p}\le C\sup\left\{\left(\sum_{s=1}^S|\langle\xi,x_s\rangle|^q\right)^{\frac1q} : \xi\in B_{X^*}\right\}.$$

The (p,p)-summing norm of T is simply called the p-summing norm of T and is denoted by $\pi_p(T)$. We say an operator is (p,q)-summing (p-summing) if its (p,q)-summing norm (p-summing norm) is finite.

We give the following well known identities.

Proposition 3.1. *Let X be a Banach space, and $x_1, x_2,\dots, x_S\in C(K)$.*

i)
$$\sup\left\{\sum_{s=1}^S|\langle\xi,x_s\rangle| : \xi\in B_{X^*}\right\}=\sup\left\{\left\|\sum_{s=1}^S\zeta_s x_s\right\| : \zeta_s=\pm1\right\}.$$

ii) *If $X=C(K)$ and $1\le q<\infty$, then*

$$\sup\left\{\left(\sum_{s=1}^S|\langle\xi,x_s\rangle|^q\right)^{\frac1q} : \xi\in B_{C(K)^*}\right\}=\left\|\left(\sum_{s=1}^S|x_s|^q\right)^{\frac1q}\right\|_\infty.$$

3.3) Type and Cotype of Operators

Definition. *Suppose $1 \leq p \leq 2 \leq q < \infty$. We say that a bounded linear operator $T\colon X \to Y$ has*

i) Rademacher (or Bernoulli) type p if for all $x_1, x_2, \ldots, x_S \in X$ we have

$$\left(\sum_{s=1}^{S} \|T(x_s)\|^p \right)^{\frac{1}{p}} \leq C\mathbf{E} \left\| \sum_{s=1}^{S} \varepsilon_s x_s \right\| ; \tag{3.1}$$

ii) Rademacher (or Bernoulli) cotype q if for all $x_1, x_2, \ldots, x_S \in X$ we have

$$\mathbf{E} \left\| \sum_{s=1}^{S} \varepsilon_s T(x_s) \right\| \leq C \left(\sum_{s=1}^{S} \|x_s\|^q \right)^{\frac{1}{q}} ; \tag{3.2}$$

iii) Gaussian type p if for all $x_1, x_2, \ldots, x_S \in X$ we have

$$\left(\sum_{s=1}^{S} \|T(x_s)\|^p \right)^{\frac{1}{p}} \leq C\mathbf{E} \left\| \sum_{s=1}^{S} \gamma_s x_s \right\| ; \tag{3.3}$$

iv) Gaussian cotype q if for all $x_1, x_2, \ldots, x_S \in X$ we have

$$\mathbf{E} \left\| \sum_{s=1}^{S} \gamma_s T(x_s) \right\| \leq C \left(\sum_{s=1}^{S} \|x_s\|^q \right)^{\frac{1}{q}} ; \tag{3.4}$$

where $C < \infty$. The Rademacher type p constant, Rademacher cotype q constant, Gaussian type p constant and Gaussian cotype q constant of the operator $T\colon X \to Y$ is the least number C in (3.1), (3.2), (3.3) and (3.4) respectively, and they are denoted by $R_p(T)$, $R^q(T)$, $G_p(T)$ and $G^q(T)$ respectively.

3.4) The (Φ, Ψ)-Summing Operators

Definition. *Suppose Φ and Ψ are normal quasi-Orlicz functions. The (Φ, Ψ)-summing norm of a bounded linear operator $T\colon X \to Y$, denoted by $\pi_{\Phi,\Psi}(T)$, is the least number C such that for all $x_1, x_2, \ldots, x_S \in X$ we have*

$$\left\| (\|T(x_s)\|)_{s=1}^S \right\|_\Phi \leq C \sup \left\{ \left\| (|\langle \xi, x_s \rangle|)_{s=1}^S \right\|_\Psi : \xi \in B_{X^*} \right\} .$$

We say an operator is (Φ, Ψ)-summing if its (Φ, Ψ)-summing norm is finite.

We will work only with the $(\Phi, 1)$-summing norm.

Chapter 1B — Properties of (p,q)-Summing Operators from $C(K)$

In this chapter, we discuss how (p,q)-summing operators factorize through certain operators and spaces. First we deal with p-summing operators, for which the theory is well known. Then we deal with the (p,q)-summing operators for $q < p$, presenting a recent result due to Pisier, and some old work due to Maurey. Finally, in Section 3, we give a finite dimensional version of Pisier's result that will be important later on.

1) The p-Summing Operators

The following result is basic to the theory of p-summing operators, and is due to Pietsch.

Theorem 1.1. *(See [P1 Thm.1.3 Rem.1.4], [Pt 17.3.2] or [J2 5.2].) Let $T: X \to Y$ be a bounded linear operator between Banach spaces, and let K be a subset of X^* that is norming ($\|x\|_X = \sup_{\xi \in K} \langle \xi, x\rangle$ for all $x \in X$), and closed (and hence compact) with respect to the weak * topology. Then for $1 \le p < \infty$ and $C < \infty$ the following are equivalent.*

i) T is p-summing with $\pi_p(T) \le C$.

ii) There is a Radon probability measure μ on K such that we have the following factorization:

$$T: X \xrightarrow{E} Z \xrightarrow{U} X,$$

where $E: X \to C(K)$ is the map $E(x) = (\xi \mapsto \langle \xi, x\rangle)$, Z is the closure of $E(X)$ in $L_p(K,\mu)$, and $\|U\| \le C$.

Corollary 1.1a. *(See [P1 Cor.1.5] or [Pt 17.3.3].) If $T: C(K) \to Y$ is a bounded linear operator, then, for $1 \le p < \infty$ and $C < \infty$, the following are equivalent.*

i) T is p-summing with $\pi_p(T) \le \infty$.

ii) There is a Radon probability measure μ on K such that we have the following factorization with $\|U\| \le C$:

$$T: C(K) \hookrightarrow L_p(\mu) \xrightarrow{U} Y.$$

Corollary 1.1b. *(See [P1 Cor.1.8] or [Pt 17.3.7].) If $T: X \to Y$ is 2-summing, then it factors through a Hilbert space, with $\gamma(T) \le \pi_2(T)$.*

For maps from $C(K)$ we have a converse to Corollary 1.1b.

Theorem 1.2. *(See [P1 Thm.5.4 and Remark following].) If $T: C(K) \to Y$ factors through Hilbert space, then it is 2-summing, with $\pi_2(T) \le 2\pi^{-\frac{1}{2}}\gamma(T)$. (The constant is $\sqrt{\frac{\pi}{2}}$ if scalars are real.)*

2) The (p,q)-Summing Operators, with $p > q$

The analogue of Pietsch's result for (p,q)-summing operators with $p > q$, is a theorem due to Pisier, which we state here.

Theorem 2.1. *(See [J2 14.1], [P3] or [P4].) Let $T\colon C(K) \to Y$ be a bounded linear operator to a Banach space, and $1 \le q < p < \infty$. Then the following are equivalent.*

i) T is (p,q)-summing.

ii) There is a Radon probability measure μ on K and a number $C < \infty$ such that for all $x \in C(K)$ we have

$$\|T(x)\| \le C \, \|x\|_{L_q(K,\mu)}^{\frac{q}{p}} \, \|x\|_\infty^{1-\frac{q}{p}} . \tag{2.1}$$

iii) There is a Radon probability measure μ on K and the factorization

$$T\colon C(K) \hookrightarrow L_{p,1}(K,\mu) \xrightarrow{U} X \tag{2.2}$$

where U is a bounded linear operator.

If we let $C_1 = \inf\{\, C : (2.1) \text{ holds for some } \mu\}$ and $C_2 = \inf\{\, \|U\| : (2.2) \text{ holds for some } \mu\}$, then

$$\pi_{p,1}(T) \le C_1 \le p^{\frac{1}{p}} \pi_{p,1}(T) \quad \text{and} \quad C_2 \le C_1 \le q^{-\frac{1}{p}} \left(\tfrac{p-1}{p-q}\right)^{1-\frac{1}{p}} C_2 .$$

Proof: For (i)⇒(ii), see the above references, or follow the methods of the proof of Theorem 1D:4.1. The implication (ii)⇒(i) is easy. For (ii)⇒(iii), use Lemma 1D:4.2.

We show (iii)⇒(ii). Suppose that $\|x\|_\infty \le 1$. Then

$$\begin{aligned} \|Tx\| &\le C \, \|x\|_{L_{p,1}(K,\mu)} \\ &= \int_0^1 (\mu\{|x| \ge s\})^{\frac{1}{p}} \, ds \\ &\le \left(q^{-\frac{1}{p-1}} \int_0^1 s^{-\frac{q-1}{p-1}} \, ds \right)^{1-\frac{1}{p}} \left(q \int_0^1 s^{q-1} \mu\{|x| \ge s\} \, ds \right)^{\frac{1}{p}} \\ &\le q^{-\frac{1}{p}} \left(\tfrac{p-1}{p-q}\right)^{1-\frac{1}{p}} \|x\|_{L_q(K,\mu)}^{\frac{q}{p}} . \end{aligned}$$

□

From this we may easily deduce the following result of Maurey.

Corollary 2.1a. *(See [Ma2 Prop.3].) If $T\colon C(K) \to Y$ is a bounded linear operator to a Banach space, and $1 \le q < p < \infty$, then T is (p,q)-summing if and only if it is $(p,1)$-summing, with*

$$\pi_{p,1}(T) \le \pi_{p,q}(T) \le \left(\tfrac{p}{q}\right)^{\frac{1}{p}} \left(\tfrac{p-1}{p-q}\right)^{1-\frac{1}{p}} \pi_{p,1}(T).$$

However this result cannot be extended to say that the $(p,1)$-summing norm is equivalent to the p-summing norm for operators from $C(K)$.

Proposition 2.2. *(See also [J1].) Let T be the map $l_\infty \hookrightarrow L_{p,1}^N$. Then $\pi_{p,1}(T) = 1$, whereas $\pi_p(T) \approx 1 + \left(\frac{1}{p}\log N\right)^{1-\frac{1}{p}}$.*

Proof: That $\pi_{p,1}(T) = 1$ follows straight away from Theorem 2.1. To show that $\pi_p(T) \le 1 + c\left(\frac{1}{p}\log N\right)^{1-\frac{1}{p}}$, notice that we have the factorization

$$l_\infty \hookrightarrow L_p^N \hookrightarrow L_{p,1}^N ,$$

where by Proposition 1A:2.5(ii) we have that

$$\left\| L_p^N \hookrightarrow L_{p,1}^N \right\| \le 1 + c\left(\tfrac{1}{p}\log N\right)^{1-\frac{1}{p}} .$$

The result then follows by Theorem 2.1.

To show the converse inequality, let $x_1,\, x_2, \ldots,\, x_N$ be the vectors

$$x_1 = \begin{pmatrix} 1^{-\frac{1}{p}} \\ 2^{-\frac{1}{p}} \\ \vdots \\ N^{-\frac{1}{p}} \end{pmatrix}, \quad x_2 = \begin{pmatrix} 2^{-\frac{1}{p}} \\ 3^{-\frac{1}{p}} \\ \vdots \\ 1^{-\frac{1}{p}} \end{pmatrix}, \ldots, \quad x_N = \begin{pmatrix} N^{-\frac{1}{p}} \\ 1^{-\frac{1}{p}} \\ \vdots \\ (N-1)^{-\frac{1}{p}} \end{pmatrix}.$$

Then

$$\left(\sum_{s=1}^{N} \|x_s\|_{L_{p,1}^N}^p \right)^{\frac{1}{p}} \approx 1 + \tfrac{1}{p}\log N,$$

whereas

$$\left\| \left(\sum_{s=1}^{N} |x_s|^p \right)^{\frac{1}{p}} \right\|_\infty \approx (1+\log N)^{\frac{1}{p}} .$$

The result follows (note that $p^{\frac{1}{p}} \approx 1$). □

Calculating the $(p,1)$-summing norm of a bounded linear operator from $C(K)$ to a Banach space is made easier by Proposition 1A:2.10.

Proposition 2.3. *(See [Ma2 Cor. to Lem.4] or [J2 Prop.14.4].) Let $T\colon C(K) \to Y$ be a bounded linear operator to a Banach space. Then*

$$\pi_{p,1}(T) = \sup \left\{ \left(\sum_{s=1}^{S} \|T(x_s)\|^p \right)^{\frac{1}{p}} \right\},$$

where the supremum is over all $x_1,\, x_2, \ldots,\, x_S \in C(K)$ with disjoint supports and $\|x_s\|_\infty \le 1$.

3) A Finite Dimensional Version of Pisier's Result

The following result, which I believe to be original, shows that if we are dealing with bounded linear operators from l_∞^N rather than from $C(K)$, then we need only consider one particular probability measure on $K = [N]$, the measure $\frac{1}{N}\,|\cdot|$. This result also works for Pietsch's Theorem. Its weakness is that it does not extend to deal with all rank N operators from $C(K)$.

Theorem 3.1. **Let α be an ideal quasi-norm, and define the sequence of positive real numbers $(\alpha_N)_{N=1}^\infty$ by*

$$\alpha_N = \alpha(l_\infty^N \hookrightarrow L_{p,1}^N).$$

Then for any bounded linear operator $T\colon l_\infty^N \to Y$ we have

$$\alpha(T) \le (2p)^{\frac{1}{p}}\, \alpha_{2N}\, \pi_{p,1}(T).$$

Proof: Suppose that the bounded linear operator $T\colon l_\infty^N \to Y$ satisfies $\pi_{p,1}(T) = C$. Then by Theorem 2.1, T factors as

$$T\colon l_\infty^N \hookrightarrow L_{p,1}([N],\mu) \xrightarrow{U} X,$$

where $\|U\| \le p^{\frac{1}{p}} C$ and μ is a probability measure on $[N]$. Define a new probability measure, ν, on $[N]$; let $\nu'(\{t\}) = \mu(\{t\})$ rounded up to the nearest multiple of $\frac{1}{2N}$, and let $\nu = \frac{1}{\nu'([N])}\nu'$. It is easy to see that $\nu \ge \frac{1}{2}\mu$, and so the map $L_{p,1}(\nu) \hookrightarrow L_{p,1}(\mu)$ has norm bounded by $2^{\frac{1}{p}}$.

Now define the maps

$$I\colon l_\infty^N \to l_\infty^{2N} \quad \text{and} \quad Q\colon L_{p,1}^{2N} \to L_{p,1}(\nu)$$

in the obvious fashion: let

$$I(e_n) = \sum_{m \epsilon A_n} e_m,$$

where

$$A_n = \left\{\, m : \nu([1,n-1]) < \tfrac{m}{2N} \le \nu([1,n]) \right\},$$

and let Q be the formal dual of I. It is easy to see that both I and Q are norm decreasing maps.

Then the following factorization of T gives the result.

$$T\colon l_\infty^N \xrightarrow{I} l_\infty^{2N} \hookrightarrow L_{p,1}^{2N} \xrightarrow{Q} L_{p,1}(\nu) \hookrightarrow L_{p,1}(\mu) \xrightarrow{U} X.$$

□

Corollary 3.1a. *Let $T\colon l_\infty^N \to Y$ be a bounded linear operator, and $p \ge 1$. Then*

$$\pi_p(T) \le c \cdot \left(1 + \left(\tfrac{1}{p}\log N\right)^{1-\frac{1}{p}}\right) \cdot \pi_{p,1}(T).$$

Proof: Use Proposition 2.2 and Theorem 3.1. □

This extends a result given in [J1] to $p \ne 2$. However, the result in [J1] also extends to all rank N operators from $C(K)$, which the above result does not.

* I would like to acknowledge a helpful suggestion made to me by G.J.O. Jameson.

Chapter 1C — Random Walks and Cotype

This chapter comes in five sections. The first two sections give a general introduction to random walks in Banach spaces and cotype of operators. Section 3 motivates the main problem of this thesis, that of the cotype 2 of operators from $C(K)$. It shows that the problem explores the 'edge' of established results. Section 4 looks at the comparison between Rademacher and Gaussian cotype, setting the second problem of this thesis. Finally, Section 5 presents solutions to these problems, deferring their proofs until Part 2.

1) Random Walks in Banach Spaces

Definition. *A (finite) random walk in a Banach space X is a random variable, $U = \sum_{s=1}^{S} \theta_s x_s$, where $S \in \mathbf{N}$, $x_1, x_2, \ldots, x_S \in X$, and $\theta_1, \theta_2, \ldots$ is either $\gamma_1, \gamma_2, \ldots$ (in which case U is call a Gaussian random walk) or $\varepsilon_1, \varepsilon_2, \ldots$ (in which case U is called a Bernoulli random walk).*

Our definition of a random walk differs from that usually given, in that we define it as a random variable and not as a sequence of random variables.

1.1) Scalar Valued Random Walks

First of all, we consider random walks in $\mathbf{C}$. The Gaussian case is easy, because $\sum_{s=1}^{S} \gamma_s x_s$ is itself a Gaussian random variable with mean 0 and standard deviation $\|(x_s)\|_2 = \left(\sum_{s=1}^{S} |x_s|^2\right)^{\frac{1}{2}}$. Thus we get the following simple results.

Proposition 1.1. *Let $x_1, x_2, \ldots, x_S \in \mathbf{C}$ and $1 \le p < \infty$. Then*

i)
$$\Pr\left(\left|\sum_{s=1}^{S} \gamma_s x_s\right| \ge t \|(x_s)\|_2\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}s^2}\, ds;$$

ii)
$$\left(\mathbf{E}\left|\sum_{s=1}^{S} \gamma_s x_s\right|^p\right)^{\frac{1}{p}} \approx \sqrt{p}\, \|(x_s)\|_2 .$$

For Bernoulli random walks, we have similar results.

Proposition 1.2. **Let $x_1, x_2, \ldots, x_S \in \mathbf{C}$ and $1 \le p < \infty$. Then*

i)
$$\Pr\left(\left|\sum_{s=1}^{S} \varepsilon_s x_s\right| \ge t \|(x_s)\|_2\right) \le c e^{-c^{-1}t^2};$$

ii)
$$c^{-1} \|(x_s)\|_2 \le \left(\mathbf{E}\left|\sum_{s=1}^{S} \varepsilon_s x_s\right|^p\right)^{\frac{1}{p}} \le \sqrt{p}\, \|(x_s)\|_2 .$$

Proof: See for example [Me Ch.II §59]. □

Corollary 1.2a. *(Khinchine's inequality.) Let $x_1, x_2, \ldots, x_S \in \mathbf{C}$.*

i) If $1 \le p \le 2$ then we have
$$\left(\mathbf{E}\left|\sum_{s=1}^{S} \varepsilon_s x_s\right|^p\right)^{\frac{1}{p}} \ge c^{-1} \|(x_s)\|_p .$$

ii) If $2 \le p < \infty$ then we have
$$\left(\mathbf{E}\left|\sum_{s=1}^{S} \varepsilon_s x_s\right|^p\right)^{\frac{1}{p}} \le c\sqrt{p}\, \|(x_s)\|_p .$$

* It may interest the reader to know that this result has applications to computer network design.

1.2) Banach Space Valued Random Walks

More generally, Gaussian and Bernoulli random walks have two properties. Firstly, the important quantities tend to be root mean squares. For example, we have the following.

Proposition 1.3. *Let X be a Banach lattice, and $x_1, x_2, \ldots, x_S \in X$.*

i) $\mathbf{E}\left\|\sum_{s=1}^{S} \theta_s x_s\right\| \geq c^{-1}\left\|\left(\sum_{s=1}^{S} |x_s|^2\right)^{\frac{1}{2}}\right\|$.

ii) If X is q-concave for some $q < \infty$, then there is a number $C < \infty$ such that

$$\left(\mathbf{E}\left\|\sum_{s=1}^{S} \theta_s x_s\right\|^q\right)^{\frac{1}{q}} \leq C\left\|\left(\sum_{s=1}^{S} |x_s|^2\right)^{\frac{1}{2}}\right\|.$$

Proof: See [L–T2 Thm.1.e.13]. □

This result (with Corollary 1.5b) completely describes the expectation of random walks in q-concave lattices. The second property is the concentration of measure phenomenon, that is, as a function of t, $\Pr\left(\left\|\sum_{s=1}^{S} \theta_s x_s\right\| \geq t\right)$ has a very fast decay rate. From this one deduces that $\left(\mathbf{E}\left\|\sum_{s=1}^{S} \theta_s x_s\right\|^p\right)^{\frac{1}{p}}$ has essentially the same value for all $1 \leq p < \infty$, and that this value is the same as its median. First, for Bernoulli random walks, we have the following results.

Theorem 1.4. *Let X be a Banach space, and $x_1, x_2, \ldots, x_S \in X$. Then*

$$\left\|\left\|\sum_{s=1}^{S} \varepsilon_s x_s\right\|\right\|_{L_{\mathrm{em}(T^2)}(\Pr)} \leq c\,\mathbf{E}\left\|\sum_{s=1}^{S} \varepsilon_s x_s\right\|.$$

Proof: See [Ka Ch.2 Thm.7]. □

Corollary 1.4a. *Let X be a Banach space, $x_1, x_2, \ldots, x_S \in X$, and $t > 0$. Then*

$$\Pr\left(\left\|\sum_{s=1}^{S} \theta_s x_s\right\| \geq t\mathbf{E}\left\|\sum_{s=1}^{S} \theta_s x_s\right\|\right) \leq ce^{-c^{-1}t^2}.$$

Proof: Use Theorem 1A:2.7(ii). □

For Gaussian random walks we have similar results.

Theorem 1.5. *Let X be a Banach space, and $x_1, x_2, \ldots, x_S \in X$. Define*

$$\mu = \mathbf{E}\left\|\sum_{s=1}^{S} \gamma_s x_s\right\| \quad \text{and} \quad \sigma = \sup\left\{\mathbf{E}\left|\left\langle \xi, \sum_{s=1}^{S} \gamma_s x_s\right\rangle\right|^2 : \xi \in B_{X^*}\right\}.$$

Then for all $t > 0$ we have

i)
$$\Pr\left(\left|\left\|\sum_{s=1}^{S} \gamma_s x_s\right\| - \mu\right| \geq t\sigma\right) \leq ce^{-c^{-1}t^2};$$

ii)
$$\Pr\left(\left\|\sum_{s=1}^{S} \gamma_s x_s\right\| \geq c^{-1}\mu + t\sigma\right) \geq c^{-1}e^{-ct^2}.$$

Part (i) of this result is better than its Bernoulli counterpart, Corollary 1.4a, because we have that $\sigma \leq c\mu$ (see below). It is a deep result due to C. Borel. Part (ii) shows that we cannot improve (i).

Proof: For (i), see [Bo] or [P2 Thm.2.1]. For (ii), we first show that $\sigma \le c\mu$. For if $\xi \in B_{X^*}$, then by Proposition 1.1(ii), we have

$$\begin{aligned}\left(\mathbf{E}\left|\left\langle \xi, \left\|\sum_{s=1}^S \gamma_s x_s\right\|\right\rangle\right|^2\right)^{\frac12} &\approx \left(\sum_{s=1}^S |\langle \xi, x_s\rangle|^2\right)^{\frac12} \\ &\approx \mathbf{E}\left|\left\langle \xi, \left\|\sum_{s=1}^S \gamma_s x_s\right\|\right\rangle\right| \\ &\le \mathbf{E}\left\|\sum_{s=1}^S \gamma_s x_s\right\|.\end{aligned}$$

Now we show that there is a universal constant $0 < \alpha < 1$ such that

$$\Pr\left(\left\|\sum_{s=1}^S \gamma_s x_s\right\| \ge \tfrac12\mu\right) \ge \alpha,$$

This is because if $\alpha' = \Pr\left(\left\|\sum_{s=1}^S \gamma_s x_s\right\| \ge \frac12\mu\right)$, then

$$\begin{aligned}\mu &= \mathbf{E}\left\|\sum_{s=1}^S \gamma_s x_s\right\| \\ &= \int_{\left(\left\|\sum_{s=1}^S \gamma_s x_s\right\| \ge \frac12\mu\right)} \left\|\sum_{s=1}^S \gamma_s x_s\right\| \, d\Pr + \int_{\left(\left\|\sum_{s=1}^S \gamma_s x_s\right\| < \frac12\mu\right)} \left\|\sum_{s=1}^S \gamma_s x_s\right\| \, d\Pr\end{aligned}$$

which, as $\sigma \le c\mu$, and by part (i) is

$$\le \int_0^{\alpha'} \left(\mu + c\mu\sqrt{\log^+ c/t}\right) dt + \tfrac12\mu.$$

But there is a universal constant $\alpha > 0$ such that for $\alpha' < \alpha$, we have

$$\int_0^{\alpha'} \left(1 + c\sqrt{\log^+ c/t}\right) dt \le \tfrac16,$$

and we are done.

Finally we prove part (ii). If $t\sigma \le \frac14\mu$, then by the above

$$\Pr\left(\left\|\sum_{s=1}^S \gamma_s x_s\right\| \ge \tfrac14\mu + t\sigma\right) \ge \alpha \ge c^{-1}e^{-ct^2}.$$

If $t\sigma > \frac14\mu$, then

$$\Pr\left(\left\|\sum_{s=1}^S \gamma_s x_s\right\| \ge \tfrac14\mu + t\sigma\right) \ge \Pr\left(\left\langle \xi, \sum_{s=1}^S \gamma_s x_s\right\rangle \ge \tfrac12 t\sigma\right),$$

where $\xi \in B_{X^*}$ is such that $\mathbf{E}\left|\left\langle \xi, \sum_{s=1}^S \gamma_s x_s\right\rangle\right|^2 = \sigma^2$, and this dominates $c^{-1}e^{-ct^2}$ by Proposition 1.1(i). $\square$

Corollary 1.5a. *Let X be a Banach space, and $x_1, x_2, \ldots, x_S \in X$. Then*

$$\left\|\left\|\sum_{s=1}^S \gamma_s x_s\right\|\right\|_{L_{\mathrm{em}(T^2)}(\Pr)} \le c\mathbf{E}\left\|\sum_{s=1}^S \gamma_s x_s\right\|.$$

Proof: Use Theorem 1A:2.7(ii). $\square$

Thus we derive the following results for both Bernoulli and Gaussian random walks.

Corollary 1.5b. *Let X be a Banach space, $x_1, x_2, \ldots, x_S \in X$, and $1 \le p < \infty$. Then*

$$\mathbf{E}\left\|\sum_{s=1}^{S}\theta_s x_s\right\| \le \left(\mathbf{E}\left\|\sum_{s=1}^{S}\theta_s x_s\right\|^p\right)^{\frac{1}{p}} \le c\sqrt{p}\,\mathbf{E}\left\|\sum_{s=1}^{S}\theta_s x_s\right\|.$$

Proof: This follows by Theorem 1.4 or Corollary 1.5a, and Proposition 1A:2.3(i), as for all $t \le 1$ we have $\sqrt{\operatorname{lm} t} \le c\sqrt{p}t^{\frac{1}{p}}$. □

Definition. *Let U be a real valued random variable, and $0 < \alpha < 1$. Define the upper α-tile of U, $M_\alpha(U)$, to be a number M such that $\Pr(U \le M) \ge 1 - \alpha$ and $\Pr(U \ge M) \ge \alpha$. Define the median to be $M_{\frac{1}{2}}(U)$.*

Corollary 1.5c. *Given $0 < \alpha < 1$, there is a number $C < \infty$ such that for all Banach spaces X, and x_1, $x_2, \ldots, x_S \in X$, we have*

$$\alpha M_\alpha\left(\left\|\sum_{s=1}^{S}\theta_s x_s\right\|\right) \le \mathbf{E}\left\|\sum_{s=1}^{S}\theta_s x_s\right\| \le C M_\alpha\left(\left\|\sum_{s=1}^{S}\theta_s x_s\right\|\right).$$

Proof: This follows straight away from Corollary 1.4a or Theorem 1.5. □

1.3) Random Walks in l_∞^N

The space l_∞^N is not q-concave, and hence we cannot apply Proposition 1.3(ii). However, we do have the following simple results.

Proposition 1.6. *Let $x_1, x_2, \ldots, x_S \in l_\infty^N$. Then*

i)
$$\mathbf{E}\left\|\sum_{s=1}^{S}\theta_s x_s\right\|_\infty \le c\sqrt{\operatorname{lm} N}\left\|\left(\sum_{s=1}^{S}|x_s|^2\right)^{\frac{1}{2}}\right\|_\infty;$$

ii)
$$\mathbf{E}\left\|\sum_{s=1}^{S}\varepsilon_s x_s\right\|_\infty \le \left\|\sum_{s=1}^{S}|x_s|\right\|_\infty.$$

Proof: Part (i) follows from Proposition 1.1(i) or Proposition 1.2(i) (See also [Ka Ch.6 Thm.1]). Part (ii) is trivial. □

These results are the best possible. For example, in part (i), if $N = 2^S$, let

$$x_s = \left(\operatorname{sign}\cos\left(2^s\pi\tfrac{n-1}{N-1}\right)\right)_{n=1}^{N} \quad \text{for } 1 \le s \le S.$$

Then $\left\|\sum_{s=1}^{S}\varepsilon_s x_s\right\|_\infty = S$, whereas $\left(\sum_{s=1}^{S}|x_s|^2\right)^{\frac{1}{2}} = \sqrt{S}$.

2) Type and Cotype of Operators and Banach Spaces

If X is isomorphic to Hilbert space, then we have the well known parallelogram law: for some number $C < \infty$ and for all $x_1, x_2, \ldots, x_S \in X$ we have

$$C^{-1}\left(\sum_{s=1}^{S}\|x_s\|_X^2\right)^{\frac{1}{2}} \leq \mathbf{E}\left\|\sum_{s=1}^{S}\varepsilon_s x_s\right\|_X \leq C\left(\sum_{s=1}^{S}\|x_s\|_X^2\right)^{\frac{1}{2}}.$$

The type and cotype measure to what extent a space or operator deviates from this law. For example, the L_p spaces obey laws weaker than the parallelogram law, as can easily be deduced from Khinchine's inequality: for all $x_1, x_2, \ldots, x_S \in L_p$ we have

$$c^{-1}\left(\sum_{s=1}^{S}\|x_s\|_p^2\right)^{\frac{1}{2}} \leq \mathbf{E}\left\|\sum_{s=1}^{S}\varepsilon_s x_s\right\|_p \leq c\left(\sum_{s=1}^{S}\|x_s\|_p^p\right)^{\frac{1}{p}}, \quad 1 \leq p \leq 2,$$

$$c^{-1}\frac{1}{\sqrt{p}}\left(\sum_{s=1}^{S}\|x_s\|_p^p\right)^{\frac{1}{p}} \leq \mathbf{E}\left\|\sum_{s=1}^{S}\varepsilon_s x_s\right\|_p \leq c\sqrt{p}\left(\sum_{s=1}^{S}\|x_s\|_p^2\right)^{\frac{1}{2}}, \quad 2 \leq p < \infty.$$

Thus L_p has type p and cotype 2 for $1 \leq p \leq 2$, and type 2 and cotype p for $2 \leq p < \infty$. Looking at the unit vectors in L_p shows that these results are the best we can do. Indeed we have the following famous result due to Maurey and Pisier.

Theorem 2.1. *(See [Ma–P] or [Mi–Sr].) Let X be a Banach space. Let*

$$p_X = \sup\{\, p : R_p(X) < \infty\} \quad , \quad q_X = \inf\{\, q : R^q(X) < \infty\},$$

and

$$S = \{\, p \in [1, \infty] : X \textit{ contains uniformly isomorphic copies of } l_p^N \}.$$

Then

$$[p_X, 2] \cup \{q_X\} \subseteq S \subseteq [p_X, q_X].$$

As the names suggest, type and cotype are in some sense dual to one another. If $T\colon X \to Y$ is a bounded linear operator, and $\frac{1}{p} + \frac{1}{p'} = 1$ where $1 < p \leq 2$, then $R^{p'}(T^*) \leq R_p(T)$ and $G^{p'}(T^*) \leq G_p(T)$. However, this duality lies only in one direction: one cannot deduce that if T has cotype p', then T^* has type p. Consider, for example $C(K)$. It has no type or cotype, indeed

$$R^p(l_\infty^N) \geq N^{\frac{1}{p}} \quad (p \geq 2),$$
$$R_p(l_\infty^N) \approx (\operatorname{lm} N)^{1-\frac{1}{p}} \quad (p \leq 2).$$

However, its dual, whose finite dimensional structure is like l_1, has cotype 2.

2.1) Kwapien's Result

The parallelogram law given above (which states that X has type 2 and cotype 2) characterizes Hilbert space. This famous result is due to Kwapien, and we state it in more generality below.

Theorem 2.2. *Let $S\colon X \to Y$ and $T\colon Y \to Z$ be bounded linear operator between Banach spaces. If S has Rademacher type 2, and T has Rademacher cotype 2, then $T \circ S$ factors through a Hilbert space, with $\gamma(T \circ S) \leq R_2(S)R^2(T)$.*

We can now deduce results about operators from L_p for $2 \leq p < \infty$.

Corollary 2.2a. *Let $2 \leq p < \infty$, and $T\colon L_p \to Y$ be a bounded linear operator to a Banach space. If T has cotype 2, then it factors through a Hilbert space.*

This suggests similar results for bounded linear operators from $C(K)$.

2.2) Grothendieck's Theorem

The space $C(K)$ does not have type 2, but it almost does — its dual has cotype 2, and indeed the type 2 constant of l_∞^N is not very big ($\sqrt{\operatorname{lm} N}$). Correspondingly, there is a slightly weaker result then Corollary 2.2a.

Theorem 2.3. *(See [P1 Thm.4.1].) Let X and Y be Banach spaces such that X^* and Y have cotype 2. Then if $T: X \to Y$ is a bounded linear operator with the approximation property (see, for example, [P1 Ch.0] or [Pt 1.3] for the definition of the approximation property; all bounded linear operators from $C(K)$ have this property), then T factors through a Hilbert space.*

Corollary 2.3a. *Let $T: C(K) \to Y$ be a bounded linear operator to a Banach space. If T factors through a space with cotype 2, then T factors through a Hilbert space (or, equivalently, is 2-summing).*

Indeed one can show that $\gamma(T) \leq cR^2(Y)\sqrt{\operatorname{lm} R^2(Y)}$, as we will show in Theorem 1D:3.1

2.3) Cotype p of Operators from $C(K)$ for $p > 2$

In this section, we present work due to Maurey, which establishes equivalent conditions for an operator from $C(K)$ to have cotype p for $p > 2$. First, it is easy to relate the cotype p of such an operator to its $(p, 1)$-summing and $(p, 2)$-summing norms as follows.

Proposition 2.4. *Let $T: C(K) \to Y$ be a bounded linear operator to a Banach space. Then for $p \geq 2$ we have that $\pi_{p,1}(T) \leq R^p(T) \leq c\, \pi_{p,2}(T)$.*

Proof: This follows from Proposition 1.3(i) and Proposition 1.6(ii). □

From this, we derive the following result.

Theorem 2.5. *(See [Ma2 Thm.2]) Let $T: C(K) \to Y$ be a bounded linear operator to a Banach space, and $p > 2$. Then the following are equivalent.*
i) T is $(p, 1)$-summing.
ii) T is $(p, 2)$-summing.
iii) T has Rademacher cotype p.
iv) T factors through a space of Rademacher cotype p.

Proof: It follows immediately from Proposition 2.4 and Corollary 1B:2.1a that (i)⇒(ii)⇒(iii)⇒(i). Clearly (iv)⇒(iii), and (i)⇒(iv) follows from the fact that $L_{p,1}$ has cotype p for $p > 2$ (see [Cr]) □

3) The Problem of Cotype 2

Now, we are ready to present the main problem of this thesis. The following question was asked by Jameson [J1].

Question 3.1. *Let $T\colon C(K) \to Y$ be a bounded linear operator to a Banach space. Do either of the following hold?*

i) If T is $(2,1)$-summing, then T has cotype 2.

ii) If T has cotype 2, then it is 2-summing.

We see from Proposition 2.4 that if T is 2-summing, then it has cotype 2, and that if T has cotype 2, then it is $(2,1)$-summing. This question can be seen as exploring what happens in Theorem 2.5 in the case $p = 2$, when, by Proposition 1B:2.2, (i) and (ii) are no longer equivalent conditions. Part (ii) would also be an obvious common extension of Corollary 2.2a and Corollary 2.3a.

Since having cotype 2, or being 2-summing or $(2,1)$-summing depends only on the finite dimensional sub-structure, the above question is equivalent to the following.

Question 3.2. *Let $T\colon l_\infty^N \to Y$ be a bounded linear operator to a Banach space. Do either of the following hold?*

i) $R^2(T) \le c\,\pi_{2,1}(T)$.

ii) $\pi_2(T) \le c\,R^2(T)$.

By Proposition 1B:2.2, we can immediately see that (i) and (ii) cannot both be true, indeed to have a negative answer to (ii), we only need that $R^2(T) \le o(\sqrt{\operatorname{lm} N})\,\pi_{2,1}(T)$.

We can also relate this question to random walks in l_∞. By Theorem 1B:2.1 and Theorem 1B:3.1, showing (i) is equivalent to finding a positive answer to the following question.

Question 3.3. *Is $R^2(l_\infty^N \hookrightarrow L_{2,1}^N) \approx 1$, that is, is it true that*

$$\mathbf{E}\left\|\sum_{s=1}^S \varepsilon_s x_s\right\|_\infty \ge c^{-1}\left(\sum_{s=1}^S \|x_s\|_{L_{2,1}^N}^2\right)^{\frac12}$$

for all $x_1, x_2, \dots, x_S \in \mathbf{C}^N$?

This question can be thought of as trying to find a partial converse to Proposition 1.6(i) (as $\|\cdot\|_{L_{2,1}^N} \le c\sqrt{\operatorname{lm} N}\,\|\cdot\|_{L_2^N}$).

We will provide a partial answer to these problems, which we will present in Section 5.1. Part (i) of Questions 3.1 and 3.2, and Question 3.3 almost hold; we have $R^2(l_\infty^N \hookrightarrow L_{2,1}^N) \le \operatorname{lm}\operatorname{lm} N$. Thus we have negative answers to part (ii) of Questions 3.1 and 3.2.

Finally, as an aside, we show that we cannot ask for any more than a positive answer to Question 3.3 would provide.

Proposition 3.4. *Let $\|\cdot\|_*$ be a symmetric norm on $\mathbf{C}^N$ such that for all $x_1, x_2, \dots, x_S \in \mathbf{C}^N$ we have*

$$\mathbf{E}\left\|\sum_{s=1}^S \varepsilon_s x_s\right\|_\infty \ge \left(\sum_{s=1}^S \|x_s\|_*^2\right)^{\frac12}.$$

Then $\|\cdot\|_* \le c\,\|\cdot\|_{L_{2,1}^N}$.

Proof: If $k \in [N]$, let $S = \lfloor N/k \rfloor$, and $x_1, x_2, \dots, x_S$ be the disjoint vectors $x_s = \chi_{[(s-1)k+1,sk]}$. Then it is easy to see that

$$\mathbf{E}\left\|\sum_{s=1}^S \varepsilon_s x_s\right\|_\infty = 1,$$

and hence $\left\|\chi_{[1,k]}\right\|_* \le c\sqrt{\frac{k}{n}}$. The result follows by Lemma 1D:4.2. $\square$

We also have a similar result for Gaussian random walks.

Proposition 3.5. *Let $\|\cdot\|_*$ be a symmetric norm on $\mathbf{C}^N$ such that for all $x_1, x_2, \ldots, x_S \in \mathbf{C}^N$ we have*

$$\mathbf{E}\left\|\sum_{s=1}^{S} \gamma_s x_s\right\|_\infty \geq \left(\sum_{s=1}^{S} \|x_s\|_*^2\right)^{\frac{1}{2}}.$$

Then $\|\cdot\|_* \leq c\,\|\cdot\|_{L^N_{T^2 \operatorname{lm} T,1}}$.

Proof: If $k \in [N]$, let $S = \lfloor N/k \rfloor$, and $x_1, x_2, \ldots, x_S$ be the disjoint vectors $x_s = \chi_{[(s-1)k+1, sk]}$. Then by Proposition 1.6(i), we have

$$\mathbf{E}\left\|\sum_{s=1}^{S} \gamma_s x_s\right\|_\infty \leq \sqrt{\operatorname{lm}(N/k)},$$

and hence $\left\|\chi_{[1,k]}\right\|_* \leq c\sqrt{(\operatorname{lm}(N/k)\frac{k}{n}}$. The result follows by Lemma 1D:4.2. ☐

We also have a similar result for random walks in L_p.

Proposition 3.6. *Let $\|\cdot\|_*$ be a 1-unconditional norm on $\mathbf{C}^N$, and $2 \leq p < \infty$. Suppose that for all x_1, $x_2, \ldots, x_S \in \mathbf{C}^N$, we have*

$$\mathbf{E}\left\|\sum_{s=1}^{S} \varepsilon_s x_s\right\|_{L_p^N} \geq \left(\sum_{s=1}^{S} \|x_s\|_*^2\right)^{\frac{1}{2}}.$$

Then there is a probability measure μ on $[N]$, and a number $C < \infty$, such that $\|\cdot\|_* \leq C\,\|\cdot\|_{L_2(\mu)}$. If $\|\cdot\|_*$ is a 1-symmetric norm, then $\|\cdot\|_* \leq \|\cdot\|_{L_2^N}$.

Proof: The hypothesis of the proposition says that the map $L_p^N \hookrightarrow (\mathbf{C}^N, \|\cdot\|_*)$ has cotype 2 constant bounded by 1. Hence, we can apply Corollary 2.2a and deduce that this map factors as $V \circ U$ through a Hilbert space H, where for some number $C < \infty$ we have

$$\left\|V\colon H \to (\mathbf{C}^N, \|\cdot\|_*)\right\| \leq C,$$

$$\left\|U\colon L_p^N \to H\right\| \leq 1.$$

Define a Euclidean norm $\|\cdot\|_a$ on $\mathbf{C}^N$ by

$$\|x\|_a = \|U(x)\|_H,$$

so that $\|\cdot\|_* \leq \|\cdot\|_a \leq C\,\|\cdot\|_{L_p^N}$. Now define the Euclidean norm $\|\cdot\|_b$ on $\mathbf{C}^N$ by

$$\|x\|_b = \left(\mathbf{E}\left\|\sum_{n=1}^{N} \varepsilon_n x(n) e_n\right\|_a^2\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{N} |x(n)|^2 \|e_n\|_a^2\right)^{\frac{1}{2}},$$

where $e_1, e_2, \ldots, e_N$ are the unit vectors of $\mathbf{C}^N$. Since $\|\cdot\|_*$ and $\|\cdot\|_{L_p^N}$ are 1-unconditional, we have $\|\cdot\|_* \leq \|\cdot\|_b \leq C\,\|\cdot\|_{L_p^N}$. So

$$\left(\sum_{n=1}^{N} \|e_n\|_a^2\right)^{\frac{1}{2}} = \left\|\sum_{n=1}^{N} e_n\right\|_b \leq C.$$

If we let μ be the probability measure on $[N]$ defined by

$$\mu\{n\} = \frac{\|e_n\|_a^2}{\sum_{n=1}^{N} \|e_n\|_a^2},$$

then we see that $\|\cdot\|_* \leq \|\cdot\|_b \leq C\,\|\cdot\|_{L_2(\mu)}$.

If we also have that $\|\cdot\|_*$ is 1-symmetric, then

$$\|x\|_* \leq C\left(\frac{1}{N!}\sum_{\sigma\in S_N}\left\|\sum_{n=1}^{N} e_n x(\sigma(n))\right\|_{L_2(\mu)}^2\right)^{\frac{1}{2}} = C\,\|x\|_{L_2^N}\,.$$

$\square$

4) Comparison of Gaussian and Rademacher Cotype

Here we introduce the second main problem of this thesis. Most interest in cotype lies only with Rademacher cotype. However for the operators that we are considering, it is much easier to calculate Gaussian cotype. If we could deduce results about Rademacher cotype from results about Gaussian cotype, we would save ourselves much work.

4.1) Comparison of Gaussian and Bernoulli Random Walks

The first result states that Bernoulli random walks do not travel as far as Gaussian random walks.

Proposition 4.1. *Let X be a Banach space, and let $\|\cdot\|_A$ be a rearrangement invariant norm on the space of all scalar valued random variables. If $x_1,\, x_2, \dots,\, x_S \in X$, and $t > 0$, then*

i)
$$\left\|\left\|\sum_{s=1}^S \varepsilon_s x_s\right\|_X\right\|_A \le c \left\|\left\|\sum_{s=1}^S \gamma_s x_s\right\|_X\right\|_A ;$$

ii)
$$\Pr\left(\left\|\sum_{s=1}^S \varepsilon_s x_s\right\|_X \ge t\right) \le c \Pr\left(\left\|\sum_{s=1}^S \gamma_s x_s\right\|_X \ge c^{-1} t\right) .$$

Part (i) is usually stated in less generality as follows: for all $1 \le p < \infty$ we have

$$\left(\mathbf{E}\left\|\sum_{s=1}^S \varepsilon_s x_s\right\|_X^p\right)^{\frac{1}{p}} \le c \left(\mathbf{E}\left\|\sum_{s=1}^S \gamma_s x_s\right\|_X^p\right)^{\frac{1}{p}} .$$

However, we require the greater generality in order to prove (ii). The proof of (ii) also depends heavily on Theorem 1.5, and so it seems to be a deep result. I suspect that there is a better proof, but I cannot find it.

Proof of i): This is a slight modification of a well known method (see, for example [Mi–Sn §9.4]). Let us write the underlying probability space as $\Omega_\gamma \times \Omega_\varepsilon$, where $\gamma_1,\, \gamma_2, \dots$ depend only on Ω_γ, and $\varepsilon_1,\, \varepsilon_2, \dots$ depend only on Ω_ε. Let $\Pr_\gamma$ be the projection of $\Pr$ onto Ω_γ, and $\Pr_\varepsilon$ be the projection of $\Pr$ onto Ω_ε, so that $\Pr = \Pr_\gamma \times \Pr_\varepsilon$.

First we establish that the operator

$$L_A(\Omega_\gamma \times \Omega_\varepsilon) \to L_A(\Omega_\varepsilon); \quad V \mapsto \int_{\Omega_\gamma} V \, d\Pr{}_\gamma$$

is a contraction, where $L_A(\Omega)$ denotes the space of measurable functions $f\colon \Omega \to \mathbf{C}$ such that $\|f\|_A < \infty$, with norm $\|\cdot\|_A$. But, this is clearly so if $\|\cdot\|_A = \|\cdot\|_1$ or $\|\cdot\|_\infty$, and the general case then follows by the interpolation result given in [L–T2 2.a.10].

Now notice that $\left\|\sum_{s=1}^S \gamma_s x_s\right\|_X$ has the same law as $\left\|\sum_{s=1}^S \varepsilon_s \left|\gamma_s\right| x_s\right\|_X$. Also notice, that as $\|\cdot\|_X$ is a norm, we have,

$$\begin{aligned}\int_{\Omega_\gamma} \left\|\sum_{s=1}^S \varepsilon_s \left|\gamma_s\right| x_s\right\|_X d\Pr{}_\gamma &\ge \left\|\sum_{s=1}^S \varepsilon_s \int_{\Omega_\gamma} \left|\gamma_s\right| \, d\Pr{}_\gamma \, x_s\right\|_X \\ &= \sqrt{2/\pi} \left\|\sum_{s=1}^S \varepsilon_s x_s\right\|_X .\end{aligned}$$

Hence

$$\begin{aligned}\left\|\left\|\sum_{s=1}^S \gamma_s x_s\right\|_X\right\|_A &\ge \left\|\int_{\Omega_\gamma} \left\|\sum_{s=1}^S \varepsilon_s \left|\gamma_s\right| x_s\right\|_X d\Pr{}_\gamma\right\|_A \\ &\ge \sqrt{2/\pi} \left\|\left\|\sum_{s=1}^S \varepsilon_s x_s\right\|_X\right\|_A .\end{aligned}$$

$\square$

Proof of ii): First we establish some notation. If U is a scalar valued random variable, let U^* be its decreasing rearrangement, and $U^{**}(t) = \int_0^t U^*(u)\,du$. Note that the map $U \mapsto U^{**}(t)$ is a rearrangement invariant norm on the space of random variables, for each $t \in (0,\infty]$.

Now let $U = \left\|\sum_{s=1}^S \varepsilon_s x_s\right\|$ and $V = \left\|\sum_{s=1}^S \gamma_s x_s\right\|$. It is sufficient to show that

$$U^*(t) \le cV^*(c^{-1}t) \quad \text{for } t \in [0,\infty].$$

However, by Theorem 1.5, we have that $c^{-1}\bigl(\mu + \sigma\sqrt{\log^+ c^{-1}/t}\bigr) \le V^*(t) \le c\bigl(\mu + \sigma\sqrt{\log^+ c/t}\bigr)$, and hence $V^{**}(t) \le cV^*(c^{-1}t)$. Furthermore, it is obvious that $U^*(t) \le U^{**}(t)$. Finally, by (i), $U^{**}(t) \le cV^{**}(t)$, and the result follows. □

The converse inequalities are weaker.

Proposition 4.2. *Let X be a Banach space, and $x_1, x_2, \ldots, x_S \in X$.*

i)
$$\mathbf{E}\left\|\sum_{s=1}^S \gamma_s x_s\right\| \le c\sqrt{\operatorname{lm} S}\,\mathbf{E}\left\|\sum_{s=1}^S \varepsilon_s x_s\right\|.$$

ii) If X has cotype q for some $q < \infty$, then for some number $C < \infty$, depending on X only, we have

$$\mathbf{E}\left\|\sum_{s=1}^S \gamma_s x_s\right\| \le C\,\mathbf{E}\left\|\sum_{s=1}^S \varepsilon_s x_s\right\|.$$

iii) If $X = l_\infty^N$, then

$$\mathbf{E}\left\|\sum_{s=1}^S \gamma_s x_s\right\| \le c\sqrt{\operatorname{lm} N}\,\mathbf{E}\left\|\sum_{s=1}^S \varepsilon_s x_s\right\|.$$

Proof: For (i) and (ii), see [Mi–Sn Append.II]. Part (iii) follows straight away from Proposition 1.3(i) and Proposition 1.6(i).

□

4.2) Comparison of Rademacher and Gaussian Cotype

For comparison of Rademacher and Gaussian cotypes we have the following results.

Proposition 4.3. *Let $T\colon X \to Y$ be a bounded linear operator between Banach spaces, and $1 \le p \le 2 \le q < \infty$.*

i) $R_p(T) \le c\,G_p(T)$.
ii) $G^q(T) \le c\,R^q(T)$.
iii) $G_p(T) \le c\,R_p(T)$.
iv) If X has cotype q_1 for some $q_1 < \infty$, then there is a number $C < \infty$, depending on X only, such that $R^q(T) \le C\,G^q(T)$.
v) If $X = l_\infty^N$, then $R^q(T) \le c\sqrt{\operatorname{lm} N}\,G^q(T)$.

Proof: Parts (i) and (ii) follow straight away from Proposition 4.1(i). For (iii), see [Mi–Sn §9.4]. Parts (iv) and (v) follow straight away from Proposition 4.2.

□

It would be of great value if one could give a better converse to part (ii) than is provided by parts (iv) or (v), even if we had to put some strong condition on the image of the operator T. Thus we formulate the following question.

Question. *Let $2 \le q < \infty$, and $T\colon C(K) \to Y$ be a bounded linear operator to a Banach space Y, where Y satisfies some suitable smoothness condition. Is there a number $C < \infty$, depending on Y only, such that $R^q(T) \le C\,G^q(T)$?*

In Section 5.2, we give a negative answer to this question, in fact we cannot improve on Proposition 4.3(v) at all.

5) The Solutions

Here we present the main results of this thesis. The whole of Part 2 is devoted to the proof of these and similar results.

5.1) The Problem of Cotype 2

We give partial answers to the questions of Section 3, and also present similar results about Gaussian cotype 2.

Theorem 5.1. *Let $x_1, x_2, \ldots, x_S \in l_\infty^N$. Then*

i)
$$\mathbf{E}\left\|\sum_{s=1}^{S} \varepsilon_s x_s\right\|_\infty \geq c^{-1}\frac{1}{\operatorname{lm}\operatorname{lm} N}\left(\sum_{s=1}^{S}\|x_s\|_{L_{2,1}^N}^2\right)^{\frac{1}{2}};$$

ii)
$$\mathbf{E}\left\|\sum_{s=1}^{S} \gamma_s x_s\right\|_\infty \geq c^{-1}\frac{1}{\sqrt{\operatorname{lm}\operatorname{lm} N}}\left(\sum_{s=1}^{S}\|x_s\|_{L_{2,1}^N}^2\right)^{\frac{1}{2}}.$$

Theorem 5.2. *Let $T\colon l_\infty^N \to Y$ be a bounded linear operator to a Banach space. Then*
i) $R^2(T) \leq c \operatorname{lm}\operatorname{lm} N\, \pi_{2,1}(T)$,
ii) $G^2(T) \leq c\sqrt{\operatorname{lm}\operatorname{lm} N}\, \pi_{2,1}(T)$.

Proof: From Theorem 5.1, we have

$$R^2(l_\infty^N \hookrightarrow L_{2,1}^N) \leq c \operatorname{lm}\operatorname{lm} N,$$

$$G^2(l_\infty^N \hookrightarrow L_{2,1}^N) \leq c\sqrt{\operatorname{lm}\operatorname{lm} N}.$$

The result now follows from Theorem 1B:3.1. □

Corollary 5.2a. *There is a bounded linear operator $T\colon C(K) \to Y$ to a Banach space that has Rademacher cotype 2, but does not factor through Hilbert space.*

5.2) Comparison of Rademacher and Gaussian Cotype

We can give a complete answer to the question of Section 4.

Theorem 5.3. *Given $2 \leq p < q < \infty$, for suitably large N, there is a bounded linear operator $T\colon l_\infty^N \to L_q$ such that*
$$R^p(T) \geq \frac{1}{\sqrt{p}}\sqrt{\operatorname{lm} N}\, G^p(T).$$

We can also show that the left hand inequality in Proposition 2.4 does not extend to the Gaussian case.

Theorem 5.4. *Given $2 < q < \infty$, for suitably large N, there is a bounded linear operator $T\colon l_\infty^N \to L_q$ such that*
$$\pi_{2,1}(T) \geq c^{-1}\sqrt{\operatorname{lm} N}\, G^2(T).$$

Chapter 1D — Miscellaneous Results

In this chapter I shall present some extra results, which although they do not fit into the main argument, are nevertheless of some interest.

1) The $p^{\frac{1}{p}}$ in Pisier's Result

Let C_p be the function of $p \in [1,\infty)$ defined to be the smallest number such that for all bounded linear operators $T\colon C(K) \to Y$ that are $(p,1)$-summing, there is a Radon probability measure μ on K such that

$$\|T(x)\| \le C_p \pi_{p,1}(T) \, \|x\|_{L_p(\mu)}^{\frac{1}{p}} \, \|x\|_\infty^{1-\frac{1}{p}} \quad \text{for all } x \in C(K).$$

Pisier's Theorem, which we presented as Theorem 1B:2.1, tells us that $C_p \le p^{\frac{1}{p}}$. It is natural to ask whether C_p is any smaller, in particular, if it is 1. We show that this is not so.

Proposition 1.1. *Let C_p be defined as above. Then*

i) $C_p \ge \left(\dfrac{32^{p-1}}{2^p+1}\right)^{\frac{1}{p}}$;

ii) $C_p \ge (c^{-1}(p-1))^{\frac{1}{p}}$.

Hence, if $p > 1$, then $C_p > 1$, and for all $1 \le p < \infty$ we have $C_p^p \approx p$.

Proof of i): Let T be the formal identity map $l_\infty^3 \to (\mathbf{C}^N, \|\cdot\|_*)$, where $\|x\|_* = x^*(1) + x^*(2)$. Then by Proposition 1B:2.3 we have

$$\pi_{p,1}(T) = \sup\left\{ \left(\sum_{s=1}^{S} \|\chi_{A_s}\|_*^p\right)^{\frac{1}{p}} : \begin{array}{l} A_1,\, A_2, \dots,\, A_S \text{ are} \\ \text{disjoint subsets of } \{1,2,3\} \end{array} \right\}.$$

By trying out all the possibilities, we see that $\pi_{p,1}(T) = (2^p+1)^{\frac{1}{p}}$. Now suppose that μ is a probability measure on $\{1,2,3\}$ such that

$$\|x\|_* \le C\pi_{p,1}(T) \, \|x\|_{L_p(\mu)}^{\frac{1}{p}} \, \|x\|_\infty^{1-\frac{1}{p}}.$$

By considering the vectors $x = (1,1,0)$, $(1,0,1)$ and $(0,1,1)$, we see that

$$2^p \le C^p(2^p+1)(\mu(i)+\mu(j)) \quad \text{for } i \ne j \in \{1,2,3\}.$$

By adding these three equations we deduce that

$$C^p \ge \frac{32^{p-1}}{2^p+1}.$$

□

Proof of ii): Let $N = K(K+1)$ where $K \in \mathbf{N}$ is sufficiently large. Let $\mathcal{A}$ be the set system consisting of the sets

$$\{\, n \in [N] : n \equiv m \,(\mathrm{mod}\, K) \text{ or } n \equiv m' \,(\mathrm{mod}\, K+1)\},$$

where $m \in [K]$ and $m' \in [K+1]$. Let $\|\cdot\|_*$ be the norm on $\mathbf{C}^N$ given by

$$\|x\|_* = \sup_{A \in \mathcal{A}} \left\{ \sum_{n \in A} |x(n)| \right\},$$

and T be the formal identity $l_\infty^N \to (\mathbf{C}^N, \|\cdot\|_*)$. First we show that for sufficiently large K we have

$$\pi_{p,1}(T) \le \left(c\tfrac{1}{p-1}\right)^{\frac{1}{p}} 2K, \tag{1.1}$$

and to show this, we need the following lemma.

Lemma 1.2. *Let $A_1, A_2, \ldots, A_S \in \mathcal{A}$. Then for $S \le K$ we have*

$$\left|\bigcup_{s=1}^{S} A_s\right| \le S(2K+1-S).$$

Proof: We may write $A_s = E_s \cup F_s$, where

$$\begin{aligned} E_s &= \{n \in [N] : n \equiv m_s \pmod K\}, \\ F_s &= \{n \in [N] : n \equiv m_s' \pmod{K+1}\}. \end{aligned}$$

For any $s_1, s_2 \in [S]$, E_{s_1} and E_{s_2} are either disjoint or equal. List the distinct E_ss as $E_1', E_2', \ldots, E_{S_1}'$ with $S_1 \le S$. Similarly, list the distinct F_ss as $F_1', F_2', \ldots, F_{S_2}'$ with $S_2 \le S$. Observe that for $s_1 \in [S_1]$, $s_2 \in [S_2]$ we have $\left|E_{s_1}' \cap F_{s_2}'\right| = 1$. Hence

$$\begin{aligned} \left|\bigcup_{s=1}^{S} A_s\right| &= \sum_{s_1=1}^{S_1} \left|E_{s_1}'\right| + \sum_{s_2=1}^{S_2} \left|F_{s_2}'\right| - \sum_{s_1=1}^{S_1} \sum_{s_2=1}^{S_2} \left|E_{s_1}' \cap F_{s_2}'\right| \\ &= S_1(K+1) + S_2 K - S_1 S_2 \\ &\le S(2K+1-S). \end{aligned}$$

$\square$

Now we proceed with finding $\pi_{p,1}(T)$. By Proposition 1B:2.3 this is given by

$$\pi_{p,1}(T) = \sup \left\{ \left(\sum_{s=1}^{S} \|\chi_{B_s}\|_*^p \right)^{\frac{1}{p}} : \begin{array}{l} B_1, B_2, \ldots, B_S \text{ are} \\ \text{disjoint subsets of } [N] \end{array} \right\}.$$

Since $\|\chi_{B_s}\|_* = \sup_{A \in \mathcal{A}} |A \cap B_s|$, we see that

$$\pi_{p,1}(T) = \sup \left\{ \left(\sum_{s=1}^{S} |B_s|^p \right)^{\frac{1}{p}} : \begin{array}{l} B_1, B_2, \ldots, B_S \text{ are disjoint subsets of } [N] \\ \text{with } B_s \subseteq A_s \text{ for some } A_s \in \mathcal{A} \end{array} \right\}.$$

Pick $B_1, B_2, \ldots, B_S$ and $A_1, A_2, \ldots, A_S$ so that the supremum is attained, and such that the B_ss are ordered so that $|B_1| \ge |B_2| \ge \ldots \ge |B_S|$. Then we have

$$\begin{aligned} |B_t| &\le 2K+1-t \quad \text{for } t \le K, \\ |B_t| &\le \frac{N}{t} \quad \text{for } K < t \le S, \end{aligned}$$

for if $t \le K$ and $|B_t| > 2K+1-t$, then $|B_1|, |B_2|, \ldots, |B_t| > 2K+1-t$. Hence

$$t(2K+1-t) < \left|\bigcup_{s=1}^{t} B_s\right| \le \left|\bigcup_{s=1}^{t} A_s\right| \le t(2K+1-t),$$

which is a contradiction. The argument is similar for $t > K$. Therefore

$$\begin{aligned} \sum_{s=1}^{S} |B_s|^p &\le \sum_{s=1}^{K} (2K+1-s)^p + \sum_{s=K+1}^{\infty} \frac{N^p}{s^p} \\ &\le c_{\frac{1}{p-1}} (2K)^{p+1}, \end{aligned}$$

for sufficiently large K, and we have shown (1.1).

Now suppose that μ is a probability measure on $[N]$ such that for all $x \in l_\infty^N$, we have $\|x\|_* \leq C\pi_{p,1}(T)\,\|x\|_{L_1(\mu)}^{\frac{1}{p}}\,\|x\|_\infty^{1-\frac{1}{p}}$. Then by considering $x = \chi_A$ where $A \in \mathcal{A}$, we see that

$$(2K)^p = |A|^p \leq C^p\big(\pi_{p,1}(T)\big)^p \sum_{m\in A} \mu(m).$$

Now notice that we may list the elements of $\mathcal{A}$ as $\{\, A + n : n \in [N]\}$, where $A \in \mathcal{A}$ and $A + n = \{\,(m+n) \bmod N : m \in A\}$. Also notice that each $n \in [N]$ appears as an element in exactly $|A| = 2K$ sets in $\mathcal{A}$. So

$$\begin{aligned} N(2K)^p &\leq C^p\big(\pi_{p,1}(T)\big)^p \sum_{n=1}^{N} \sum_{m\in A+n} \mu(m) \\ &= C^p\big(\pi_{p,1}(T)\big)^p\,|A|\,. \end{aligned}$$

Therefore

$$C^p \geq \frac{N(2K)^{p-1}}{\big(\pi_{p,1}(T)\big)^p} \geq c^{-1}(p-1).$$

$\square$

2) The Space $L_{2,1}$ is not $(2,1)$-Summing

The usual way to see that a space is $(p,1)$-summing is to deduce this from the following result.

Proposition 2.1. *Let $T\colon X \to Y$ be a bounded linear operator between Banach spaces, and $2 \le p < \infty$. If T has Rademacher cotype p, then it is $(p,1)$-summing, with*

$$\pi_{p,1}(T) \le R^p(T).$$

Proof: This follows from Proposition 1A:3.1(i). □

Hence $L_{p,1}$ is $(p,1)$-summing for $p > 2$, and L_q is $(2,1)$-summing for $1 \le q \le 2$. Now the map $C(K) \hookrightarrow L_{2,1}(K,\mu)$ is $(2,1)$-summing, and so it is natural to ask whether the space $L_{2,1}$ itself is $(2,1)$-summing. However $L_{2,1}$ does not have cotype 2, and so the above argument will not work. We show that $L_{2,1}$ is not $(2,1)$-summing, and note that this adds support to the conjecture that all spaces that are $(2,1)$-summing have cotype 2 (see for example [P1 Ch.6] or [Ma2 Rem.2]).

Proposition 2.2. *For any $N \ge 2$, we have*

$$\pi_{2,1}(l_{2,1}^{N2^N}) \ge c^{-1}\sqrt{\operatorname{lm} N}.$$

Lemma 2.3. *The space $L_{2,1}$ is q-concave for $q > 2$.*

Proof: See [Cr]. □

Proof of Proposition 2.2: Identify $[N2^N]$ with the space $[N] \times \{1,-1\}^N$. Define the functions a_1, $a_2, \ldots$, a_N, ζ_1, $\zeta_2, \ldots$, ζ_N by

$$\begin{aligned} a_n(m, s_1, s_2, \ldots, s_N) &= \frac{1}{\sqrt{(m+n) \bmod N}}, \\ \zeta_n(m, s_1, s_2, \ldots, s_N) &= s_n. \end{aligned}$$

Then for any signs $s'_n = \pm 1$ we have that $\sum_{n=1}^N s'_n \zeta_n a_n$ has the same distribution as $\sum_{n=1}^N \zeta_n a_n$. Using Proposition 1A:3.1(i) and Proposition 1C:1.3(ii), we see that

$$\begin{aligned} \sup\left\{ \sum_{n=1}^N |\langle \xi, \zeta_n a_n\rangle| : \xi \in B_{\left(L_{2,1}^{N2^N}\right)^*} \right\} &= \sup\left\{ \left\| \sum_{n=1}^N s_n \zeta_n a_n \right\|_{2,1} : s_n = \pm 1 \right\} \\ &= \mathbf{E} \left\| \sum_{n=1}^N \varepsilon_n \zeta_n a_n \right\|_{2,1} \\ &\le c \left\| \left(\sum_{n=1}^N |\zeta_n a_n|^2 \right)^{\frac12} \right\|_{2,1} \\ &\le c\, N\, 2^N \sqrt{\operatorname{lm} N}. \end{aligned}$$

But

$$\left(\sum_{n=1}^N \|\zeta_n a_n\|_{2,1}^2 \right)^{\frac12} \approx N\, 2^N \operatorname{lm} N,$$

and the result follows. □

3) A Proof of Grothendieck's Theorem via Pisier's Result

Theorem 3.1. *Let $T\colon C(K) \to Y$ be a bounded linear operator to a Banach space. If Y has cotype 2, then T is 2-summing, with*

$$\pi_2(T) \le c\, R^2(Y) \sqrt{\operatorname{lm} R^2(Y)}.$$

I do not know whether this result is new. The methods given in [P1 Ch.4] only show that $\pi_2(T) \le c\, R^2(Y) \operatorname{lm} R^2(Y)$. The argument given below takes many of its ideas from [P1 Ch.9]. Its original feature is in its use of $L_{\mathrm{em}(T^2)}$ ([P1 Ch.9] uses L_4).

Lemma 3.2. *Let μ be a Radon probability measure on K. Then for all $x_1, x_2, \dots, x_S \in C(K)$, we have*

$$\mathbf{E} \left\| \sum_{s=1}^S \varepsilon_s x_s \right\|_{\mathrm{em}(T^2)} \le c \left\| \left(\sum_{s=1}^S |x_s|^2 \right)^{\frac12} \right\|_\infty .$$

Proof: First note for any $x \in C(K)$ that

$$\|x\|_{\mathrm{em}(T^2)} \le \int e^{|x|^2} \, d\mu,$$

for if $\int e^{|x|^2} \, d\mu < e$, then $\int \mathrm{em}(|x|^2) \, d\mu < 1$ and so $\|x\|_{\mathrm{em}(T^2)} \le 1$. But as we always have $\int e^{|x|^2} \, d\mu \ge 1$, the result follows. If, on the other hand, we have $C = \int e^{|x|^2} \, d\mu \ge e$, then

$$\int \left(e^{\left|\frac{x}{C}\right|^2} - 1 \right) d\mu \le \tfrac{1}{C^2} \int \left(e^{|x|^2} - 1 \right) d\mu \le 1,$$

and hence

$$\int e^{\left|\frac{x}{C}\right|^2 - 1} \, d\mu \le 2e^{-1} < 1,$$

that is, $\|x\|_{\mathrm{em}(T^2)} \le C$.

Now suppose that $\left\| \sum_{s=1}^S |x|^2 \right\|_\infty = 1$. It follows from Proposition 1C:1.2(i) that for some number $C < \infty$ that for all $t > 0$ and $w \in K$ we have

$$\Pr\left(\left| \sum_{s=1}^S \varepsilon_s x_s(w) \right| \ge t \right) \le C e^{-C^{-1} t^2}.$$

Hence

$$\begin{aligned} \mathbf{E} \left\| \sum_{s=1}^S \varepsilon_s x_s \right\|_{\mathrm{em}(T^2)} &\le \sqrt{2C}\, \mathbf{E} \int e^{\left| \frac{1}{\sqrt{2C}} \sum \varepsilon_s x_s \right|^2} \, d\mu \\ &\le \sqrt{2C} \int_K \int_{t=0}^\infty e^{\frac12 C^{-1} t^2} \frac{d}{dt} \left(C e^{-C^{-1} t^2} \right) dt \, d\mu \\ &\le c, \end{aligned}$$

and the result follows. $\square$

Proof of Theorem 3.1: First note that by Proposition 2.1, T is $(2,1)$-summing with $\pi_{2,1}(T) \leq R^2(Y)$. Hence there is a Radon probability measure μ on K such that for all $x \in C(K)$ we have

$$\|T(x)\| \leq \sqrt{2} R^2(Y) \, \|x\|_{L_{2,1}(\mu)} .$$

Now we show that for all $x \in C(K)$ we have

$$\|T(x)\| \leq c\sqrt{\operatorname{lm} R^2(Y)} \, \|x\|_{L_{\mathrm{em}(T^2)}(\mu)} . \tag{3.1}$$

For suppose that $\|x\|_{\mathrm{em}(T^2)} < 1$. Then there is a positive simple function y such that $|x| < y$ and $\|y\|_{\mathrm{em}(T^2)} \leq 1$. By Theorem 1A:2.7(ii), $\|y\|_{\mathrm{em}(T^2),\infty} \leq c$, that is, $y^*(t) \leq c/\sqrt{\operatorname{lm} t}$. Hence

$$y = \sum_{n=1}^{N} a_n \chi_{A_n},$$

where $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_N$ are measurable subsets of K, and $a_1, a_2, \ldots, a_N \geq 0$ are such that $\sum_{n=M}^{N} a_n \leq c\big/\sqrt{\operatorname{lm} \mu(A_M)}$. Let N_0 be the largest natural number such that for $n \leq N_0$ we have $\mu(A_n) \leq 1/(R^2(Y))^2$, and write $x = x_1 + x_2$ where

$$x_1 = \frac{x}{y} \sum_{n=1}^{N_0} a_n \chi_{A_n}$$

$$x_2 = \frac{x}{y} \sum_{n=N_0+1}^{N} a_n \chi_{A_n} .$$

Now $\|T(x)\| \leq \|T(x_1)\| + \|T(x_2)\|$. We have that

$$\|T(x_1)\| \leq \sqrt{2} R^2(Y) \, \|x_1\|_{L_{2,1}(\mu)} .$$

Since

$$x_1^*(t) \leq c \, \frac{\chi_{[0,1/(R^2(Y))^2]}(t)}{\sqrt{\operatorname{lm}(t)}},$$

we have

$$\|x_1\|_{L_{2,1}(\mu)} \leq c \int_0^{1/(R^2(Y))^2} \tfrac{1}{2\sqrt{t \operatorname{lm} t}} \, dt \leq c\sqrt{\operatorname{lm} R^2(Y)}\Big/ R^2(Y).$$

Also

$$\|T(x_2)\| \leq \|x_2\|_\infty \leq \sum_{n=N_0+1}^{N} a_n \leq c\sqrt{\operatorname{lm} R^2(Y)}.$$

Thus (3.1) follows.

Therefore, for $x_1, x_2, \ldots, x_S \in C(K)$, we have

$$\begin{aligned} \left(\sum_{s=1}^{S} \|T(x_s)\|_Y^2 \right)^{\frac{1}{2}} &\leq R^2(Y) \, \mathbf{E} \left\| \sum_{s=1}^{S} \varepsilon_s T(x_s) \right\|_Y \\ &\leq c \, R^2(Y) \sqrt{\operatorname{lm} R^2(Y)} \, \mathbf{E} \left\| \sum_{s=1}^{S} \varepsilon_s T(x_s) \right\|_{\mathrm{em}(T^2)} \\ &\leq c \, R^2(Y) \sqrt{\operatorname{lm} R^2(Y)} \left\| \left(\sum_{s=1}^{S} |x_s|^2 \right)^{\frac{1}{2}} \right\|_\infty . \end{aligned}$$

□

4) The $(\Phi,1)$-Summing Norm

It is very easy to extend Pisier's result to cover $(\Phi,1)$-summing operators for suitable normal quasi-Orlicz functions Φ. I believe this result is new, but the proof is only a slight modification of that given in [P3], [P4] or [J2 14.1].

Theorem 4.1. *Let $T\colon C(K) \to Y$ be a bounded linear operator to a Banach space, and Φ be a convex normal quasi-Orlicz function satisfying the Δ_2-condition. Then the following are equivalent.*

i) T is $(\Phi,1)$-summing.

ii) There is a Radon probability measure μ on K and a number $C < \infty$ such that for all $x \in C(K)$ with $\|x\|_\infty \le 1$ we have

$$\|T(x)\| \le C\Phi^{-1}\big(\|x\|_{L_q(K,\mu)}\big).$$

iii) There is a Radon probability measure μ on K and the factorization

$$T\colon C(K) \hookrightarrow L_{\tilde\Phi,1}(K,\mu) \xrightarrow{U} X$$

where U is a bounded linear operator.

Lemma 4.2. *Let $\|\cdot\|_s$ be a semi-norm on $C(K)$, Φ a normal quasi-Orlicz function, and μ a Radon probability measure on K such that for all $x \in C(K)$ and measurable $A \subseteq K$ with $|x| \le \chi_A$ we have*

$$\|x\|_s \le \Phi^{-1}\big(\mu(A)\big).$$

Then for all $x \in C(K)$ we have

$$\|x\|_s \le \|x\|_{L_{\tilde\Phi,1}(K,\mu)} .$$

Proof: Suppose $x \in C(K)$ with $\|x\|_{L_{\Phi,1}} < 1$. Then there is a positive simple function $y\colon C(K) \to \mathbf{C}$ with $|x| \le y$ and $\|y\|_{L_{\Phi,1}} < 1$. Write $y = \sum_{n=1}^N a_n\chi_{A_n}$, where $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_N$ are measurable subsets of K, and $a_n \ge 0$. Then

$$\begin{aligned}\|x\|_s &\le \sum_{n=1}^N \left\| \frac{x}{y}\, a_n\chi_{A_n}\right\|_s \\ &\le \sum_{n=1}^N a_n\Phi^{-1}\big(\mu(A_n)\big) \\ &= \|y\|_{L_{\Phi,1}} \\ &< 1.\end{aligned}$$

□

Proof of i)⇒ii): By Proposition 1A:2.1, there is a convex function Ψ, a number $C < \infty$, and a number $1 \le q < \infty$, such that $\Psi^{-1} \overset{C}{\approx} \Phi^{-1}$ and $\Psi \circ T^{\frac{1}{q}}$ is concave. Since, by Proposition 1A:2.3(i) we have that $\|\cdot\|_\Psi \overset{C}{\approx} \|\cdot\|_\Phi$, it follows that T is $(\Psi,1)$-summing.

So we may suppose that $\pi_{\Psi,1}(T) = 1$, that is,

$$\sup\left\{ \left\|\big(\|T(x_s)\|\big)_{s=1}^S\right\|_\Psi : \left\|\sum_{s=1}^S |x_s|\right\|_\infty \le 1 \right\} = 1.$$

For each $\epsilon > 0$, choose $x_1, x_2, \ldots, x_S \in C(K)$ such that

$$\left\|\sum_{s=1}^S |x_s|\right\|_\infty \le 1+\epsilon \quad\text{and}\quad \left\|\big(\|T(x_s)\|\big)_{s=1}^S\right\|_\Psi = 1.$$

Since Ψ is convex, l_Ψ is a norm. Hence we can choose $\zeta_1, \zeta_2, \dots, \zeta_S \in Y^*$ such that

$$\left\| \left(\|\zeta_s\|\right)_{s=1}^S \right\|_\Psi^* = 1 \quad \text{and} \quad \sum_{s=1}^S \langle \zeta_s, T(x_s) \rangle = 1.$$

Define $\varphi_\epsilon \colon C(K) \to \mathbf{C}$ to be

$$\varphi_\epsilon(y) = \sum_{s=1}^S \langle \zeta_s, T(y x_s) \rangle.$$

We list three properties of φ_ϵ.

$$\|\varphi_\epsilon\| \le 1 + \epsilon, \tag{4.1}$$

$$\varphi_\epsilon(1) = 1, \tag{4.2}$$

$$\Psi\left(\tfrac{\|T(x)\|}{(1+\epsilon)^2}\right) \le 1 - \left(\tfrac{1}{(1+\epsilon)^2} \varphi_\epsilon\big(1 - |x|\big)\right)^q \quad \text{for } \|x\|_\infty \le 1. \tag{4.3}$$

Properties (4.1) and (4.2) are easily established. For property (4.3), if $\|x\|_\infty \le 1$, let $y = 1 - |x|$, $z_0 = x$, and $z_s = y x_s$ for $s \in [S]$. Then $\left\|\sum_{s=0}^S |z_s|\right\|_\infty \le 1 + \epsilon$, and hence

$$\left\| \left(\|T(z_s)\|\right)_{s=0}^S \right\|_\Psi \le 1 + \epsilon.$$

Therefore

$$\sum_{s=0}^S \Psi\left(\tfrac{\|T z_s\|}{(1+\epsilon)^2}\right) \le 1.$$

Now we show that

$$\sum_{s=1}^S \Psi\left(\tfrac{\|T(z_s)\|}{(1+\epsilon)^2}\right) \ge \left\| \left(\tfrac{\|T(z_s)\|}{(1+\epsilon)^2}\right)_{s=1}^S \right\|_\Psi^q,$$

for if $\left\| \left(\tfrac{\|T(z_s)\|}{(1+\epsilon)^2}\right)_{s=1}^S \right\|_\Psi > u$, then $\sum_{s=1}^S \Psi\left(\tfrac{\|T(z_s)\|}{(1+\epsilon)^2 u}\right) \ge 1$. As $\Psi \circ T^{\frac{1}{q}}$ is concave, we have $\sum_{s=1}^S \Psi\left(\tfrac{\|T(z_s)\|}{(1+\epsilon)^2}\right) \ge u^q$, and the result follows.

Therefore

$$\begin{aligned}
\Psi\left(\tfrac{\|T(x)\|}{(1+\epsilon)^2}\right) &= \Psi\left(\tfrac{\|T(z_0)\|}{(1+\epsilon)^2}\right) \\
&\le 1 - \sum_{s=1}^S \Psi\left(\tfrac{\|T(z_s)\|}{(1+\epsilon)^2}\right) \\
&\le 1 - \left\| \left(\tfrac{\|T(z_s)\|}{(1+\epsilon)^2}\right)_{s=1}^S \right\|_\Psi^q \\
&\le 1 - \left(\tfrac{1}{(1+\epsilon)^2} \sum_{s=1}^S \langle \zeta_s, T(z_s) \rangle\right)^q \\
&= 1 - \left(\tfrac{1}{(1+\epsilon)^2} \varphi_\epsilon\big(1 - |x|\big)\right)^q,
\end{aligned}$$

and property (4.3) is established.

Now let φ be a weak * limit point of φ_ϵ as $\epsilon \to 0$. Then, by (4.1) and (4.2), φ is a positive functional of norm 1, and hence by the Riesz representation theorem, there is a Radon probability measure μ such that $\varphi(x) = \int_K x \, d\mu$ for all $x \in C(K)$.

Also, from (4.3) we deduce that, if $\|x\|_\infty \le 1$, then

$$\begin{aligned}
\Psi\big(\|T(x)\|\big) &\le 1 - \Big(\varphi\big(1 - |x|\big)\Big)^q \\
&\le q\, \varphi(|x|).
\end{aligned}$$

Therefore

$$\begin{aligned}\|T(x)\| &\le \Psi^{-1}\big(q\varphi(|x|)\big)\\ &\le q\,\Psi^{-1}\big(\varphi(|x|)\big)\\ &\le Cq\,\Phi^{-1}\big(\varphi(|x|)\big),\end{aligned}$$

as Ψ is convex, and as $\Psi \overset{C}{\approx} \Phi$. □

Proof of ii)⇒iii): This follows by Lemma 4.2. □

Proof of iii)⇒i): Suppose that $x_1, x_2, \dots, x_S \in C(K)$ have disjoint supports $A_1, A_2, \dots, A_S$ on K, and that $\|x_s\|_\infty < 1$ for each $s \in [S]$. Then $\|T(x_s)\| \le C\,\|x_s\|_{L_{\tilde{\Phi},1}} < C\Phi^{-1}\big(\mu(A_s)\big)$, where $C = \|U\|$. Hence

$$\sum_{s=1}^{S} \Phi\left(\frac{\|T(x_s)\|}{C}\right) < \sum_{s=1}^{S} \mu(A_s) \le 1,$$

and so $\left\|\big(\|Tx_s\|\big)_{s=1}^{S}\right\|_\Phi \le C$. By Proposition 1A:2.10, this is sufficient to show that $\pi_{\Phi,1}(T) \le C$. □

Part 2 — Lower Bounds for Random Walks in l_∞^N

Contents

Introduction to Part 2

The purpose of Part 2 is to prove the results of Section 1C:5. All of this part is devoted to calculating the cotype p constants of various operators from l_∞^N. That is, we prove results of the form: for all x_1, $x_2, \ldots,$ $x_S \in \mathbf{C}^N$ we have

$$\mathbf{E}\left\|\sum_{s=1}^{S}\theta_s x_s\right\|_\infty \geq \frac{1}{f(N)}\left(\sum_{s=1}^{S}\|x_s\|_*^p\right)^{\frac{1}{p}},$$

where θ_1, $\theta_2, \ldots$ are either normalised Gaussians or Bernoulli random variables, $\|\cdot\|_*$ is a 1-symmetric norm on $\mathbf{C}^N$, $2 \leq p < \infty$, and $f\colon \mathbf{N} \to \mathbf{R}_+$ is some function.

Briefly, Theorems 1C:5.1(i) and 1C:5.2(i) are proven in Chapter 2A, Theorems 1C:5.1(ii) and 1C:5.2(ii) are proven in Chapter 2C, and Theorems 1C:5.3 and 1C:5.4 are proven in Chapter 2D (Chapter 2B is taken up with a discussion of the techniques used in Chapter 2A).

Chapter 2A — The Problem of Rademacher Cotype 2

This chapter proves the main result of the thesis, Theorem 1C:5.1(i), which we restate here.

Theorem 0.1. *Let $T\colon l_\infty^N \to Y$ be a bounded linear operator to a Banach space. Then*

$$R^2(T) \le c \,\operatorname{lm}\operatorname{lm} N \, \pi_{2,1}(T).$$

As we showed in Section 5.1, by Theorem 1B:3.1, it is sufficient to prove Theorem 1C:5.2(i), that is, $R^2(l_\infty^N \hookrightarrow L_{2,1}^N) \le c \,\operatorname{lm}\operatorname{lm} N$. The whole of this chapter is devoted to proving this result, and for convenience, we restate it here.

Theorem 0.2. *Let $x_1, x_2, \dots, x_S \in l_\infty^N$. Then*

$$\mathbf{E} \left\| \sum_{s=1}^S \varepsilon_s x_s \right\|_\infty \ge c^{-1} \frac{1}{\operatorname{lm}\operatorname{lm} N} \left(\sum_{s=1}^S \|x_s\|_{L_{2,1}^N}^2 \right)^{\frac{1}{2}} .$$

This seems to be difficult to prove, and so as a prelude, we first prove a weaker result.

Theorem 0.3. *Let $x_1, x_2, \dots, x_S \in l_\infty^N$. Then*

$$\mathbf{E} \left\| \sum_{s=1}^S \gamma_s x_s \right\|_\infty \ge c^{-1} \frac{1}{\operatorname{lm}\operatorname{lm} N} \left(\sum_{s=1}^S \|x_s\|_{L_{2,1}^N}^2 \right)^{\frac{1}{2}} .$$

I do not know whether the $\operatorname{lm}\operatorname{lm} N$ factor of Theorems 0.1 and 0.2 is the best possible. However, Theorem 0.3 will be superseded in Chapter 2C, where we will replace the $\operatorname{lm}\operatorname{lm} N$ factor by a $\sqrt{\operatorname{lm}\operatorname{lm} N}$ factor.

Outline of the Proof of Theorem 0.3

We are concerned with finding lower bounds for the l_∞ norm of a random walk. The first thing we do is to treat the random walk as a process. Much work has already been done on finding lower bounds for suprema of Gaussian processes, and this is discussed in Chapters 2B and 2C. However, I present my own way of handling this situation in Section 2, which also deals with Bernoulli processes. There, we reduce the problem to finding the expectation of suprema of independent processes.

Then we find formulae for lower bounds of suprema of independent processes. For Gaussian processes, the problem is easy, and we deal with this in Section 3. The Bernoulli case is more difficult and more interesting, and amongst other things, requires an interesting result describing the tail of a Bernoulli random walk. This we deal with in Section 5.

After these, an averaging argument takes over. The Gaussian case is discussed in Section 4. Two variations of the argument are presented. The first argument is simpler, but the second argument is needed as it generalizes to the Bernoulli case.

The Bernoulli case, given in Section 6, is a more difficult argument with many technical details.

1) Notation

For the rest of this chapter, and indeed for Chapters 2B, 2C and 2D, we will work with a fixed $x = (x_1, x_2, \ldots, x_S) \in (\mathbf{C}^N)^S$ (here $(\mathbf{C}^N)^S$ denotes the space of S-tuples of N-vectors). We associate with each x_s an ordering permutation $\pi_s \colon [N] \to [N]$, so that we have

$$\|x_s\|_{2,1} \approx \sum_{n=1}^{N} \frac{1}{\sqrt{\pi_s(n)}} \left| x_s(n) \right| .$$

Our main motivating example will be when x is *cyclic* on a vector $a \in \mathbf{C}^N$, that is, we have $a_1 \geq a_2 \geq \ldots \geq a_N \geq 0$, $S = N$, and

$$x_s(n) = a((s+n) \bmod N).$$

We write $\Gamma(x)$, or simply Γ (as no confusion can arise) for the Gaussian process $(\,\Gamma_n : n \in [N])$ where

$$\Gamma_n = \sum_{s=1}^{S} \gamma_s x_s(n).$$

Similarly, we write $E(x)$ or E for the Bernoulli process $(\,E_n : n \in [N])$ where

$$E_n = \sum_{s=1}^{S} \varepsilon_s x_s(n).$$

Thus

$$\mathbf{E} \left\| \sum_{s=1}^{S} \gamma_s x_s \right\|_\infty = \mathbf{E} \sup_{n \in [N]} |\Gamma_n|,$$

and

$$\mathbf{E} \left\| \sum_{s=1}^{S} \varepsilon_s x_s \right\|_\infty = \mathbf{E} \sup_{n \in [N]} \left| E_n \right| .$$

It will be useful to write θ_1, $\theta_2, \ldots$ to denote either γ_1, $\gamma_2, \ldots$ or ε_1, $\varepsilon_2, \ldots$, and $\Theta(x)$ for $\Gamma(x)$ or $E(x)$.

We will say that the process $\Theta = (\,\Theta_n : n \in [N])$ is *independent* if the random variables Θ_1, $\Theta_2, \ldots$, Θ_N are independent, and that it has *size* k if $|\{\, n \in [N] : \Theta_n \neq 0\}| = k$.

2) Reducing to Independent Processes

Our problem is to find a lower bound to $\mathbf{E}\sup_{n\in[N]}|\Theta_n|$, where, as stated before, $\Theta_n = \sum_{s=1}^S \theta_s x_s(n)$. This is a difficult problem as $\Theta_1, \Theta_2, \ldots, \Theta_N$ are not necessarily independent. The main result of this section is to replace $\Theta_1, \Theta_2, \ldots, \Theta_N$ by independent random variables.

Theorem 2.1. *Let $\sigma_1, \sigma_2, \ldots, \sigma_N$ be disjoint subsets of $[S]$. Let Θ^σ be the process*

$$\Theta_n^\sigma = \sum_{s\in\sigma_n} \theta_s x_s(n),$$

so that $\Theta_1^\sigma, \Theta_2^\sigma, \ldots, \Theta_N^\sigma$ are independent. Then

$$\mathbf{E}\sup_{n\in[N]}|\Theta_n| \geq \tfrac{1}{4}\mathbf{E}\sup_{n\in[N]}|\Theta_n^\sigma|.$$

At first sight this result appears to throw away far too much information. However this is not the case, although it may well account for the $\operatorname{lm}\operatorname{lm} N$ factor (this will be discussed in Section 2B:1).

Before proving this result, we introduce some notation which will allow us to apply it more easily.

Definition. *A system of strips is a N-tuple $\sigma = (\sigma_n : n \in [N])$ of disjoint subsets of $[S]$. It is said to be of size k if exactly k of the subsets $\sigma_1, \sigma_2, \ldots, \sigma_N$ are non-empty. If σ is a N-tuple of subsets of $[S]$, not necessarily disjoint, then we say σ is almost a system of strips.*

If $\Theta = (\Theta_n : n \in [N])$ is a process with $\Theta_n = \sum_{s=1}^S \theta_s x_s(n)$, then $\Theta^\sigma = (\Theta_n^\sigma : n \in [N])$ is the process $\Theta_n^\sigma = \sum_{s\in\sigma_n}\theta_s x_s(n)$.

If $x_1, x_2, \ldots, x_S \in l_\infty^N$ and $\sigma \subseteq [S]$, we write $x_\sigma(n)$ for the vector in $\mathbf{C}^S$ given by

$$x_\sigma(n)(s) = \begin{cases} x_s(n) & \text{if } s\in\sigma \\ 0 & \text{if } s\notin\sigma. \end{cases}$$

For $y \in (\mathbf{C}^N)^S$, and any system of strips σ, we write (with a slight abuse of the usual notation)

$$\operatorname{supp} x \subseteq \sigma,$$

if for all $n\in[N]$ and $s\in[S]$, we have $x_s(n)\neq 0$ only if $s\in\sigma_n$.

Since Theorem 2.1 seemingly throws away so much information, we will have to choose our strips wisely and make sure that they pick out the larger elements of $x_1, x_2, \ldots, x_S$. First we illustrate this in the case when x is cyclic on a as follows. For each $k\in[N]$, let $\sigma(k) = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ be the system of strips of size k given by

$$\sigma_n = \begin{cases} [N-n+1, N-n+\lfloor N/k\rfloor] & \text{if } \lfloor N/k\rfloor \big| n \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathbf{E}\sup_{n\in[N]}|\Theta_n^{\sigma(k)}|$ is the expectation of the supremum of k independent copies of $\left|\sum_{s=1}^{\lfloor N/k\rfloor}\theta_s a(s)\right|$. If x is not cyclic things are more tricky. The first natural system of strips that one would choose is not a tuple of disjoint subsets, that is, it is only almost a systems of strips.

Definition. *Given $\nu\in[N]^{(k)}$, so that $\nu = \{n_1, n_2, \ldots, n_k\}$ with $1\leq n_1 < n_2 < \ldots < n_k \leq N$, we define $\sigma'(\nu) = (\sigma_1', \sigma_2', \ldots, \sigma_N')$ to be*

$$\sigma_n' = \begin{cases} \{\, s\in[S] : \pi_s(n) \leq \lfloor N/k\rfloor\} & \text{if } n\in\nu \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus $\sigma'(\nu)$ is almost a system of strips.

The required system of strips is given by the following definition.

Definition. *Given $\nu \in [N]^{(k)}$, we define the system of strips $\sigma(\nu) = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ by*

$$\sigma_n(\nu) = \sigma'_n(\nu) \setminus \bigcup_{n' \neq n} \sigma'_{n'}(\nu).$$

These systems of strips will be referred to as the *standard systems of strips.* The following result shows that the standard systems of strips inherit many of the properties of the $\sigma'(\nu)$.

Proposition 2.2. *Let $s \in [S]$, $n \in [N]$ and $k \in [N/3]$. Then*

$$\left|\{\, \nu \in [N]^{(k)} : s \in \sigma_n(\nu)\}\right| \geq c^{-1} \left|\{\, \nu \in [N]^{(k)} : s \in \sigma'_n(\nu)\}\right|.$$

and

$$\left|\{\, \nu \in [N]^{(k)} : s \in \sigma'_n(\nu)\}\right| = \begin{cases} \frac{k}{N}\left|[N]^{(k)}\right| & \text{if } \pi_s(n) \leq \lfloor N/k \rfloor \\ 0 & \text{otherwise.} \end{cases}$$

Proof: The second equation is easy, for we have that

$$\left|\{\, \nu \in [N]^{(k)} : n \in \nu\}\right| = \frac{k}{N}\left|[N]^{(k)}\right|,$$

and that if $n \in \nu$, then $s \in \sigma'_n(\nu)$ if and only if $\pi_s(n) \leq \lfloor N/k \rfloor$.

So let us derive the first equation. We suppose that $\pi_n(s) \leq \lfloor N/k \rfloor$ (otherwise it is clearly true). We are interested in showing that the proportion of those $\nu \in [N]^{(k)}$ with $n \in \nu$ that are such that $\pi_s(n') > \lfloor N/k \rfloor$ for $n' \in \nu \setminus \{n\}$, is bounded by a universal constant. In calculating this quantity, we may assume without loss of generality that $n = 1$ and $\pi_s = \mathrm{Id}_{[N]}$. Then we are finding what proportion of $\nu' \in [2, N]^{(k-1)}$ have $\nu' \cap [1, \lfloor N/k \rfloor] = \emptyset$.

For $k = 1$ this is 1. Otherwise, it is

$$\frac{(N - \lfloor N/k \rfloor)(N - \lfloor N/k \rfloor - 1) \cdots (N - \lfloor N/k \rfloor - k + 2)}{(N-1)(N-2)\cdots(N-k+1)}.$$

Remembering that $k \leq N/3$, we see that this is greater than

$$\begin{aligned} &\frac{(N - N/k)(N - N/k - 1) \cdots (N - N/k - k + 2)}{(N-1)(N-2)\cdots(N-k+1)} \\ &= \frac{(N - N/k)}{(N-k+1)} \left(1 - \frac{N}{(N-1)k}\right)\left(1 - \frac{N}{(N-2)k}\right) \cdots \left(1 - \frac{N}{(N-k+2)k}\right) \\ &\geq \frac{\frac{1}{2}N}{N}\left(1 - \frac{3}{2}\frac{1}{k}\right)^{k-1} \\ &\geq c^{-1}, \end{aligned}$$

as desired. ☐

Now we prove Theorem 2.1. What follows is well known, and combines a lemma of P. Lévy with the so called reflection principle.

Lemma 2.3. *Let $v_1, v_2, \ldots, v_S$ be independent symmetric random variables taking values in a Banach space X. Define the quantities*

$$U = \left\|\sum_{s=1}^{S} v_s\right\|,$$

$$V = \sup_{S' \in [S]} \left\|\sum_{s=1}^{S'} v_s\right\|,$$

$$W = \sup_{1 \leq S_1 \leq S_2 \leq S} \left\|\sum_{s=S_1}^{S_2} v_s\right\|.$$

Then we have the following inequalities for all $r > 0$:

i) $\Pr(U > r) \geq \frac{1}{2}\Pr(V > r)$;

ii) $2V \geq W$.

Proof of i): See [Ka, Ch.2, Lem.1]. ☐

Proof of ii): This follows as

$$\left\| \sum_{s=S_1}^{S_2} v_i \right\| \leq \left\| \sum_{s=1}^{S_1-1} v_i \right\| + \left\| \sum_{s=1}^{S_2} v_i \right\| \leq 2V.$$

$\square$

Proposition 2.4. *Let σ be a system of strips. Then for all $r > 0$ we have*

$$\Pr\left(\sup_{n\in[N]} |\Theta_n| \geq r\right) \geq \tfrac{1}{2} \Pr\left(\sup_{n\in[N]} |\Theta_n^\sigma| \geq \tfrac{1}{2} r\right).$$

Proof: Without loss of generality we may assume that $\sigma_n = [S_n, S'_n]$ for $t \leq k$, where $1 \leq S_1 \leq S'_1 < S_2 \leq S'_2 < \ldots < S_k \leq S'_k \leq S$ is a sequence of integers, and that $\sigma_n = \emptyset$ for $t > k$. Then

$$\sup_{n\in[N]} \left| \sum_{s\in\sigma_n} \theta_s x_s(n) \right| \leq \sup_{n\in[N]} \left\| \sum_{s\in\sigma_n} \theta_s x_s \right\|_\infty \leq \sup_{1\leq S_1\leq S_2\leq N} \left\| \sum_{s=S_1}^{S_2} \theta_s x_s \right\|_\infty$$

and the result follows immediately from Lemma 2.3. $\square$

Proof of Theorem 2.1: This follows straight away from Proposition 2.4. $\square$

3) Independent Gaussian Processes

The main result of this section is easy to prove.

Proposition 3.1. *Let $\Gamma = (\Gamma_n : n \in [N])$ be an independent Gaussian process of size at most k such that Γ_n has standard deviation a_n. Then for $1 \le p < \infty$ we have*

$$\mathbf{E} \sup_{n \in [N]} |\Gamma_n| \ge c^{-1} \frac{1}{\sqrt{p}} \frac{\sqrt{\operatorname{lm} k}}{k^{\frac{1}{p}}} \left\| (a_n)_{n\in[N]} \right\|_p .$$

The proof of this theorem falls into two easy parts.

Proposition 3.2. *Let $\Gamma = (\Gamma_n : n \in [N])$ be an independent Gaussian process such that Γ_n has standard deviation a_n. Then*

$$\mathbf{E} \sup_{n \in [N]} |\Gamma_n| \ge c^{-1} \left\| (a_n)_{n\in[N]} \right\|_{\operatorname{em}(T^2)} .$$

To prove this, we require the following lemma.

Lemma 3.3. *Given $C_1 > 0$, there is a $C_2 < \infty$ such that for all $0 \le x_1, x_2 \dots, x_N \le 1$ we have that if $\prod_{n=1}^N (1 - x_n) \ge C_1$, then $\sum_{n=1}^N x_n \le C_2$.*

Proof: This follows by 'A'-level calculus. □

Proof of Proposition 3.2: This is essentially a finite version of the second Borel–Cantelli Lemma.

Suppose that $\mathbf{E} \sup_{n\in[N]} |\Gamma_n| \le 1$. Then $\Pr(\sup_{n\in[N]} |\Gamma_n| \ge 2) \le \frac{1}{2}$, that is,

$$\prod_{n=1}^N (1 - \Pr(|\Gamma_n| \ge 2)) \ge \tfrac{1}{2}.$$

By Proposition 1C:1.1, this implies that

$$\prod_{n=1}^N \left(1 - e^{-c^{-1}/a_n^2}\right) \ge c^{-1},$$

and by Lemma 3.3, this implies that

$$\sum_{n=1}^N e^{-c^{-1}/a_n^2} \le c.$$

From this, we deduce

$$\sum_{n=1}^N \operatorname{em}(c^{-1} a_n^2) < 1,$$

that is, $\left\| (a_n)_{n\in[N]} \right\|_{\operatorname{em}(T^2)} \le c$. □

Proposition 3.4. *If $a \in \mathbf{C}^k$, then*

$$\|a\|_{\operatorname{em}(T^2)} \ge c^{-1} \frac{1}{\sqrt{p}} \frac{\sqrt{\operatorname{lm} k}}{k^{\frac{1}{p}}} \|a\|_p .$$

Proof: This follows from Proposition 1A:2.3(ii), and the fact that for $1 \le t \le k$ we have

$$\sqrt{\operatorname{lm} t} \ge c^{-1} \frac{1}{\sqrt{p}} \frac{\sqrt{\operatorname{lm} k}}{k^{\frac{1}{p}}} t^{\frac{1}{p}} .$$

□

We also have a converse to Proposition 3.2.

Proposition 3.5. *Let* $\Gamma = (\Gamma_n : n \in [N])$ *be a Gaussian process such that* Γ_n *has standard deviation* a_n*. Then*

$$\mathbf{E} \sup_{n\in[N]} |\Gamma_n| \le c \left\| (a_n)_{n\in[N]} \right\|_{\mathrm{em}(T^2)} .$$

Proof: This is somewhat like the proof of the first Borel–Cantelli lemma.

Suppose that $\left\| (a_n)_{n\in[N]} \right\|_{\mathrm{em}(T^2)} \le 1$, that is

$$\sum_{n=1}^{N} e^{-1/a_n^2} \le e^{-1}.$$

Then, by Proposition 1C:1.1, we have

$$\Pr\left(\sup_{n\in[N]} |\Gamma_n| \ge 1 \right) \le \sum_{n=1}^{N} \Pr\big(|\Gamma_n| \ge 1 \big) \le c.$$

By Corollary 1C:1.5c, it follows that $\mathbf{E} \sup_{n\in[N]} |\Gamma_n| \le c$. ☐

4) The Averaging Argument in the Gaussian Case

In this section, we establish Theorem 0.3. We give two arguments, both of them based on taking suitable averages.

4.1) The Cyclic Case

First of all we give the argument for when x is cyclic. This will illustrate many of the techniques that we shall use.

Proposition 4.1. *Suppose that x is cyclic on a. Then*

$$\mathbf{E}\left\|\sum_{s=1}^{S}\gamma_s x_s\right\|_\infty \geq c^{-1}\frac{1}{\operatorname{lm}\operatorname{lm} N}\left\|a\right\|_{2,1}.$$

We split the proof of this proposition up into a series of smaller results. The purpose of this is not to make it easier to understand, but rather to illustrate some of the main steps that will be used in future proofs.

Proposition 4.2. *Suppose that x is cyclic on a. Then there is a $k\in[N/3]$ such that*

$$\sqrt{\operatorname{lm} k}\left\|a\big|_{[N/k]}\right\|_2 \geq c^{-1}\frac{1}{\operatorname{lm}\operatorname{lm} N}\left\|a\right\|_{2,1}.$$

Proposition 4.3. *Suppose that x is cyclic on a. Then there is a $k\in[N/3]$ such that*

$$\sqrt{\operatorname{lm} k}\left\|a\big|_{[N/k]}\right\|_2 \geq c^{-1}\frac{1}{\sqrt{\operatorname{lm}\operatorname{lm} N}}\sqrt{N}\left\|a\right\|_{L^N_{T^2\operatorname{lm} T,2}}.$$

Proposition 4.4. *Suppose that x is cyclic on a. Let $W_k=\dfrac{1}{k\operatorname{lm} k}$. Then*

$$\left(\sum_{k=1}^{N/3} W_k\operatorname{lm} k\left\|a\big|_{[N/k]}\right\|_2^2\right)^{\frac{1}{2}} \geq c^{-1}\sqrt{N}\left\|a\right\|_{L^N_{T^2\operatorname{lm} T,2}}.$$

Proposition 4.5. *Let $a\in\mathbf{C}^N$. Then*

$$\left\|a\right\|_{L^N_{T^2\operatorname{lm} T,2}} \geq c^{-1}\frac{1}{\sqrt{\operatorname{lm}\operatorname{lm} N}}\left\|a\right\|_{L^N_{2,1}}.$$

Proof of Proposition 4.5: By Proposition 1A:2.6(ii), we have

$$\left\|a\right\|_{L^N_{T^2\operatorname{lm} T,2}} \approx \frac{1}{\sqrt{N}}\left(\sum_{n=1}^{N}\operatorname{lm}(N/\pi(n))\left|a(n)\right|^2\right)^{\frac{1}{2}},$$

where π is an ordering permutation for a, and by the Cauchy–Schwartz inequality, we have

$$\begin{aligned}
\left\|a\right\|_{L^N_{2,1}} &\approx \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\frac{1}{\sqrt{\pi(n)}}\left|a(n)\right| \\
&\leq \frac{1}{\sqrt{N}}\left(\sum_{n=1}^{N}\frac{1}{\pi(n)\operatorname{lm}(N/\pi(n))}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{N}\operatorname{lm}(N/\pi(n))\left|a(n)\right|^2\right)^{\frac{1}{2}} \\
&\approx c^{-1}\sqrt{\operatorname{lm}\operatorname{lm} N}\left\|a\right\|_{L^N_{T^2\operatorname{lm} T,2}}.
\end{aligned}$$

□

Proof of Proposition 4.4:

$$
\begin{aligned}
\sum_{k=1}^{N/3} W_k \operatorname{lm} k \left\| a|_{[N/k]} \right\|_2^2 &= \sum_{k=1}^{N/3} \frac{1}{k} \sum_{n=1}^{N/k} |a(n)|^2 \\
&\approx \sum_{n=1}^{N} \sum_{k=1}^{N/n} \frac{1}{k} |a(n)|^2 \\
&\approx \sum_{n=1}^{N} \operatorname{lm}(N/n) |a(n)|^2 \\
&\approx \sqrt{N} \, \|a\|_{L^N_{T^2 \operatorname{lm} T, 2}} .
\end{aligned}
$$

□

Proof of Proposition 4.3: It is easy to see that

$$\sum_{k=1}^{N/3} W_k \approx \operatorname{lm} \operatorname{lm} N,$$

and hence

$$\sup_{k \in [N/3]} \sqrt{\operatorname{lm} k} \left\| a|_{[N/k]} \right\|_2 \geq c^{-1} \frac{1}{\sqrt{\operatorname{lm} \operatorname{lm} N}} \left(\sum_{k=1}^{N/3} W_k \operatorname{lm} k \left\| a|_{[N/k]} \right\|_2^2 \right)^{\frac{1}{2}} .$$

Now apply Proposition 4.4. □

Proof of Proposition 4.2: This follows straight away from Propositions 4.3 and 4.5. □

Proof of Proposition 4.1: Let $\sigma(k)$ be the system of strips defined in Section 2. By Theorem 2.1, we have

$$\mathbf{E} \left\| \sum_{s=1}^{S} \gamma_s x_s \right\|_\infty \geq \tfrac{1}{4} \mathbf{E} \sup_{n \in [N]} |\Gamma_n^{\sigma(k)}(x)|.$$

By Proposition 3.1, the right hand side of this dominates

$$c^{-1} \sqrt{\operatorname{lm} k} \left(\frac{1}{k} \sum_{n=1}^{N} \left\| x_{\sigma_n(k)}(n) \right\|_2^2 \right)^{\frac{1}{2}} \geq c^{-1} \sqrt{\operatorname{lm} k} \left\| a|_{[N/k]} \right\|_2 .$$

Finally, the result follows from Proposition 4.2. □

The choice of the weighting W_k seems somewhat arbitary, but it is going to be fundamental to the rest of this chapter. In Chapter 2B we will show how it occurs naturally as a consequence of Fernique's Theorem. We will also consider the effect of other weightings.

4.2) The L_2 Averaging Argument

Now we prove Theorem 0.3 when x is not cyclic. First note that, by Proposition 4.5, it is sufficient to prove the following.

Proposition 4.6. *We have*

$$\mathbf{E}\left\|\sum_{s=1}^{S}\gamma_s x_s\right\|_\infty \geq c^{-1}\frac{1}{\sqrt{\operatorname{lm}\operatorname{lm} N}}\left(\sum_{s=1}^{S}\|x_s\|_{L^N_{T^2\operatorname{lm}T,2}}\right)^{\frac{1}{2}}.$$

Proof: If $\nu \in [N]^{(k)}$, then by Theorem 2.1. we have

$$\mathbf{E}\left\|\sum_{s=1}^{S}\gamma_s x_s\right\|_\infty \geq \tfrac{1}{4}\mathbf{E}\sup_{n\in[N]}|\Gamma_n^{\sigma(\nu)}(x)|,$$

and by Proposition 3.1, this is greater than

$$c^{-1}\sqrt{\operatorname{lm} k}\left(\frac{1}{k}\sum_{n=1}^{N}\|x_{\sigma_n}(n)\|_2^2\right)^{\frac{1}{2}}.$$

Now we take a L_2 average over all standard collections of strips of size k, and deduce

$$\mathbf{E}\left\|\sum_{s=1}^{S}\gamma_s x_s\right\|_\infty \geq c^{-1}\left(\frac{1}{\left|[N]^{(k)}\right|}\sum_{\nu\in[N]^{(k)}}\frac{\operatorname{lm} k}{k}\sum_{n=1}^{N}\|x_{\sigma_n}(n)\|_2^2\right)^{\frac{1}{2}},$$

which is easily seen to be

$$= c^{-1}\left(\frac{\operatorname{lm} k}{k}\sum_{s=1}^{S}\sum_{n=1}^{N}|x_s(n)|^2\,\frac{\left|\{\,\nu\in[N]^{(k)} : s\in\sigma_n(\nu)\}\right|}{\left|[N]^{(k)}\right|}\right)^{\frac{1}{2}},$$

which by Proposition 2.2

$$\geq c^{-1}\left(\frac{\operatorname{lm} k}{N}\sum_{n=1}^{N}\sum_{\{s:\pi_s(n)\leq N/k\}}|x_s(n)|^2\right)^{\frac{1}{2}}.$$

Now we take the L_2 average over all $k \in [N/3]$ with the weighting $W_k = \dfrac{1}{k\operatorname{lm} k}$, remembering that $\sum_{k=1}^{N/3} W_k \approx \operatorname{lm}\operatorname{lm} N$, and derive the following.

$$\begin{aligned}
\mathbf{E}\left\|\sum_{s=1}^{S}\gamma_s x_s\right\|_\infty &\geq c^{-1}\left(\frac{1}{\operatorname{lm}\operatorname{lm} N}\sum_{k=1}^{N/3}\frac{1}{Nk}\sum_{n=1}^{N}\sum_{\{s:\pi_s(n)\leq N/k\}}|x_s(n)|^2\right)^{\frac{1}{2}} \\
&\approx c^{-1}\frac{1}{\sqrt{\operatorname{lm}\operatorname{lm} N}}\left(\sum_{n=1}^{N}\frac{1}{N}\sum_{s=1}^{S}\sum_{k=1}^{N/\pi_s(n)}\frac{1}{k}\,|x_s(n)|^2\right)^{\frac{1}{2}} \\
&\approx c^{-1}\frac{1}{\sqrt{\operatorname{lm}\operatorname{lm} N}}\left(\sum_{n=1}^{N}\frac{1}{N}\sum_{s=1}^{S}\operatorname{lm}(N/\pi_s(n))\,|x_s(n)|^2\right)^{\frac{1}{2}} \\
&\approx \frac{1}{\sqrt{\operatorname{lm}\operatorname{lm} N}}\left(\sum_{n=1}^{N}\|x_s\|_{L^N_{T^2\operatorname{lm}T,2}}\right)^{\frac{1}{2}},
\end{aligned}$$

as desired. □

4.3) The Duality Argument

Here we give a different proof of Theorem 0.3. It can be presented as an averaging argument, where the average over k is an L_1 average, or as a duality argument. The arguments involve a parameter β, but this can be chosen arbitarily as long as it is some number greater than $\frac{1}{2}$. Throughout, W_k will be the weighting equal to $\dfrac{1}{k \operatorname{lm} k}$.

As before, we first illustrate the argument in the cyclic case.

Proposition 4.7. *Let x be cyclic on a. Then*

$$\sum_{k=1}^{N/3} W_k \mathbf{E} \sup_{n \in [N]} |\Gamma_n^{\sigma(k)}(x)| \geq c^{-1} \|a\|_{2,1} .$$

Proof: As before,

$$\mathbf{E} \sup_{n \in [N]} |\Gamma_n^{\sigma(k)}(x)| \geq c^{-1} \sqrt{\operatorname{lm} k} \left\| a|_{[N/k]} \right\|_2 .$$

Now, let

$$w^k(n) = \begin{cases} \dfrac{1}{\sqrt{n}} \left(\dfrac{\operatorname{lm} k}{\operatorname{lm}(N/n)} \right)^\beta & \text{if } n \leq N/k \\ 0 & \text{otherwise} \end{cases}$$

so that $\left\| w^k \right\|_2 \leq c \left(1 \vee \frac{1}{\sqrt{2\beta - 1}} \right) \sqrt{\operatorname{lm} k}$ for $\beta > \frac{1}{2}$. Then by the Cauchy–Schwartz inequality we have

$$\sqrt{\operatorname{lm} k} \left\| a|_{[N/k]} \right\|_2 \geq c^{-1} (1 \wedge \sqrt{2\beta - 1}) \sum_{n=1}^{N} w^k(n) a(n).$$

Hence

$$\sum_{k=1}^{N/3} W_k \mathbf{E} \sup_{n \in [N]} |\Gamma_n^{\sigma(k)}(x)| \geq c^{-1} (1 \wedge \sqrt{2\beta - 1}) \sum_{k=1}^{N/3} \frac{1}{k \operatorname{lm} k} \sum_{n=1}^{N/k} w^k(n) a(n)$$

$$= c^{-1} (1 \wedge \sqrt{2\beta - 1}) \sum_{n=1}^{N} \frac{1}{\sqrt{n}} (\operatorname{lm}(N/n))^{-\beta} \sum_{k=1}^{N/n} \frac{1}{k} (\operatorname{lm} k)^{\beta - 1}$$

$$\geq c^{-1} \frac{1 \wedge \sqrt{2\beta - 1}}{\beta} \|a\|_{2,1} .$$

If we choose $\beta = 1$, say, the result follows. □

Now we give the argument for the non-cyclic case. However, by rewriting the argument in a dual formulation, we concentrate on the choice of weightings.

Proposition 4.8. *Let $\|\cdot\|_\Gamma$ be a norm on $(\mathbf{C}^N)^S$ defined by*

$$\|x\|_\Gamma = \sup_{k \in [N/3]} \sup_{\nu \in [N]^{(k)}} \sqrt{\operatorname{lm} k} \left(\frac{1}{k} \sum_{n=1}^{N} \left\| x_{\sigma_n(\nu)}(n) \right\|_2^2 \right)^{\frac{1}{2}} ,$$

and let $\|\cdot\|_\Gamma^*$ be its dual norm with respect to the duality

$$\langle w, x \rangle = \sum_{s=1}^{S} \sum_{n=1}^{N} w_s(n) x_s(n).$$

Then

$$\|w\|_\Gamma^* = \inf \left\{ \sum_{k=1}^{N/3} \sum_{\nu\in[N]^{(k)}} \frac{k}{\sqrt{\operatorname{lm} k}} \left(\frac{1}{k} \sum_{n=1}^{N} \left\| w^\nu_{\sigma_n(\nu)}(n) \right\|_2^2 \right)^{\frac{1}{2}} \right\},$$

where the infimum is over the set

$$\left\{ (w^\nu \in (\mathbf{C}^N)^S)_{\nu\in[N]^{(k)},k\in[N/3]} : \operatorname{supp} w^\nu \subseteq \sigma(\nu),\ \sum_\nu w^\nu \geq w \right\}.$$

Proof: Let Z be the space $\{(x^\nu \in (\mathbf{C}^N)^S)_{\nu\in[N]^{(k)},k\in[N/3]}\}$ with norm

$$\|(x^\nu)\|_Z = \sup_{k\in[N/3]} \sup_{\nu\in[N]^{(k)}} \sqrt{\frac{\operatorname{lm} k}{k}} \left(\sum_{n=1}^{N} \left\| x^\nu_{\sigma_n(\nu)}(n) \right\|_2^2 \right)^{\frac{1}{2}}.$$

Then $((\mathbf{C}^N)^S, \|\cdot\|_\Gamma)$ embeds isometrically into Z via the diagonal map $I\colon x \mapsto (x)_\nu$. The dual of I is the map

$$\begin{aligned} I^*\colon Z^* &\to ((\mathbf{C}^N)^S, \|\cdot\|_\Gamma^*) \\ (w^\nu)_\nu &\mapsto \sum_\nu w^\nu, \end{aligned}$$

where the induced quotient map $I^*\colon Z^*/\ker(I^*) \to ((\mathbf{C}^N)^S, \|\cdot\|_\Gamma^*)$ is an isometry. But

$$\|(w^\nu)\|_Z^* = \sum_{k=1}^{N/3} \sum_{\nu\in[N]^{(k)}} \sqrt{\frac{k}{\operatorname{lm} k}} \left(\left\| w^\nu_{\sigma_n(\nu)}(n) \right\|_2^2 \right)^{\frac{1}{2}},$$

and the result follows. $\square$

Now, if $w \in (\mathbf{C}^N)^S$ is given by

$$w_s(n) = \overline{\operatorname{sign}(x_s(n))} \frac{1}{\sqrt{\pi_s(n)}} u_s, \tag{4.1}$$

where $u_s = \|x_s\|_{2,1} \Big/ \left(\sum_{s=1}^S \|x_s\|_{2,1}^2 \right)^{\frac{1}{2}}$, then it is easy to see that

$$\left(\sum_{s=1}^{S} \|x_s\|_{2,1}^2 \right)^{\frac{1}{2}} \approx \sum_{s=1}^{S} \sum_{n=1}^{N} w_s(n) x_s(n). \tag{4.2}$$

Now Theorem 2.1 can be stated as follows: for all $x \in (\mathbf{C}^N)^S$ we have

$$\mathbf{E} \left\| \sum_{s=1}^{S} \gamma_s x_s \right\|_\infty \geq c^{-1} \|x\|_\Gamma.$$

Hence it is sufficient to show that

$$\|x\|_\Gamma \geq c^{-1} \frac{1}{\operatorname{lm}\operatorname{lm} N} \frac{1}{\sqrt{N}} \left(\sum_{s=1}^{S} \|x_s\|_{2,1}^2 \right)^{\frac{1}{2}}.$$

But then, by Proposition 4.8 and equation (4.2), it is sufficient to prove the following proposition, to whose proof we devote the rest of this section.

Proposition 4.9. *If $w \in (\mathbf{C}^N)^S$ is the vector given by (4.1), then*

$$\|w\|_\Gamma^* \leq c\sqrt{N}\operatorname{lm}\operatorname{lm} N.$$

Proof: We first note that it is sufficient to prove this in the case that $w \geq 0$. The proof now proceeds by choosing a suitable sequence (w^ν) and checking that it has the correct properties.

For $\nu \in [N]^{(k)}$, we let

$$w_s^\nu(n) = \begin{cases} w_s(n) W_k \dfrac{1}{\frac{k}{N}\left|[N]^{(k)}\right|}\left(\dfrac{\operatorname{lm} k}{\operatorname{lm}(N/\pi_s(n))}\right)^\beta & \text{if } s \in \sigma_n(\nu) \\ 0 & \text{otherwise.} \end{cases}$$

We need to show

i) $$\sum_\nu w^\nu \geq c^{-1} w,$$

and

ii) $$\sum_{k=1}^{N/3} \sum_{\nu\in[N]^{(k)}} \frac{k}{\sqrt{\operatorname{lm} k}} \left(\frac{1}{k}\sum_{n=1}^N \left\|w^\nu_{\sigma_n(\nu)}(n)\right\|_2^2\right)^{\frac{1}{2}} \leq c\sqrt{N}\operatorname{lm}\operatorname{lm} N.$$

To show (i) we note that

$$\begin{aligned}
&\sum_{k=1}^{N/3} \sum_{\nu\in[N]^{(k)}} w_s^\nu(n) \\
&= w_s(n) \sum_{k=1}^{N/3} W_k \frac{1}{\frac{k}{N}\left|[N]^{(k)}\right|}\left(\frac{\operatorname{lm} k}{\operatorname{lm}(N/\pi_s(n))}\right)^\beta \left|\{\, \nu \in [N]^{(k)} : s \in \sigma_n(\nu)\}\right| \\
&\geq w_s(n) \sum_{k=1}^{N/\pi_s(n)} W_k \left(\frac{\operatorname{lm} k}{\operatorname{lm}(N/\pi_s(n))}\right)^\beta \\
&= w_s(n)(\operatorname{lm}(N/\pi_s(n)))^{-\beta} \sum_{k=1}^{N/\pi_s(n)} \frac{1}{k}(\operatorname{lm} k)^{\beta-1} \\
&\approx \tfrac{1}{\beta} w_s(n).
\end{aligned}$$

To show (ii) we have the following chain of inequalities. The quantity

$$\sum_{k=1}^{N/3} \sum_{\nu\in[N]^{(k)}} \frac{k}{\sqrt{\operatorname{lm} k}} \left(\frac{1}{k}\sum_{n=1}^N \left\|w^\nu_{\sigma_n(\nu)}(n)\right\|_2^2\right)^{\frac{1}{2}}$$

is, by the Cauchy–Schwartz inequality, less than

$$\begin{aligned}
&\sum_{k=1}^{N/3} \frac{k}{\sqrt{\operatorname{lm} k}} \left(\left|[N]^{(k)}\right| \sum_{\nu\in[N]^{(k)}} \frac{1}{k}\sum_{n=1}^N \left\|w^\nu_{\sigma_n(\nu)}(n)\right\|_2^2\right)^{\frac{1}{2}} \\
&= \sum_{k=1}^{N/3} \frac{k}{\sqrt{\operatorname{lm} k}} \Bigg(\left|[N]^{(k)}\right| \sum_{s=1}^S \Bigg[\frac{1}{k}\sum_{n=1}^N w_s(n)^2 W_k^2 \times \\
&\qquad \frac{1}{\left(\frac{k}{N}\left|[N]^{(k)}\right|\right)^2}\left(\frac{\operatorname{lm} k}{\operatorname{lm}(N/\pi_s(n))}\right)^{2\beta} \left|\{\, \nu \in [N]^{(k)} : s \in \sigma_n(\nu)\}\right|\Bigg]\Bigg)^{\frac{1}{2}}.
\end{aligned}$$

This, by Proposition 2.2, (and collecting terms) is approximately equal to

$$\sqrt{N}\sum_{k=1}^{N/3} W_k \frac{1}{\sqrt{\operatorname{lm} k}} \left(\sum_{s=1}^{S} u_s^2 \sum_{\substack{n=1 \\ \pi_s(n) \le N/k}}^{N} \frac{1}{\pi_s(n)} \left(\frac{\operatorname{lm} k}{\operatorname{lm}(N/\pi_s(n))} \right)^{2\beta} \right)^{\frac{1}{2}} .$$

Now $\sum_{s=1}^{S} u_s^2 = 1$, and hence the above is equal to

$$\sqrt{N}\sum_{k=1}^{N/3} W_k \frac{1}{\sqrt{\operatorname{lm} k}} \left(\sum_{n=1}^{N/k} \frac{1}{n} \left(\frac{\operatorname{lm} k}{\operatorname{lm}(N/n)} \right)^{2\beta} \right)^{\frac{1}{2}}$$

which is approximately less than $\left(1 \vee \frac{1}{\sqrt{2\beta-1}}\right) \sqrt{N} \operatorname{lm}\operatorname{lm} N$.

$\square$

5) Independent Bernoulli Processes

We now turn our attention to Bernoulli processes, first dealing with the independent case. In contrast to the Gaussian case, this problem is rather difficult. Firstly we are required to know something about the tail of the distribution of a Bernoulli random walk. Secondly, the formula so obtained does not fit into an argument like that given for Proposition 3.2, and a new approach is needed.

5.1) The Tail of the Distribution of a Bernoulli Random Walk

Here we are concerned with finding lower bounds for

$$\Pr\left(\left|\sum_{s=1}^{S} \varepsilon_s a(s)\right| \geq r\right)$$

where $a \in \mathbf{C}^S$ and $r > 0$. To simplify things, we note that it is sufficient to consider $\Pr\left(\sum_{s=1}^{S} \varepsilon_s a(s) \geq r\right)$ when $a \in \mathbf{R}^S$, for we have the following result.

Proposition 5.1. *(See [Ka Ch.2 §6].) Let $a \in \mathbf{C}^S$, and $r > 0$. Then*

$$c^{-1} \Pr\left(\sum_{s=1}^{S} \varepsilon_s |a(s)| \geq cr\right) \leq \Pr\left(\left|\sum_{s=1}^{S} \varepsilon_s a(s)\right| \geq r\right) \leq c \Pr\left(\sum_{s=1}^{S} \varepsilon_s |a(s)| \geq c^{-1} r\right).$$

Furthermore, to save writing, we write E_a for the random variable $\sum_{s=1}^{S} \varepsilon_s a(s)$.

First, we consider upper bounds. We have, by Proposition 1C:1.2(i), that

$$\Pr(E_a \geq t \left\|a\right\|_2) \leq c e^{-c^{-1} t^2}. \tag{5.1}$$

However, this cannot provide the lower bound, because we also have

$$\Pr(E_a > \left\|a\right\|_1) = 0. \tag{5.2}$$

Finding conditions on a for which (5.1) has a reverse inequality has received a great deal of attention (see, for example [Ch Ch.7]). However we take a different approach.

If we combine equations (5.1) and (5.2), we see that whenever $a = a' + a''$, we have

$$\Pr(E_a > \left\|a\right\|_1 + t \left\|a\right\|_2) \leq c e^{-c^{-1} t^2}.$$

Hence

$$\Pr(E_a > K(a, t, \left\|\cdot\right\|_1, \left\|\cdot\right\|_2)) \leq c e^{-c^{-1} t^2},$$

where $K(\cdot, t, \left\|\cdot\right\|_1, \left\|\cdot\right\|_2)$ is the interpolation norm given in Section 1A:2.5. The surprising fact is that this is the best result.

Theorem 5.2. *If $a \in \mathbf{R}^S$ and $t > 0$, then*

$$\Pr(E_a \geq K(a, t, \left\|\cdot\right\|_1, \left\|\cdot\right\|_2)) \geq c^{-1} e^{-ct^2}.$$

Lemma 5.3. *(Paley–Zygmund, see [Ka Ch.3 Thm.3].) If $a \in \mathbf{R}^S$, then*

$$\Pr(E_a \geq \tfrac{1}{2} \left\|a\right\|_2) \geq \tfrac{3}{32}.$$

Proof of Theorem 5.2: First note that it is sufficient to prove this theorem when t is a perfect square, say $t = M^2$. By Proposition 1A:2.9, we need only show

$$\Pr(E_a \geq c^{-1} \|a\|_{P_{1,2}(M)}) \geq e^{-cM},$$

where $\|\cdot\|_{P_{1,2}(M)}$ is the $(1,2)$-M-partition norm described in Section 1A:2.5. But if $\beta_1, \beta_2, \ldots, \beta_M$ is a M-partition of $[S]$ such that

$$\|a\|_{P_{1,2}(M)} = \sum_{m=1}^{M} \left\| a|_{\beta_m} \right\|_2,$$

then by Lemma 5.3, we have

$$\Pr(E_a \geq \tfrac{1}{2} \|a\|_{P_{1,2}(M)}) \geq \prod_{m=1}^{M} \Pr\left(E_{a|_{\beta_m}} \geq \tfrac{1}{2} \left\| a|_{\beta_m} \right\|_2 \right) \geq \left(\tfrac{3}{32}\right)^M .$$

□

5.2) Suprema of Independent Bernoulli Processes

Here we prove the Bernoulli analogue of Proposition 3.1.

Theorem 5.4. *Let $E = (E_n : n \in [N])$ be an independent Bernoulli process of size k, where $E_n = E_{a_n}$ for some $a_n \in \mathbf{C}^S$. Then for $1 \leq p < \infty$ we have*

$$\left(\mathbf{E} \sup_{n \in [N]} |E_n|^p \right)^{\frac{1}{p}} \geq c^{-1} \frac{1}{\sqrt{p}} \frac{1}{k^{\frac{1}{p}}} \left\| \left(K(a_n, \sqrt{\operatorname{lm} k}, \|\cdot\|_1, \|\cdot\|_2) \right)_{n=1}^{N} \right\|_p .$$

(Note that by Corollary 1C:1.5b we have

$$\mathbf{E} \sup_{n \in [N]} |E_n| \geq c^{-1} \frac{1}{\sqrt{p}} \left(\mathbf{E} \sup_{n \in [N]} |E_n|^p \right)^{\frac{1}{p}}$$

for any $1 \leq p < \infty$.)

What is this result saying? If we let $K_n = K(a_n, \sqrt{\operatorname{lm} k}, \|\cdot\|_1, \|\cdot\|_2)$, so that $\Pr(E_n \geq c^{-1} K_n) \geq c^{-1} \frac{1}{k}$, then this result says that the expected supremum of the E_ns dominates the L_p average of the K_ns. Now this is not obvious. It *is* obvious that the expected supremum is greater than the median of the non-zero K_ns. For if we rearrange the a_ns so that the non-zero K_ns are arranged $0 < K_1 \leq K_2 \leq \ldots \leq K_k$, then we see that

$$\begin{aligned} \Pr\left(\sup_{n \in [N]} |E_n| \geq c^{-1} K_{k/2} \right) &\geq \Pr\left(\sup_{k \in [k/2, k]} |E_n| \geq c^{-1} K_{k/2} \right) \\ &= 1 - \prod_{k=k/2}^{k} \left(1 - \Pr(|E_n| \geq c^{-1} K_{k/2}) \right) \\ &\geq 1 - \left(1 - c^{-1} \frac{1}{k} \right)^{k/2} \\ &\geq c^{-1}. \end{aligned}$$

However, there is no reason why the median, $K_{k/2}$, should come anywhere near the desired L_p average.

But we can do better than this. We can show that the expected supremum is larger than the upper $\frac{1}{\sqrt{k}}$-tile, that is, $K_{k-\sqrt{k}}$. This is because $\operatorname{lm} k \approx \operatorname{lm} \sqrt{k}$, and hence $\Pr(E_n \geq c^{-1} K_n) \geq c^{-1} \frac{1}{\sqrt{k}}$. Therefore, following the same argument as above, we have

$$\begin{aligned} \Pr\left(\sup_{n \in [N]} |E_n| \geq c^{-1} K_{k-\sqrt{k}} \right) &\geq 1 - \left(1 - c^{-1} \frac{1}{\sqrt{k}} \right)^{\sqrt{k}} \\ &\geq c^{-1}. \end{aligned}$$

This turns out to be sufficient, as we now show. To make the argument clearer, we abstract Theorem 5.4 into the following result.

Theorem 5.5. *Let $\Theta_1, \Theta_2, \ldots, \Theta_k$ be independent random variables, $M_1, M_2, \ldots, M_k$, $K_1, K_2, \ldots, K_k \geq 0$ and $1 \leq p < \infty$. Suppose we have the following for each $n \in [k]$:*

i)
$$\Pr(|\Theta_n| > K_n) \geq c^{-1}\frac{1}{\sqrt{k}},$$

ii)
$$\mathbf{E}(|\Theta_n|^p)^{\frac{1}{p}} \geq c^{-1}M_n,$$

iii)
$$K_n \leq c\sqrt{\operatorname{lm} k}M_n.$$

Then
$$\left(\mathbf{E}\sup_{n\in[N]}|\Theta_n|^p)\right)^{\frac{1}{p}} \geq c^{-1}\frac{1}{\sqrt{p}}\left\|(K_n)_{n=1}^k\right\|_{L_p^k}.$$

Proof: Suppose, first, that $K_1 \leq K_2 \leq \ldots \leq K_k$. From the above arguments we see that if $K_{k-\sqrt{k}}^p \geq \frac{1}{2}\left\|(K_n)_{n=1}^k\right\|_{L_p^k}^p$, then we are done. Furthermore the theorem is easy to prove when $k < 16$. So let us assume that none of these hold.

First we show that
$$\left\|(K_n)_{n=k-\sqrt{k}}^k\right\|_{L_p^k}^p \geq \frac{1}{4}\left\|(K_n)_{n=1}^k\right\|_{L_p^k}^p.$$

For, we have
$$\frac{1}{k-\sqrt{k}}\sum_{n=1}^{k-\sqrt{k}}K_n^p \leq \frac{1}{2k}\sum_{n=1}^{k}K_n^p,$$

which implies
$$\frac{1}{k-\sqrt{k}}\sum_{n=k-\sqrt{k}}^{k}K_n^p \geq \left(\frac{1}{k-\sqrt{k}}-\frac{1}{2k}\right)\sum_{n=1}^{k}K_n^p,$$

which, as $k \geq 16$, gives the result.

So to prove the theorem, we see that

$$\begin{aligned}
\mathbf{E}\sup_{n\in[N]}|\Theta_n|^p &\geq \mathbf{E}\left(\frac{1}{\sqrt{k}}\sum_{n=k-\sqrt{k}}^{k}|\Theta_n|^p\right)\\
&\geq c^{-p}\frac{1}{\sqrt{k}}\sum_{n=k-\sqrt{k}}^{k}M_n^p\\
&\geq c^{-p}\frac{1}{\sqrt{k}(\operatorname{lm} k)^{\frac{p}{2}}}\sum_{n=k-\sqrt{k}}^{k}K_n^p\\
&\geq c^{-p}\frac{\sqrt{k}}{(\operatorname{lm} k)^{\frac{p}{2}}}\frac{1}{k}\sum_{n=1}^{k}K_n^p.
\end{aligned}$$

Since
$$\frac{\sqrt{k}}{(\operatorname{lm} k)^{\frac{p}{2}}} \geq \frac{1}{(cp)^{\frac{p}{2}}} \quad \text{for all } k \geq 1,$$

the result follows. $\square$

6) The Averaging Argument in the Bernoulli Case

Here we prove Theorem 0.2. The arguments here take up the process we followed in Section 4.3. Once again, we first illustrate the argument in the cyclic case. However the non-cyclic case contains many more difficulties, which we elaborate on later.

As in Section 4.3, we work with a parameter β, which will be a number, usually greater than $\frac{1}{2}$ (we will indicate at what point this limitation is necessary). To save much writing, as in Section 1A:1.1, we will write c_β to denote the phrase 'a continuous function of $\beta \in (\frac{1}{2}, \infty)$, taking positive real values'.

The cyclic case is as follows.

Proposition 6.1. *Let x be cyclic on a. If W_k is the weighting equal to $\dfrac{1}{k \operatorname{lm} k}$ then*

$$\sum_{k=1}^{N/3} W_k \mathbf{E} \sup_{n\in[N]} |E_n^{\sigma(k)}(x)| \geq c^{-1} \|a\|_{2,1}.$$

Proof: By Theorem 5.2, we have

$$\mathbf{E} \sup_{n\in[N]} |E_n^{\sigma(k)}(x)| \geq c^{-1} K(a\big|_{[N/k]}, \sqrt{\operatorname{lm} k}, \|\cdot\|_1, \|\cdot\|_2).$$

Now we use the duality given in Proposition 1A:2.8. Let

$$w^k(n) = \begin{cases} \dfrac{1}{\sqrt{n}} \left(\dfrac{\operatorname{lm} k}{\operatorname{lm}(N/n)} \right)^\beta & \text{for } n \leq N/k \\ 0 & \text{otherwise.} \end{cases}$$

Then $J\big(w^k, \frac{1}{\sqrt{\operatorname{lm} k}}, \|\cdot\|_\infty, \|\cdot\|_2\big) \leq c(1 \vee \sqrt{2\beta - 1})$ for $\beta > \frac{1}{2}$. So

$$K\big(a\big|_{[N/k]}, \sqrt{\operatorname{lm} k}, \|\cdot\|_1, \|\cdot\|_2\big) \geq c_\beta^{-1} \sum_{n=1}^{N} w^k(n) a(n).$$

Now the argument proceeds as in the proof of Proposition 4.7. $\square$

In the non-cyclic case, we use a duality argument as in Section 4.3. However, whereas in Section 4.3 we use Proposition 3.1 with $p = 2$, here we need to use Theorem 5.4 with $p > 2$ (we will indicate when this is needed). So for the rest of Section 6, we fix $p = 3$, and $p' = \frac{3}{2}$ so that $\frac{1}{p} + \frac{1}{p'} = 1$.

Proposition 6.2. *Let $\|\cdot\|_E$ be a norm on $(\mathbf{C}^N)^S$ defined by*

$$\|x\|_E = \sup_{k\in[N/3]} \sup_{\nu\in[N]^{(k)}} \left(\frac{1}{k} \sum_{n=1}^{N} K(x_{\sigma_n(\nu)}(n), \sqrt{\operatorname{lm} k}, \|\cdot\|_1, \|\cdot\|_2)^p \right)^{\frac{1}{p}},$$

and let $\|\cdot\|_E^*$ be its dual norm with respect to the duality

$$\langle w, x \rangle = \sum_{s=1}^{S} \sum_{n=1}^{N} w_s(n) x_s(n).$$

Then

$$\|w\|_E^* = \inf \left\{ \sum_{k=1}^{N/3} \sum_{\nu\in[N]^{(k)}} k \left(\frac{1}{k} \sum_{n=1}^{N} J(w^\nu_{\sigma_n(\nu)}(n), \frac{1}{\sqrt{\operatorname{lm} k}}, \|\cdot\|_\infty, \|\cdot\|_2)^{p'} \right)^{\frac{1}{p'}} \right\},$$

where the infimum is over the set

$$\left\{ (w^\nu \in (\mathbf{C}^N)^S)_{\nu\in[N]^{(k)}, k\in[N/3]} : \operatorname{supp} w^\nu \subseteq \sigma(\nu), \sum_\nu w^\nu \geq w \right\}.$$

Proof: As Proposition 4.8. $\square$

As in Section 4.2, we let $w \in (\mathbf{C}^N)^S$ be given by

$$w_s(n) = \overline{\operatorname{sign}(x_s(n))} \frac{1}{\sqrt{\pi_s(n)}} u_s, \tag{6.1}$$

where $u_2 = \|x_s\|_{2,1} \Big/ \left(\sum_{s=1}^S \|x_s\|_{2,1}^2\right)^{\frac{1}{2}}$, and note that it is sufficient to prove the following.

Proposition 6.3. *If $w \in (\mathbf{C}^N)^S$ is the vector given by* (6.1)*, then*

$$\|w\|_E^* \le c\sqrt{N} \operatorname{lm} \operatorname{lm} N.$$

6.1) The Proof of Proposition 6.3

This proof is long and involved, and so it is split up into several steps. It is presented roughly in the order in which I thought of it, that is, as a sequence of attempts to prove the result, each successive attempt being more refined than the previous attempt, until eventually the result is established. At the end, I shall recap briefly, indicating where the main thread of the 'correct' argument may be found.

The main motivating example will be when all the x_ss have the same $L_{2,1}$ norm, that is, when $u_1 = u_2 = \ldots = u_S = \frac{1}{\sqrt{S}}$. These examples will be presented in 'preludes to steps'. Thus, for example, Step 5 will be illustrated with this example in Prelude to Step 5.

Step 1: Setting the Scene

Since $\|\cdot\|_E$ is a lattice norm, we may assume that $w \ge 0$. We are required to find a sequence (w^ν) that satisfies the required properties:

i)
$$\sum_\nu w^\nu \ge c^{-1} w,$$

and

ii)
$$\sum_{k=1}^{N/3} \sum_{\nu \in [N]^{(k)}} k \left(\frac{1}{k} \sum_{n=1}^N J(w^\nu_{\sigma_n(\nu)}(n), \frac{1}{\sqrt{\operatorname{lm} k}}, \|\cdot\|_\infty, \|\cdot\|_2)^{p'} \right)^{\frac{1}{p'}} \le c\sqrt{N} \operatorname{lm} \operatorname{lm} N.$$

It is easy to see that showing (ii) is equivalent to showing both of

ii′)
$$\sum_{k=1}^{N/3} \sum_{\nu \in [N]^{(k)}} \frac{k}{\sqrt{\operatorname{lm} k}} \left(\frac{1}{k} \sum_{n=1}^N \left\| w^\nu_{\sigma_n(\nu)}(n) \right\|_2^{p'} \right)^{\frac{1}{p'}} \le c\sqrt{N} \operatorname{lm} \operatorname{lm} N,$$

and

ii″)
$$\sum_{k=1}^{N/3} \sum_{\nu \in [N]^{(k)}} k \left(\frac{1}{k} \sum_{n=1}^N \left\| w^\nu_{\sigma_n(\nu)}(n) \right\|_\infty^{p'} \right)^{\frac{1}{p'}} \le c\sqrt{N} \operatorname{lm} \operatorname{lm} N.$$

Step 2: The First Guess at (w^ν)

For the rest of this section, we set $W_k = \dfrac{1}{k \operatorname{lm} k}$ (so that $\sum_{k=1}^{N/3} W_k \approx \operatorname{lm} \operatorname{lm} N$). Let us first consider the w^ν considered in Section 4, that is, let

$$w_s^{(k)}(n) = w_s(n) W_k \frac{1}{\frac{k}{N}\left|[N]^{(k)}\right|} \left(\frac{\operatorname{lm} k}{\operatorname{lm}(N/\pi_s(n))} \right)^\beta,$$

and for $\nu \in [N]^{(k)}$, let

$$w_s^\nu(n) = \begin{cases} w_s^{(k)}(n) & \text{if } s \in \sigma_n(\nu) \\ 0 & \text{otherwise.} \end{cases}$$

Step 3: Establishing Conditions (i) and (ii′)

Condition (i) holds, exactly as in the proof of Proposition 4.9. To show (ii′), we note that the quantity

$$\left(\frac{1}{k} \sum_{n=1}^{N} \left\| w^{\nu}_{\sigma_n(\nu)}(n) \right\|_2^{p'} \right)^{\frac{1}{p'}}$$

is a $L_{p'}$ average (because only at most k of the terms $\left\| w^{\nu}_{\sigma_n(\nu)}(n) \right\|_2$ are non-zero), and hence, by Hölder's inequality, it dominates

$$\left(\frac{1}{k} \sum_{n=1}^{N} \left\| w^{\nu}_{\sigma_n(\nu)}(n) \right\|_2^{2} \right)^{\frac{1}{2}} .$$

Hence condition (ii′) also holds as in the proof of Proposition 4.9.

Step 4: Setting the Scene for Condition (ii′′)

Condition (ii′′) is quite different. We have the following chain of inequalities. The quantity

$$\sum_{k=1}^{N/3} \sum_{\nu \in [N]^{(k)}} k \left(\frac{1}{k} \sum_{n=1}^{N} \left\| w^{\nu}_{\sigma_n(\nu)}(n) \right\|_\infty^{p'} \right)^{\frac{1}{p'}}$$

is, by Hölder's inequality, less than

$$\sum_{k=1}^{N/3} k \left| [N]^{(k)} \right| \left(\frac{1}{\left| [N]^{(k)} \right|} \sum_{\nu \in [N]^{(k)}} \frac{1}{k} \sum_{n=1}^{N} \left\| w^{\nu}_{\sigma_n(\nu)}(n) \right\|_\infty^{p'} \right)^{\frac{1}{p'}} .$$

With a bit of thought, we see that this is at most

$$\sum_{k=1}^{N/3} k \left| [N]^{(k)} \right| \left(\frac{1}{N} \sum_{n=1}^{N} \left\| w^{(k)}_{\rho_{n,k}}(n) \right\|_\infty^{p'} \right)^{\frac{1}{p'}} , \tag{6.2}$$

where $\rho_{n,k} = \bigcup_{\nu \in [N]^{(k)}} \sigma_n(\nu)$. This, in turn, is dominated by

$$\sum_{k=1}^{N/3} k \left| [N]^{(k)} \right| \left(\frac{1}{N} \sum_{n=1}^{N} \left(w^{(k)}_{s_{n,k}}(n) \right)^{p'} \right)^{\frac{1}{p'}}$$

for some $s_{n,k} \in \rho_{n,k}$. This is the same as

$$N^{\frac{1}{p}} \sum_{k=1}^{N/3} W_k \left(\sum_{n=1}^{N} \left(u_{s_{n,k}} \frac{1}{\sqrt{\pi_{s_{n,k}}(n)}} \left(\frac{\operatorname{lm} k}{\operatorname{lm}(N/\pi_{s_{n,k}}(n))} \right)^{\beta} \right)^{p'} \right)^{\frac{1}{p'}} ,$$

and this is less than

$$N^{\frac{1}{p}} \sum_{k=1}^{N/3} W_k M_k^{\frac{1}{p'}} , \tag{6.3}$$

where M_k is the sum of the N largest elements of

$$\left(\left(u_s \frac{1}{\sqrt{n}} \left(\frac{\operatorname{lm} k}{\operatorname{lm}(N/n)} \right)^{\beta} \right)^{p'} : 1 \le s \le S, 1 \le n \le N \right) .$$

So in considering condition (ii′′), we will concentrate on the quantity (6.3).

Prelude to Step 5: Attempting to Establish Condition (ii$''$)

So consider the case when $u_1 = u_2 = \ldots = u_S = \frac{1}{\sqrt{S}}$. For sufficiently large n (depending only on β), the quantity $\sqrt{n}(\operatorname{lm}(N/n))^\beta$ increases as n increases. Thus it is easy to see that M_k is dominated by

$$\begin{aligned} & c_\beta \, S \sum_{n=1}^{N/S} \left(\frac{1}{\sqrt{S}} \frac{1}{\sqrt{n}} \left(\frac{\operatorname{lm} k}{\operatorname{lm}(N/n)} \right)^\beta \right)^{p'} \qquad (6.4) \\ & \le c_\beta N^{1-\frac{p'}{2}} \left(\frac{\operatorname{lm} k}{\operatorname{lm} S} \right)^{\beta p'} . \end{aligned}$$

Hence, the right hand side of (ii$''$) is dominated by

$$c_\beta \sqrt{N} \left(\frac{\operatorname{lm} N}{\operatorname{lm} S} \right)^\beta .$$

Remembering that we insist that $\beta > \frac{1}{2}$ (for (ii$'$) to work), we see that, unless S is large (a power of N), we do not get the desired result.

We also have our first indication that we require $p > 2$. For if we had $p = 2$, then (6.4) would be dominated only by

$$c_\beta \frac{(\operatorname{lm} k)^{2\beta}}{(\operatorname{lm} S)^{2\beta-1}},$$

and so the right hand side of (ii$''$) would be dominated by

$$c_\beta \sqrt{N} \frac{(\operatorname{lm} N)^\beta}{(\operatorname{lm} S)^{\beta-1}}.$$

Even if S were a power of N, this would be too big by a factor $\sqrt{\operatorname{lm} N}$.

Step 5: Attempting to Establish Condition (ii$''$)

The above suggests that if $\sup_{s\in[S]} u_s$ is small, then we should get something reasonable. This is the case, as the following result shows.

Lemma 6.4. *Let M be the sum of the N largest elements of the set*

$$\left(\left(u_s \frac{1}{\sqrt{n}} \left(\frac{\operatorname{lm} k}{\operatorname{lm}(N/n)} \right)^\beta \right)^{p'} : 1 \le s \le S, 1 \le n \le N \right),$$

where u_1, $u_2, \ldots, u_S \ge 0$ with $\sum_{s=1}^S u_s^2 = 1$. If $\sup_{s\in[S]} u_s \le \frac{1}{\sqrt{K}}$ then

$$M \le c \, N^{1-\frac{p'}{2}} \left(\frac{\operatorname{lm} k}{\operatorname{lm} K} \right)^{\beta p'} .$$

Proof: First note that $\sqrt{n}\,(\operatorname{lm}(N/n))^\beta$ increases with n, for n large enough (depending on β). Therefore M must have the form given by

$$M = (\operatorname{lm} k)^{\beta p'} \sum_{s=1}^S u_s^{p'} f(d_s),$$

where

$$\begin{aligned} f(d) &= \sum_{n=1}^d \left(\frac{1}{\sqrt{n}\big(\operatorname{lm}(N/n)\big)^\beta} \right)^{p'} \\ &\le c_\beta \frac{d^{1-\frac{p'}{2}}}{(\operatorname{lm}(N/d))^{\beta p'}} \end{aligned}$$

and $d_1,\ d_2,\ldots,\ d_S$ are non-negative integers such that $\sum_{s=1}^S d_s = N$. (If we had $p=2$, the last line would only be $c_\beta \frac{1}{(\operatorname{lm}(N/d))^{2\beta-1}}$, which would lose a factor $\sqrt{\operatorname{lm} K}$ in the final result.) Also notice that, as f is approximately equal to a sum of a decreasing sequence, it must be approximately equal to a concave function (here the constants of approximation depend on β only).

From the duality described in Proposition 1A:2.8 we have

$$\sum_{s=1}^S u_s^{p'} f(d_s) \le J\Big((u_s^{p'}), K^{\frac{p'}{2}}, \|\cdot\|_{\frac{2}{p'}}, \|\cdot\|_\infty\Big) \cdot K\Big((f(d_s)), K^{-\frac{p'}{2}}, \|\cdot\|_r, \|\cdot\|_1\Big),$$

where $\frac{1}{r} + \frac{p'}{2} = 1$. The first term is easy, as the hypotheses of the lemma say

$$J\Big((u_s^{p'}), K^{\frac{p'}{2}}, \|\cdot\|_{\frac{2}{p'}}, \|\cdot\|_\infty\Big) \le 1.$$

For the second term, suppose without loss of generality that $d_1 \ge d_2 \ge \ldots \ge d_S$. Then

$$K\Big((f(d_s)), K^{-\frac{p'}{2}}, \|\cdot\|_r, \|\cdot\|_1\Big) \le K^{-\frac{p'}{2}} \left\|(f(d_s))_{s=1}^K\right\|_1 + \left\|(f(d_s))_{s=K+1}^S\right\|_r.$$

By Jensen's inequality we have

$$\left\|(f(d_s))_{s=1}^K\right\|_1 \le K f(N/K) \le c\, K^{\frac{p'}{2}} \frac{N^{1-\frac{p'}{2}}}{(\operatorname{lm} K)^{\beta p'}}.$$

Also, a simple argument shows that if $s > K$ then $d_s \le N/K$. Hence

$$\begin{aligned}\left\|(f(d_s))_{s=K+1}^S\right\|_r^r &\le c_\beta \sum_{s=K+1}^S \left(\frac{d_s^{1-\frac{p'}{2}}}{(\operatorname{lm}(N/d_s))^{\beta p'}}\right)^r \\ &\le \frac{1}{(\operatorname{lm} K)^{\beta r p'}} \sum_{s=1}^S d_s \\ &= \frac{N}{(\operatorname{lm} K)^{\beta r p'}}.\end{aligned}$$

The result follows. □

So, if $\sup_{s\in[S]} u_s \le \frac{1}{\sqrt{K}}$, then

$$N^{\frac{1}{p}} \sum_{k=1}^{N/3} W_k M_k^{\frac{1}{p}} \le \sqrt{N} \left(\frac{\operatorname{lm} N}{\operatorname{lm} K}\right)^\beta,$$

and we have established a weaker form of condition (ii$''$). Thus for $\beta > \frac{1}{2}$, we have

$$\mathbf{E}\left\|\sum_{s=1}^S \varepsilon_s x_s\right\|_\infty \ge c_\beta^{-1} \left(\operatorname{lm}\operatorname{lm} N + \left(\frac{\operatorname{lm} N}{\operatorname{lm} K}\right)^\beta\right)^{-1} \left(\sum_{s=1}^S \|x_s\|_{L_{2,1}^N}^2\right)^{\frac{1}{2}}.$$

Step 6: Our First Non-trivial Result about Rademacher Cotype

We now have sufficient to obtain our first non-trivial result about the Rademacher cotype of bounded linear operators from $C(K)$.

Theorem 6.5. *Let T be the formal identity $l_\infty^N \hookrightarrow L_{2,1}^N$. Then for $\beta > \frac{1}{2}$, we have*

$$R^2(T) \leq c_\beta \left(\sqrt{\operatorname{lm} K} \operatorname{lm}\operatorname{lm} N + \left(\frac{\operatorname{lm} N}{\operatorname{lm} K} \right)^\beta \right).$$

In particular, if we set $K = N^{(\log N)^{1/2}}$, we obtain

$$R^2(T) \leq C_\delta (\operatorname{lm} N)^{\frac{1}{4}+\delta},$$

where C_δ depends on $\delta > 0$ only.

Proof: Let $[S] = \mathcal{S}_1 \cup \mathcal{S}_2$, where

$$\mathcal{S}_1 = \{ s \in [S] : u_s \leq \frac{1}{\sqrt{K}} \},$$
$$\mathcal{S}_2 = \{ s \in [S] : u_s > \frac{1}{\sqrt{K}} \}.$$

From the above, we see that

$$\mathbf{E} \left\| \sum_{s \in \mathcal{S}_1} \varepsilon_s x_s \right\|_\infty \geq c_\beta^{-1} \left(\operatorname{lm}\operatorname{lm} N + \left(\frac{\operatorname{lm} N}{\operatorname{lm} K} \right)^\beta \right)^{-1} \left(\sum_{s \in \mathcal{S}_1} \|x_s\|_{L_{2,1}^N}^2 \right)^{\frac{1}{2}}.$$

Also, by Proposition 1C:4.2(ii) and Theorem 0.3 (and noting that $|\mathcal{S}_2| \leq K$) we have

$$\begin{aligned}
\mathbf{E} \left\| \sum_{s \in \mathcal{S}_2} \varepsilon_s x_s \right\|_\infty &\geq c^{-1} \frac{1}{\sqrt{\operatorname{lm} |\mathcal{S}_2|}} \mathbf{E} \left\| \sum_{s \in \mathcal{S}_2} \gamma_s x_s \right\|_\infty \\
&\geq c^{-1} \frac{1}{\sqrt{\operatorname{lm} K}} \frac{1}{\operatorname{lm}\operatorname{lm} N} \left(\sum_{s \in \mathcal{S}_2} \|x_s\|_{L_{2,1}^N}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Now we argue as follows.

$$\begin{aligned}
\mathbf{E} \left\| \sum_{s=1}^S \varepsilon_s x_s \right\|_\infty &= \frac{1}{2} \left(\mathbf{E} \left\| \sum_{s \in \mathcal{S}_1} \varepsilon_s x_s + \sum_{s \in \mathcal{S}_2} \varepsilon_s x_s \right\|_\infty \right. \\
&\qquad \left. + \mathbf{E} \left\| \sum_{s \in \mathcal{S}_1} \varepsilon_s x_s - \sum_{s \in \mathcal{S}_2} \varepsilon_s x_s \right\|_\infty \right) \\
&\geq \frac{1}{2} \max \left\{ \mathbf{E} \left\| \sum_{s \in \mathcal{S}_1} \varepsilon_s x_s \right\|_\infty, \mathbf{E} \left\| \sum_{s \in \mathcal{S}_2} \varepsilon_s x_s \right\|_\infty \right\} \\
&\geq c^{-1} \left(\sqrt{\operatorname{lm} K} \operatorname{lm}\operatorname{lm} N + \left(\frac{\operatorname{lm} N}{\operatorname{lm} K} \right)^\beta \right)^{-1} \\
&\qquad \max \left\{ \left(\sum_{s \in \mathcal{S}_1} \|x_s\|_{L_{2,1}^N}^2 \right)^{\frac{1}{2}}, \left(\sum_{s \in \mathcal{S}_2} \|x_s\|_{L_{2,1}^N}^2 \right)^{\frac{1}{2}} \right\} \\
&\geq c^{-1} \left(\sqrt{\operatorname{lm} K} \operatorname{lm}\operatorname{lm} N + \left(\frac{\operatorname{lm} N}{\operatorname{lm} K} \right)^\beta \right)^{-1} \left(\sum_{s=1}^S \|x_s\|_{L_{2,1}^N}^2 \right)^{\frac{1}{2}}
\end{aligned}$$

as desired. $\square$

This result is certainly good enough to show Corollary 1C:5.2a.

Step 7: The Second Guess at (w^ν)

However, we can do better than this. Suppose in equation (6.1) that we could say that $\rho_{n,k} \subseteq \{\, s : u_s \le \frac{1}{\sqrt{k}} \}$. Then Lemma 6.4 would give $M_k \le c N^{1-\frac{p'}{2}}$, which would easily be sufficient. This could be achieved by insisting that $w_s^{(k)}(n) = 0$ if $u_s > \frac{1}{\sqrt{K}}$. However, we then have to modify the other values of $w_s^{(k)}(n)$ so that (i) still holds. Let

$$w_s^{(k)}(n) = \begin{cases} w_s(n) W_k \dfrac{1}{\frac{k}{N}\left|[N]^{(k)}\right|} \left(\dfrac{\operatorname{lm} k}{\operatorname{lm}((N/\pi_s(n)) \wedge (1/u_s^2))} \right)^\beta & \text{f } u_s \le \frac{1}{\sqrt{k}} \\ 0 & \text{otherwise,} \end{cases}$$

and define w^ν from $w^{(k)}$ as before:

$$w_s^\nu(n) = \begin{cases} w_s^{(k)}(n) & \text{if } s \in \sigma_n(\nu) \\ 0 & \text{otherwise.} \end{cases}$$

Step 8: Establishing Conditions (i) and (ii$''$)

The argument for showing (i) is almost exactly as in the proof of Proposition 4.9:

$$\begin{aligned} &\sum_{k=1}^{N/3} \sum_{\nu \in [N]^{(k)}} w_s^\nu(n) \\ &= w_s(n) \sum_{k=1}^{N \wedge (1/u_s^2)} W_k \frac{1}{\frac{k}{N}\left|[N]^{(k)}\right|} \\ &\qquad \left(\frac{\operatorname{lm} k}{\operatorname{lm}((N/\pi_s(n)) \wedge (1/u_s^2))} \right)^\beta \left| \{\, \nu \in [N]^{(k)} : s \in \sigma_n(\nu) \} \right| \\ &\ge w_s(n) \sum_{k=1}^{(N/\pi_s(n)) \wedge (1/u_s^2)} W_k \left(\frac{\operatorname{lm} k}{\operatorname{lm}((N/\pi_s(n)) \wedge (1/u_s^2))} \right)^\beta \\ &\approx c_\beta^{-1} w_s(n). \end{aligned}$$

Also condition (ii$''$) follows very much as in Step 4. We see that the quantity

$$\sum_{k=1}^{N/3} \sum_{\nu \in [N]^{(k)}} k \left(\frac{1}{k} \sum_{n=1}^{N} \left\| w_{\sigma_n(\nu)}^\nu(n) \right\|_\infty^{p'} \right)^{\frac{1}{p'}}$$

is dominated by

$$N^{\frac{1}{p}} \sum_{k=1}^{N/3} W_k M_k^{\frac{1}{p'}},$$

where M_k is the sum of the N largest elements of

$$\left(\left(u_s \frac{1}{\sqrt{n}} \left(\frac{\operatorname{lm} k}{\operatorname{lm}((N/n) \wedge (1/u_s^2))} \right)^\beta \right)^{p'} : 1 \le s \le S, 1 \le n \le N, u_s \le \frac{1}{\sqrt{k}} \right).$$

Then condition (ii$''$) follows from the next lemma.

Lemma 6.6. *Let M_k be the quantity just described. Then*

$$M_k \le c_\beta N^{1-\frac{p'}{2}}.$$

Proof: This follows as Lemma 6.4, with only a very few changes. ∎

Step 9: Setting the Scene for Condition (ii′)

However condition (ii′) is now much more difficult to prove. First let us establish the preliminary inequalities, so that we may identify more easily where the difficulties lie. The quantity

$$\sum_{k=1}^{N/3} \sum_{\nu \in [N]^{(k)}} \frac{k}{\sqrt{\operatorname{lm} k}} \left(\frac{1}{k} \sum_{n=1}^{N} \left\| w^{\nu}_{\sigma_n(\nu)}(n) \right\|_2^{p'} \right)^{\frac{1}{p'}}$$

is, by Hölder's inequality, less than

$$\sum_{k=1}^{N/3} \frac{k}{\sqrt{\operatorname{lm} k}} \left| [N]^{(k)} \right| \left(\frac{1}{\left| [N]^{(k)} \right|} \sum_{\nu \in [N]^{(k)}} \frac{1}{k} \sum_{n=1}^{N} \left\| w^{\nu}_{\sigma_n(\nu)}(n) \right\|_2^{p'} \right)^{\frac{1}{p'}}$$

$$= N \sum_{k=1}^{N/3} W_k \frac{1}{\sqrt{\operatorname{lm} k}} \left(\frac{1}{\left| [N]^{(k)} \right|} \sum_{\nu \in [N]^{(k)}} \frac{1}{k} \sum_{n=1}^{N} \left(\sum_{\substack{s \in \sigma_n(\nu) \\ u_s \le \frac{1}{\sqrt{k}}}} V(s,n,k)^2 \right)^{\frac{p'}{2}} \right)^{\frac{1}{p'}},$$

where $V(s,n,k) = w_s(n) \left(\frac{\operatorname{lm} k}{\operatorname{lm}((N/\pi_s(n)) \wedge (1/u_s^2))} \right)^{\beta}$. This in turn is dominated by

$$N \sum_{k=1}^{N/3} W_k \frac{1}{\sqrt{\operatorname{lm} k}} \left(\frac{1}{N} \sum_{n=1}^{N} \left(\sum_{s \in \sigma_{n,k}} V(s,n,k)^2 \right)^{\frac{p'}{2}} \right)^{\frac{1}{p'}}, \tag{6.5}$$

where $\sigma_{n,k} = \{ s : \pi_s(n) \le N/k \text{ and } u_s \le \frac{1}{\sqrt{k}} \}$. It is the quantity (6.5) that we will concentrate on.

Prelude to Step 10: Establishing Condition (ii′)

We illustrate the arguments required in the case $u_1 = u_2 = \ldots = u_S = \frac{1}{\sqrt{S}}$. Now quantity (6.5) becomes

$$N \sum_{k=1}^{N \wedge S} W_k \frac{1}{\sqrt{\operatorname{lm} k}} \left(\frac{1}{N} \sum_{n=1}^{N} \left(\sum_{\{s : \pi_s(n) \le N/k\}} V(s,n,k)^2 \right)^{\frac{p'}{2}} \right)^{\frac{1}{p'}}. \tag{6.6}$$

Now, this quantity is the sum of a $L_{p'}$ average of a l_2 sum. Our first attempt is to dominate this by a sum of a L_2 average of a l_2 sum, to get

$$N \sum_{k=1}^{N \wedge S} W_k \frac{1}{\sqrt{\operatorname{lm} k}} \left(\frac{1}{N} \sum_{n=1}^{N} \sum_{\{s : \pi_s(n) \le N/k\}} V(s,n,k)^2 \right)^{\frac{1}{2}}$$

$$\approx \sqrt{N} \sum_{k=1}^{N \wedge S} W_k \frac{1}{\sqrt{\operatorname{lm} k}} \left(\sum_{s=1}^{S} u_s^2 \sum_{n=1}^{N/k} \frac{1}{n} \left(\frac{\operatorname{lm} k}{\operatorname{lm}((N/n) \wedge S)} \right)^{2\beta} \right)^{\frac{1}{2}}, \tag{6.7}$$

where $\sum_{s=1}^{S} u_s^2 = 1$. Now, the innermost sum evaluates to

$$\sum_{n=1}^{N/S} \frac{1}{n} \left(\frac{\operatorname{lm} k}{\operatorname{lm} S} \right)^{2\beta} + \sum_{n=N/S+1}^{N/k} \frac{1}{n} \left(\frac{\operatorname{lm} k}{\operatorname{lm}(N/n)} \right)^{2\beta}$$

$$\overset{c_\beta}{\approx} (\operatorname{lm} N/S)\left(\frac{\operatorname{lm} k}{\operatorname{lm} S}\right)^{2\beta} + (\operatorname{lm} k)^{2\beta} - (\operatorname{lm} S)\left(\frac{\operatorname{lm} k}{\operatorname{lm} S}\right)^{2\beta}.$$

If S is, for example, 1, then this is approximately equal to $\sqrt{\operatorname{lm} N}$, and the quantity (6.7) is dominated only by $\sqrt{N}\sqrt{\operatorname{lm} N}$, and the attempt has failed. (Thus we have another indication that we need $p > 2$).

The solution is to dominate the $L_{p'}$ average of the l_2 sum by an L_{q_k} average of an l_{q_k} sum, where q_k depends on k only, and $p' \le q_k \le 2$. Then (6.6) becomes

$$N^{\left(1-\frac{1}{q_k}\right)} \sum_{k=1}^{N\wedge S} W_k \frac{1}{\sqrt{\operatorname{lm} k}} \left(\sum_{s=1}^{S} u_s^{q_k} \sum_{n=1}^{N/k} \frac{1}{n^{\frac{q_k}{2}}} \left(\frac{\operatorname{lm} k}{\operatorname{lm}((N/n)\wedge S)}\right)^{q_k\beta}\right)^{\frac{1}{q_k}}.$$

Now it is sufficient to note that

a) for $n \le N/k$ we have $\operatorname{lm}((N/n)\wedge S) \ge \operatorname{lm} k$;

b) by Hölder's inequality we have

$$\left(\sum_{s=1}^{S} u_s^{q_k}\right)^{\frac{1}{q_k}} \le S^{\left(\frac{1}{q_k}-\frac{1}{2}\right)} \left(\sum_{s=1}^{S} u_s^{2}\right)^{\frac{1}{2}} \le S^{\left(\frac{1}{q_k}-\frac{1}{2}\right)};$$

c) $\displaystyle\sum_{n=1}^{N/k} \frac{1}{n^{\frac{q_k}{2}}} \le c\left(\frac{1}{q_k}-\frac{1}{2}\right)^{-1} (N/k)^{\left(1-\frac{q_k}{2}\right)}$.

Thus the above is dominated by

$$c\sqrt{N} \sum_{k=1}^{N\wedge S} W_k \frac{1}{\sqrt{\operatorname{lm} k}} (S/k)^{\left(\frac{1}{q_k}-\frac{1}{2}\right)} \left(\frac{1}{q_k}-\frac{1}{2}\right)^{-\frac{1}{q_k}}.$$

Now let q_k be such that $\frac{1}{q_k} - \frac{1}{2} \approx \frac{1}{\operatorname{lm} S}$. Then this is approximately less than

$$\sqrt{N}\sqrt{\operatorname{lm} S} \sum_{k=1}^{N\wedge S} W_k \frac{1}{\sqrt{\operatorname{lm} k}} \le c\sqrt{N} \left(\frac{\operatorname{lm} S}{\operatorname{lm}(N\wedge S)}\right)^{\frac{1}{2}},$$

and we are done in the case when all the u_ss are the same, and $S \le N$.

Step 10: Establishing Condition (ii′)

Now we come to what is probably the most difficult part of the thesis. We give the complete argument for when the u_ss are different. This argument is essentially batching together the u_ss that are roughly the same size. Our starting point is the quantity (6.5). To save space, we will write

$$V'(s,n) = \frac{V(s,n,k)}{(\operatorname{lm} k)^\beta} = w_s(n) \frac{1}{(\operatorname{lm}((N/\pi_s(n))\wedge(1/u_s^2)))^\beta}.$$

First, we split the outer sum of (6.5) to get

$$N \sum_{l=1}^{\log(1+\log(1+\frac{N}{3}))} \sum_{m=e^{l-1}}^{e^l-1} \sum_{k=e^{m-1}}^{e^m-1} W_k \frac{1}{\sqrt{\operatorname{lm} k}} \left(\frac{1}{N}\sum_{n=1}^{N}\left(\sum_{s\in\sigma_{n,k}} V(s,n,k)^2\right)^{\frac{p'}{2}}\right)^{\frac{1}{p'}}$$

$$= N \sum_{l=1}^{\log(1+\log(1+\frac{N}{3}))} \sum_{m=e^{l-1}}^{e^l-1} \sum_{k=e^{m-1}}^{e^m-1} \frac{(\operatorname{lm} k)^{(\beta-\frac{1}{2}-1)}}{k} \left(\frac{1}{N}\sum_{n=1}^{N}\left(\sum_{s\in\sigma_{n,k}} V'(s,n)^2\right)^{\frac{p'}{2}}\right)^{\frac{1}{p'}}$$

$$\leq c\,N \sum_{l=1}^{\log(1+\log(1+\frac{N}{3}))} \sum_{m=e^{l-1}}^{e^l-1} m^{(\beta-\frac{1}{2}-1)} \left(\frac{1}{N} \sum_{n=1}^{N} \left(\sum_{s\in\sigma_{n,e^{m-1}}} V'(s,n)^2 \right)^{\frac{p'}{2}} \right)^{\frac{1}{p'}}$$

$$\leq c_\beta\,N \sum_{l=1}^{\log(1+\log(1+\frac{N}{3}))} e^{l(\beta-\frac{1}{2})} \left(\frac{1}{N} \sum_{n=1}^{N} \left(\sum_{s\in\sigma_{n,\exp(e^{l-1}-1)}} V'(s,n)^2 \right)^{\frac{p'}{2}} \right)^{\frac{1}{p'}}.$$

Now, let $\tau_{n,l} = \sigma_{n,\exp(e^{l-1}-1)} \setminus \sigma_{n,\exp(e^{l}-1)}$, so that $\sigma_{n,\exp(e^{l-1}-1)} = \bigcup_{l'=l}^{\mathrm{lm}\,\mathrm{lm}\,N} \tau_{n,l'}$. Then by the triangle inequality for the $L_{p'}$ and l_2 norms, the above is approximately less than

$$N \sum_{l=1}^{\log(1+\log(1+\frac{N}{3}))} e^{l(\beta-\frac{1}{2})} \sum_{l'=l}^{\mathrm{lm}\,\mathrm{lm}\,N} \left(\frac{1}{N} \sum_{n=1}^{N} \left(\sum_{s\in\tau_{n,l'}} V'(s,n)^2 \right)^{\frac{p'}{2}} \right)^{\frac{1}{p'}}$$

which, rearranging sums, is the same as

$$N \sum_{l'=1}^{\mathrm{lm}\,\mathrm{lm}\,N} \sum_{l=1}^{l'\wedge\log(1+\log(1+N))} e^{l(\beta-\frac{1}{2})} \left(\frac{1}{N} \sum_{n=1}^{N} \left(\sum_{s\in\tau_{n,l'}} V'(s,n)^2 \right)^{\frac{p'}{2}} \right)^{\frac{1}{p'}}. \tag{6.8}$$

The inner sum is given by

$$\sum_{l=1}^{l'\wedge\log(1+\log(1+N))} e^{l(\beta-\frac{1}{2})} \overset{c_\beta}{\approx} e^{l'(\beta-\frac{1}{2})}. \tag{6.9}$$

So now we will concentrate on the term

$$\left(\frac{1}{N} \sum_{n=1}^{N} \left(\sum_{s\in\tau_{n,l'}} V'(s,n)^2 \right)^{\frac{p'}{2}} \right)^{\frac{1}{p'}}. \tag{6.10}$$

First note that if $s \in \tau_{n,l'}$, then

$$\mathrm{lm}((N/\pi_s(n)) \wedge (1/u_s^2)) \geq c^{-1}e^{l'},$$

and so (6.10) is dominated by

$$c_\beta e^{-\beta l'} \left(\frac{1}{N} \sum_{n=1}^{N} \left(\sum_{s\in\tau_{n,l'}} \big(w_s(n)\big)^2 \right)^{\frac{p'}{2}} \right)^{\frac{1}{p'}}.$$

Now

$$\left(\frac{1}{N} \sum_{n=1}^{N} \left(\sum_{s\in\tau_{n,l'}} \big(w_s(n)\big)^2 \right)^{\frac{p'}{2}} \right)^{\frac{1}{p'}}$$

$$\leq \left(\frac{1}{N} \sum_{n=1}^{N} \left(\sum_{s \in \tau'_{n,l'}} \big(w_s(n)\big)^2 \right)^{\frac{p'}{2}} \right)^{\frac{1}{p'}} \tag{6.11}$$

$$+ \left(\frac{1}{N} \sum_{n=1}^{N} \left(\sum_{s \in \tau''_{n,l'}} \big(w_s(n)\big)^2 \right)^{\frac{p'}{2}} \right)^{\frac{1}{p'}}, \tag{6.12}$$

where

$$\tau'_{n,l'} = \left\{ s : \begin{array}{c} \frac{N}{\exp(e^{l'}-1)} \leq \pi_s(n) \leq \frac{N}{\exp(e^{l'-1}-1)} \\ \text{and } u_s^2 \leq \frac{1}{\exp(e^{l'-1}-1)} \end{array} \right\},$$

and

$$\tau''_{n,l'} = \left\{ s : \begin{array}{c} \pi_s(n) \leq \frac{N}{\exp(e^{l'-1}-1)} \text{ and} \\ \frac{1}{\exp(e^{l'}-1)} \leq u_s^2 \leq \frac{1}{\exp(e^{l'-1}-1)} \end{array} \right\},$$

so that $\tau_{n,l'} = \tau'_{n,l'} \cup \tau''_{n,l'}$.

First we estimate (6.11). We dominate the $L_{p'}$ average of the l_2 sum by a L_2 average of a l_2 sum:

$$\begin{aligned}
&\left(\frac{1}{N} \sum_{n=1}^{N} \sum_{s \in \tau'_{n,l'}} u_s^2 \frac{1}{\pi_s(n)} \right)^{\frac{1}{2}} \\
&= \left(\frac{1}{N} \sum_{s=1}^{S} u_s^2 \sum_{n : s \in \tau'_{n,l'}} \frac{1}{\pi_s(n)} \right)^{\frac{1}{2}} \\
&\leq \left(\frac{1}{N} \int_{N/\exp(e^{l'}-1)}^{N/\exp(e^{l'-1}-1} (1/t)\, dt \right)^{\frac{1}{2}} \\
&\leq \tfrac{1}{\sqrt{N}} e^{\frac{1}{2} l'}.
\end{aligned}$$

Next, we dominate (6.12) by a L_q average of a l_q sum, where q, depending on l' only, is such that $p' \leq q \leq 2$, to get

$$\begin{aligned}
&\left(\frac{1}{N} \sum_{n=1}^{N} \sum_{s \in \tau'_{n,l'}} u_s^q \frac{1}{\big(\pi_s(n)\big)^{\frac{q}{2}}} \right)^{\frac{1}{q}} \\
&\leq \left(\frac{1}{N} \left(\sum_{\substack{s=1 \\ u_s^2 \geq \frac{1}{\exp(e^{l'}-1)}}}^{S} u_s^q \right) \left(\sum_{n=1}^{N} \frac{1}{n^{\frac{q}{2}}} \right) \right)^{\frac{1}{q}} \\
&\leq \left(\tfrac{1}{N} \left(\exp(e^{l'}-1) \right)^{\left(1-\frac{q}{2}\right)} \left(\sum_{s=1}^{S} u_s^2 \right) \left(1-\tfrac{q}{2}\right)^{-1} (N+1)^{\left(1-\frac{q}{2}\right)} \right)^{\frac{1}{q}} \\
&\leq c \tfrac{1}{\sqrt{N}} \exp\left(\left(\tfrac{1}{q} - \tfrac{1}{2} \right) e^{l'} \right) \left(1-\tfrac{q}{2}\right)^{\left(\frac{1}{2}-\frac{1}{q}\right)} \left(1-\tfrac{q}{2}\right)^{-\frac{1}{2}}.
\end{aligned}$$

Now choose q so that $\frac{1}{q}-\frac{1}{2}=e^{-l'}$ (and note that $q\approx 2$). Then this is dominated by

$$c\frac{1}{\sqrt{N}}e\left(e^{-l'}\right)^{-e^{-l'}}e^{\frac{1}{2}l'}\approx\frac{1}{\sqrt{N}}e^{\frac{1}{2}l'}.$$

Hence (6.10) is dominated by $ce^{l'(\frac{1}{2}-\beta)}$.

Now we substitute this estimate for (6.10), and also (6.9), into (6.8), and we see that (6.8) is approximately less than

$$\sqrt{N}\sum_{l'=1}^{\operatorname{lm}\operatorname{lm}N}1\approx\sqrt{N}\operatorname{lm}\operatorname{lm}N,$$

as desired. Hence condition (ii$'$) follows.

Step 11: Cry Victory!

Thus Proposition 6.3 is proved. Now we will recap the main steps, picking out the 'correct' argument. First of all, in Step 1, we establish three conditions, called (i), (ii$'$) and (ii$''$), that a suitable sequence $\left(w^\nu\in(\mathbf{C}^N)^S\right)$ has to satisfy. We choose the sequence in Step 7. Condition (i) is easily shown in Step 8. Condition (ii$''$) is established in Steps 4 and 8 (with the help of Step 5). Finally condition (ii$'$) is proved in Steps 9 and 10. □

Chapter 2B — A Discussion of the Methods of Chapter 2A

In this chapter, we look at the various methods used in Chapter 2A, and see to what extent they are the best we can do. We also modify the methods to find the cotype 2 constant of other operators from l_∞^N.

1) The Reduction to Independent Processes

First we look at Theorem 2A:2.1. As we pointed out before, this result seems to throw away a lot of information. Indeed it is surprising that we seem to only pick up, at worst, a $\operatorname{lm}\operatorname{lm} N$ factor. We compare this theorem (in the Gaussian case) to other well known results that find lower bounds for suprema of Gaussian processes, specifically the theorems of Sudakov and Fernique. (There is also a recent theorem due to Talagrand, which we discuss in the next chapter.)

We avoid a few technicalities by only quoting the theorems for finite Gaussian processes, since that is all we are interested in. First we introduce some definitions.

Definition. *Let $\Gamma = (\Gamma_n : n \in [N])$ be a Gaussian process. Define the semi-metric $d = d_\Gamma$ on $[N]$ by*

$$d(n,m) = \left(\mathbf{E}\,|\Gamma_n - \Gamma_m|^2\right)^{\frac{1}{2}}.$$

For $\epsilon > 0$, let $\mathcal{N}(\epsilon)$ be the size of the largest collection of disjoint d-balls of radius ϵ. (Thus $\mathcal{N}(\epsilon)$ is a decreasing integer valued function of ϵ.)

We say that Γ is stationary if there is a group operation $\circ$ on $[N]$ (not necessarily abelian) such that for $n, m, l \in [N]$ we have

$$d(n,m) = d(n \circ l, m \circ l).$$

Theorem 1.1. *(Sudakov, see [F 2.3.1].) Let $\Gamma = (\Gamma_n : n \in [N])$ be a Gaussian process. Then for any $S \subseteq [N]$ we have*

$$\mathbf{E} \sup_{n\in[N]} |\Gamma_n| \ge c^{-1}\sqrt{\log|S|} \inf_{n\ne m\in[S]} d(n,m).$$

Thus

$$\mathbf{E} \sup_{n\in[N]} |\Gamma_n| \ge c^{-1} \sup_{\epsilon>0} \epsilon \sqrt{\log \mathcal{N}(\epsilon)}.$$

Theorem 1.2. *(Fernique, see [F 7.2.2] or [Ka Ch.15 §5].) Let $\Gamma = (\Gamma_n : n \in [N])$ be a stationary Gaussian process. Then*

$$\mathbf{E} \sup_{n\in[N]} |\Gamma_n| \ge c^{-1} \int_0^\infty \sqrt{\log \mathcal{N}(\epsilon)}\, d\epsilon.$$

Clearly Fernique's theorem implies Sudakov's theorem for stationary processes.

1.1) The Cyclic Case

Here we shed light on the cyclic case using Sudakov's and Fernique's theorems. Sudakov's theorem doesn't give us any new information, it simply allows us to rederive the results we have. However Fernique's theorem shows us why the weighting $W_k = \dfrac{1}{k \operatorname{lm} k}$, used so often in Chapter 2A, is correct.

When x is cyclic on a, Theorem 2A:2.1 states that

$$\mathbf{E} \sup_{n\in[N]} |\Gamma_n(x)| \geq \tfrac{1}{4} \sup_{k\in[N/3]} \mathbf{E} \sup_{n\in[N]} |\Gamma_n^{\sigma(k)}(x)|. \tag{1.1}$$

In Section 2A:4.2, we bounded this below by a L_2 average, with weighting $W_k = \dfrac{1}{k \operatorname{lm} k}$:

$$\mathbf{E} \sup_{n\in[N]} |\Gamma_n(x)| \geq c^{-1} \left(\frac{1}{\operatorname{lm}\operatorname{lm} N} \sum_{k=1}^{N/3} W_k \left(\mathbf{E} \sup_{n\in[N]} |\Gamma_n^{\sigma(k)}(x)| \right)^2 \right)^{\frac{1}{2}},$$

and from this we obtained our result. However, in Section 2A:4.3, we used an even weaker lower bound, the L_1 average with the same weighting:

$$c\mathbf{E} \sup_{n\in[N]} |\Gamma_n| \geq c^{-1} \frac{1}{\operatorname{lm}\operatorname{lm} N} \sum_{k=1}^{N/3} W_k \mathbf{E} \sup_{n\in[N]} |\Gamma_n^{\sigma(k)}(x)|.$$

With the help of Fernique's theorem we shall be able to derive the same result without the $\operatorname{lm}\operatorname{lm} N$ factor.

To apply Sudakov's and Fernique's theorems, we need to calculate $\mathcal{N}(\epsilon)$. This requires the following technical lemmas.

Lemma 1.3. *Let x be cyclic on a, and x' be cyclic on $a\big|_{[N/2]}$. Then*

$$\mathbf{E} \left\| \sum_{s=1}^{S} \gamma_s x_s \right\|_\infty \leq c\mathbf{E} \left\| \sum_{s=1}^{S} \gamma_s x'_s \right\|_\infty.$$

Proof:

$$\begin{aligned}
\mathbf{E} \left\| \sum_{s=1}^{S} \gamma_s x'_s \right\|_\infty &\leq \mathbf{E} \left(\sup_{1\leq S_1\leq S_2<S_3\leq S_4\leq S} \left\| \sum_{s=S_1}^{S_2} \gamma_s x_s + \sum_{s=S_3}^{S_4} \gamma_s x_s \right\|_\infty \right) \\
&\leq 2\mathbf{E} \left(\sup_{1\leq S_1\leq S_2\leq S} \left\| \sum_{s=S_1}^{S_2} \gamma_s x_s \right\|_\infty \right),
\end{aligned}$$

which by Lemma 2A:2.3, is

$$\leq 8\mathbf{E} \left\| \sum_{s=1}^{S} \gamma_s x_s \right\|_\infty.$$

□

Lemma 1.4. *Suppose x is cyclic on a, where $a = a\big|_{[N/2]}$. Then for $2 \leq k \leq N$ we have*

$$\mathcal{N}\left(\left\| a\big|_{[N/k]} \right\|_2 \right) \geq \tfrac{k}{2}.$$

Proof: First notice that for $m \leq N/2$, we have

$$\begin{aligned}
d(1,m) &= d\big((n+1) \bmod N, (m+n) \bmod N\big) \\
&= \left(\sum_{n=1}^{N} |a(n) - a((n+m) \bmod N)|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore $d(1,m)$ is an increasing function of m, and also $d(1,m) \geq \left\| a\big|_{[m]} \right\|_2$. Now we may observe that if $2 \leq k \leq N$, then $d(1, N/k) \geq \epsilon \Rightarrow \mathcal{N}(\epsilon) \geq \frac{k}{2}$, and the result follows. □

Now we show how to derive (1.1) from Sudakov's result. By Lemma 1.3, we may assume that $a = a\big|_{[N/2]}$. Hence, by Lemma 1.4, we have

$$\mathbf{E} \sup_{n\in[N]} |\Gamma_n(x)| \geq \sup_{2\leq k\leq N} \left\| a\big|_{[N/k]} \right\|_2 \sqrt{\log(k/2)}$$
$$\approx \sup_{k\in[N/3]} \left\| a\big|_{[N/k]} \right\|_2 \sqrt{\operatorname{lm} k}.$$

Hence (1.1) follows from Proposition 2A:3.5.

Applying Fernique's theorem is not much harder.

Proposition 1.5. *Let x be cyclic on a, and $W_k = \dfrac{1}{k \operatorname{lm} k}$. Then*

$$\mathbf{E} \sup_{n\in[N]} |\Gamma_n(x)| \geq c^{-1} \sum_{k=1}^{N/3} W_k \mathbf{E} \sup_{n\in[N]} |\Gamma_n^{\sigma(k)}(x)|.$$

Proof: By Lemma 1.3 we may assume that $a = a\big|_{[N/2]}$. Furthermore, as x is cyclic, $\Gamma(x)$ is stationary with respect to the group structure of $\mathbf{Z}/N\mathbf{Z}$. So, by Lemma 1.4, we have

$$\int_0^\infty \sqrt{\log \mathcal{N}(\epsilon)}\, d\epsilon \geq \sum_{k=3}^{N} \sqrt{\log((k-1)/2)} \left(\left\| a\big|_{[N/(k-1)]} \right\|_2 - \left\| a\big|_{[N/k]} \right\|_2 \right)$$
$$= \sum_{k=3}^{N} \left(\sqrt{\log(k/2)} - \sqrt{\log((k-1)/2)} \right) \left\| a\big|_{[N/k]} \right\|_2$$
$$\approx \sum_{k=1}^{N/3} \frac{1}{k\sqrt{\operatorname{lm} k}} \left\| a\big|_{[N/k]} \right\|_2 .$$

Apply Proposition 2A:3.5, and the result follows. □

Thus Proposition 2A:4.7 provides us with the following corollary.

Corollary 1.5a. *Let x be cyclic. Then*

$$\mathbf{E} \left\| \sum_{s=1}^{S} \gamma_s x_s \right\|_\infty \geq c^{-1} \left(\sum_{s=1}^{S} \|x_s\|_{L^N_{2,1}}^2 \right)^{\frac{1}{2}} .$$

This suggests that Theorem 2A:2.1 may not be the best possible result.

1.2) Lack of Unconditionality of Processes

Here we look at Theorem 2A:2.1 in a completely different way, and consider what would be a natural generalization if it were true.

False Conjecture. *Let $\Gamma^{(i)} = (\Gamma_n^{(i)} = \sum_{s=1}^S \gamma_s x_s^{(i)} : n \in [N])$, where $i = 1, 2$, be two Gaussian processes such that for all $s \in [S]$ we have $|x_s^{(2)}| \leq |x_s^{(1)}|$. Then*

$$\mathbf{E} \sup_{n\in[N]} |\Gamma_n^{(2)}| \leq c\mathbf{E} \sup_{n\in[N]} |\Gamma_n^{(1)}|.$$

Counterexample: Let $S = 2^N$, and for $n \in [N]$ and $s \in [S]$, let

$$x_s^{(1)}(n) = 1 \quad \text{and} \quad x_s^{(2)}(n) = (n-1)\text{th binary digit of } s.$$

Then clearly

$$\mathbf{E} \sup_{n\in[N]} |\Gamma_n^{(1)}| \approx \sqrt{N},$$

but it follows straight away from Theorem 1.1 that

$$\mathbf{E} \sup_{n\in[N]} |\Gamma_n^{(2)}| \geq c^{-1} \sqrt{\log N} \sqrt{N}.$$

□

2) The Averaging Arguments of Sections 4 and 6

In Section 4, we proved that

$$\sup_\sigma \left(\mathbf{E} \sup_{n\in[N]} |\Gamma_n^\sigma(x)| \right) \ge c^{-1} \frac{1}{\operatorname{lm}\operatorname{lm} N} \left(\sum_{s=1}^S \|x_s\|^2_{L^N_{2,1}} \right)^{\frac12},$$

where the outer supremum is over all systems of strips σ. Section 6 gave a similar result for Bernoulli random variables. Our purpose here is to show that this is a best possible result; we cannot remove the $\operatorname{lm}\operatorname{lm} N$ factor. We only do this in the Gaussian case, the Bernoulli case then follows by Proposition 1C:4.1(i).

We are looking for vectors $x_1, x_2, \dots, x_S \in l^N_\infty$ such that

$$\sup_\sigma \left(\mathbf{E} \sup_{n\in[N]} |\Gamma_n^\sigma(x)| \right) \le c \frac{1}{\operatorname{lm}\operatorname{lm} N} \left(\sum_{s=1}^S \|x_s\|^2_{L^N_{2,1}} \right)^{\frac12}. \tag{2.1}$$

We conduct our search amongst the cyclic x, and so we look for conditions for equality in Propositions 2A:4.2 to 2A:4.5. Proving Proposition 2A:4.5 is just a simple application of the Cauchy–Schwartz inequality, and equality is granted when

$$a(n) = \frac{1}{\sqrt{n}(\operatorname{lm}(N/n))}.$$

In the proof of Proposition 2A:4.4 all the inequalities are actually approximate identities. In the proof of Proposition 2A:4.3, we bound a supremum below by a L_2 average, and we have approximate equality if all the $\frac{1}{k\sqrt{\operatorname{lm} k}} \left\| a|_{[N/k]} \right\|_2$ are approximately the same for all k. This is also granted by the same a as above. Thus we know where to look.

Proposition 2.1. *Let x be cyclic on $a = \left(\frac{1}{\sqrt{n}(\operatorname{lm}(N/n))} : n \in [N] \right)$. Then (2.1) holds.*

Proof: It is immediate that the right hand side of (2.1) is approximately equal to 1. As for the left hand side, $\mathbf{E}\sup_{n\in[N]} |\Gamma_n^\sigma(x)|$ is maximised when σ is a strip such that

$$x_{\sigma_n}(n) = \text{some rearrangement of } \left(\frac{1}{\sqrt{s}(\operatorname{lm}(N/s))} \right)_{s=1}^{d_n},$$

where $d_1, d_2, \dots, d_N$ are non-negative integers such that $\sum_{n=1}^N d_n = N$. It is easy to see that $\|x_{\sigma_n}(n)\|_2 \approx \frac{1}{\sqrt{\operatorname{lm}(N/d_n)}}$, and hence, by Proposition 2A:3.5, we have

$$\mathbf{E} \sup_{n\in[N]} |\Gamma_n^\sigma(x)| \le c \left\| \left(\frac{1}{\sqrt{\operatorname{lm}(N/d_n)}} \right)_{n=1}^N \right\|_{\operatorname{em}(T^2)} \approx 1.$$

□

3) Cotype 2 Constants of Other Operators from l_∞^N

Here we consider the effect of choosing weightings other than $W_k = \dfrac{1}{k \operatorname{lm} k}$.

Theorem 3.1. *Let $0 < \alpha < 1$. Then*

$$G^2(l_\infty^N \hookrightarrow L^N_{T^2(\operatorname{lm} T)^\alpha,2}) \le c \tfrac{1}{\sqrt{\alpha(1-\alpha)}}.$$

Proof: Follow the proof of Proposition 2A:4.6 with the single exception of changing W_k to $\dfrac{1}{k(\operatorname{lm} k)^{2-\alpha}}$. □

Proposition 3.2. *Let $1 < q < 2$, and $\frac{2}{q} - 1 < \alpha < 2$. Then for all $a \in \mathbf{C}^N$ we have*

$$\|a\|_{L^N_{2,q}} \le \left(\tfrac{1}{\alpha - \frac{2}{q} + 1}\right)^{\left(\frac{1}{q}-\frac{1}{2}\right)} \|a\|_{L^N_{T^2(\operatorname{lm} T)^\alpha,2}}.$$

Proof: Let π be an ordering permutation for a, and let $\frac{1}{r} + \frac{1}{2} = \frac{1}{q}$. Then by Hölder's inequality, we have

$$\begin{aligned}
\|a\|_{L^N_{2,q}} &\approx \frac{1}{\sqrt{N}} \left(\sum_{n=1}^N (\pi(n))^{\frac{q}{2}-1} |a(n)|^q \right)^{\frac{1}{q}} \\
&\le \frac{1}{\sqrt{N}} \left(\sum_{n=1}^N \frac{1}{\pi(n)(\operatorname{lm}(N/\pi(n)))^{r\alpha/2}} \right)^{\frac{1}{r}} \left(\sum_{n=1}^N (\operatorname{lm}(N/\pi(n)))^\alpha |a(n)|^2 \right)^{\frac{1}{2}} \\
&\le c \Big(\tfrac{1}{\alpha - \frac{2}{q} + 1}\Big)^{\left(\frac{1}{q}-\frac{1}{2}\right)} \|a\|_{L^N_{T^2(\operatorname{lm} T)^\alpha,2}}.
\end{aligned}$$

□

Corollary 3.1a. *Let $1 < q \le 2$. Then the formal identity $l_\infty^N \hookrightarrow L^N_{2,q}$ has its Gaussian cotype 2 constant uniformly bounded for all N.*

Chapter 2C — Applications of Talagrand's Theorem

In this chapter we prove the following result.

Theorem 0.1. *Let $x_1, x_2, \ldots, x_S \in l_\infty^N$. Then*

$$\mathbf{E}\left\|\sum_{s=1}^{S}\gamma_s x_s\right\|_\infty \geq c^{-1}\left(\sum_{s=1}^{S}\|x_s\|^2_{L^N_{T^2 \operatorname{lm} T,2}}\right)^{\frac{1}{2}}.$$

Corollary 0.1a. *Let μ be a Radon measure on a compact topological space K. Then the formal identity map $C(K) \hookrightarrow L_{T^2 \operatorname{lm} T,2}(K,\mu)$ has Gaussian cotype 2.*

As a digression, this raises the following question, for which I do not know the answer.

Question. *Is it true that all bounded linear operators $T\colon C(K) \to Y$ with cotype 2 factor as*

$$C(K) \hookrightarrow L_{T^2 \operatorname{lm} T,2}(K,\mu) \xrightarrow{U} Y,$$

where μ is some Radon probability measure on K, and U is a bounded operator?

However, for our purposes the following corollary is more important, as it is Theorem 1C:5.2(ii) restated.

Corollary 0.1b. *Let $x_1, x_2, \ldots, x_S \in l_\infty^N$. Then*

$$\mathbf{E}\left\|\sum_{s=1}^{S}\gamma_s x_s\right\|_\infty \geq c^{-1}\frac{1}{\sqrt{\operatorname{lm}\operatorname{lm} N}}\left(\sum_{s=1}^{S}\|x_s\|^2_{L^N_{2,1}}\right)^{\frac{1}{2}}.$$

In Section 2, we will investigate this corollary further, investigating whether the $\sqrt{\operatorname{lm}\operatorname{lm} N}$ factor is necessary, and giving other necessary and sufficient conditions for its existence.

All the work of this chapter depends on a recent result due to Talagrand.

Theorem 0.2. *(see [T]) Let $(\Gamma_n : n \in [N])$ be a Gaussian process.*

i) Let $V_1 = \mathbf{E}\sup_{n\in[N]}|\Gamma_n|$.

ii) Let V_2 be the infimum of

$$\left(\sup_{t\in\mathbf{N}}\sqrt{\operatorname{lm} t}\left(\mathbf{E}\,|Y_t|^2\right)^{\frac{1}{2}}\right)\left(\sup_{n\in[N]}\sum_{t=1}^{\infty}|\alpha_t(n)|\right)$$

over all Gaussian processes $(Y_t : t \in \mathbf{N})$ and all collections of numbers $(\alpha_t(n) : t \in \mathbf{N}, n \in [N])$ such that $\Gamma_n = \sum_{t=1}^\infty \alpha_t(n) Y_t$.

Then $V_1 \approx V_2$.

If we rewrite this result for the Gaussian process we are interested in, that is, $\Gamma_n = \sum_{s=1}^S \gamma_s x_s(n)$ (and set $Y_t = \sum_{s=1}^S \gamma_s y_t(s)$), we obtain the following corollary.

Corollary 0.2a. *For $x_1, x_2, \ldots, x_S \in l_\infty^T$ we have the following.*

i) Let

$$V_1 = \mathbf{E}\left\|\sum_{s=1}^{S}\gamma_s x_s\right\|_\infty.$$

ii) Let V_2 be the infimum of

$$\left(\sup_{t\in\mathbf{N}}\sqrt{\operatorname{lm} t}\,\|y_t\|_{l_2^S}\right)\left\|\sum_{t=1}^{\infty}|\alpha_t|\right\|_{l_\infty^N}$$

over all collections of numbers $(y_t(s) : t \in \mathbf{N}, s \in [S])$ and $(\alpha_t(n) : t \in \mathbf{N}, n \in [N])$ such that $x_s(n) = \sum_{t=1}^\infty \alpha_t(n) y_t(s)$.

Then $V_1 \approx V_2$.

1) The Gaussian Cotype 2 Constant of $l_\infty^N \hookrightarrow L^N_{T^2 \operatorname{lm} T,2}$

In this section we establish Theorem 0.1. To aid us, we define the following spaces.

Definition. *Let A, Y, $\tilde{Y}$, $X_{T^2 \operatorname{lm} T,2}$ and $X_{T \operatorname{lm} T,1}$ be the vector spaces*

$$\begin{aligned}
A &= \{\, (\alpha_t \in l_\infty^N)_{t=1}^\infty : \alpha_t = 0 \text{ for all but finitely many } t\},\\
Y &= \{\, (y_t \in l_2^S)_{t=1}^\infty : y_t = 0 \text{ for all but finitely many } t\},\\
\tilde{Y} &= \{\, (y_t \in l_1^S)_{t=1}^\infty : y_t = 0 \text{ for all but finitely many } t\},\\
X_{T^2 \operatorname{lm} T,2} &= \{\, (x_s \in L^N_{T^2 \operatorname{lm} T,2})_{s=1}^S\},\\
X_{T \operatorname{lm} T,1} &= \{\, (x_s \in L^N_{T \operatorname{lm} T,1})_{s=1}^S\},
\end{aligned}$$

with norms

$$\begin{aligned}
\|(\alpha_t)\|_A &= \left\| \sum_{t=1}^\infty |\alpha_t| \right\|_\infty ,\\
\|(y_t)\|_Y &= \sup_{t\in\mathbf{N}} \sqrt{\operatorname{lm} t}\, \|y_t\|_{l_2^S} ,\\
\|(\tilde{y}_t)\|_{\tilde{Y}} &= \sup_{t\in\mathbf{N}} (\operatorname{lm} t)\, \|\tilde{y}_t\|_{l_1^S} ,\\
\|(x_s)\|_{X_{T^2 \operatorname{lm} T,2}} &= \left(\sum_{s=1}^S \|x_s\|_{L^N_{T^2 \operatorname{lm} T,2}} \right)^{\frac12} ,\\
\|(\tilde{x}_s)\|_{X_{T \operatorname{lm} T,1}} &= \left(\sum_{s=1}^S \|\tilde{x}_s\|_{L^N_{T \operatorname{lm} T,1}} \right)^{\frac12} .
\end{aligned}$$

For each $T \in \mathbf{N}$, define the subspaces of Y and $\tilde{Y}$:

$$\begin{aligned}
Y_T &= \{\, (y_t) \in Y : y_t = 0 \text{ for } t > T\},\\
\tilde{Y}_T &= \{\, (\tilde{y}_t) \in \tilde{Y} : \tilde{y}_t = 0 \text{ for } t > T\}.
\end{aligned}$$

Proof of Theorem 0.1:

Step 1: Let $m\colon A \times Y \to X_{T^2 \operatorname{lm} T,2}$ be the bilinear map

$$m((\alpha_t),(y_t)) = (x_s),$$

where $x_s = (\,\sum_{t=1}^\infty \alpha_t(n) y_t(s) : n \in [N])$. Then by Corollary 0.2a, it follows that Theorem 0.1 holds if and only if m is a bounded map for all S and N.

Step 2: By Proposition 2A:2.10 it follows that, in order to calculate $\|m\|$, it is sufficient to consider only those (α_t) of the form $\alpha_t = \chi_{I_t}$, where $I_1, I_2, \ldots, I_T$ are disjoint subsets of $[N]$. Put another way, $\|m\|$ is the smallest number C such that for all disjoint subsets $I_1, I_2, \ldots, I_T$ of $[N]$, if $(y_t) \in Y_T$, $\|(y_t)\|_Y \le 1$, then $\|(x_s)\|_{X_{T^2 \operatorname{lm} T,2}} \le C$, where $x_s(n) = y_t(s)$ if $n \in I_t$ for some $t \in [T]$, and $x_s(n) = 0$ otherwise.

Step 3: So now let us fix $I_1, I_2, \ldots, I_T$ as disjoint subsets of $[N]$. For each $(y_t) \in Y_T$, let $(\tilde{y}_t) \in \tilde{Y}_T$ be defined by $\tilde{y}_t(s) = |y_t(s)|^2$. Thus $\|(y_t)\|_Y^2 = \|(\tilde{y}_t)\|_{\tilde{Y}}$. Notice that the extreme points of the unit ball of $\tilde{Y}_T$ are vectors of the form

$$y_t(s) = \frac{1}{\operatorname{lm} t} \delta_{s_t}(s) \qquad (t \le T),$$

where $\delta_{s_t}(s) = 1$ if $s = s_t$ and 0 otherwise, and $s_t \in [S]$ depends upon t only. For convenience later on, we write $J_s = \{\, t \in [T] : s_t = s\}$, so that $J_1, J_2, \ldots, J_S$ are disjoint subsets of $[T]$.

Step 4: Now let $\tilde{x}_s(n) = |x_s(n)|^2$ so that $\tilde{x}_s(n) = \tilde{y}_t(s)$ if $n \in I_t$ for some $t \in [T]$, and $x_s(n) = 0$ otherwise, and also so that $\|(x_s)\|^2_{X_{T^2 \operatorname{lm} T,2}} \approx \|(\tilde{x}_s)\|_{X_{T \operatorname{lm} T,1}}$. We see that to show Theorem 0.1, we need only show that the map $\tilde{m}\colon \tilde{Y} \to X_{T \operatorname{lm} T,1}$, $(\tilde{y}_t) \mapsto (\tilde{x}_s)$ is bounded.

Step 5: Since $\|\cdot\|_{X_{T\,\mathrm{lm}\,T,1}}$ is a norm, it follows by the Krein–Milman theorem (See [Ru 3.2.1]) that in order to calculate $\|\tilde{m}\|$, we need only look at $\|\tilde{m}((\tilde{y}_t))\|_{X_{T\,\mathrm{lm}\,T,1}}$ where $(\tilde{y}_t)$ is an extreme point of the unit ball of $\tilde{Y}_T$. So let $(\tilde{y}_t)$ have the form given in Step 3, and notice that then

$$\tilde{x}_s = \sum_{t \in J_s} \frac{1}{\mathrm{lm}\, t} \chi_{I_t}.$$

Hence, by the triangle inequality for $\|\cdot\|_{T\,\mathrm{lm}\,T,1}$, we have

$$\begin{aligned}
\|(\tilde{x}_s)\|_{X_{T\,\mathrm{lm}\,T,1}} &= \sum_{s=1}^{S} \|\tilde{x}_s\|_{L^N_{T\,\mathrm{lm}\,T,1}} \\
&\le \sum_{s=1}^{S} \sum_{t \in J_s} \frac{1}{\mathrm{lm}\, t} \|\chi_{I_t}\|_{L^N_{T\,\mathrm{lm}\,T,1}} \\
&\le c \sum_{t=1}^{T} \frac{1}{\mathrm{lm}\, t} \frac{|I_t|}{N} \Big/ \mathrm{lm}\left(\frac{|I_t|}{N}\right) \\
&\le c \left\| \left(\frac{|I_t|}{N} \Big/ \mathrm{lm}\left(\frac{|I_t|}{N}\right) \right) \right\|_{l^T_{T\,\mathrm{lm}\,T,1}} \\
&\approx \left\| \left(\frac{|I_t|}{N} \Big/ \mathrm{lm}\left(\frac{|I_t|}{N}\right) \right) \right\|_{l^T_{T\,\mathrm{lm}\,T}} \\
&\approx 1,
\end{aligned}$$

where the second to last approximate identity is by Theorem 1A:2.7(i). The result follows. $\square$

2) The Gaussian Cotype 2 Constant of $l_\infty^N \hookrightarrow L_{2,1}^N$

In this section we investigate the necessity of the $\sqrt{\operatorname{lm}\operatorname{lm} N}$ factor in Corollary 0.1b. Although we do not come to any conclusions, we do find directions in which to search, and conclude that it is a hard problem. The method is to find functions of N equivalent to, or bounding the function $N \mapsto G^2(l_\infty^N \hookrightarrow L_{2,1}^N)$.

First we define some spaces.

Definition. *Let A and Y be defined as in Section 1, and let Y' and $X_{2,1}$ be the vector spaces defined by*

$$\begin{aligned} Y' &= \{\, (\xi_t \in l_2^S)_{t=1}^\infty : \xi_t = 0 \text{ for all but finitely many } t\}, \\ X_{2,1} &= \{\, (x_s \in L_{2,1}^N)_{s=1}^S\}, \end{aligned}$$

with norms

$$\begin{aligned} \|(\xi_n)\|_{Y'} &= \sum_{t=1}^\infty \frac{1}{\sqrt{\operatorname{lm} t}} \|\xi_t\|_{l_2^S}\,, \\ \|(x_s)\|_{X_{2,1}} &= \left(\sum_{s=1}^S \|x_s\|_{L_{2,1}^N}^2\right)^{\frac{1}{2}}. \end{aligned}$$

Definition. *If μ is a probability measure on S_N, let Z_μ be $\mathbf{C}^N$ with norm*

$$\|\alpha\|_{Z_\mu} = \left(\int_{S_N} \left(\frac{1}{\sqrt{N}} \sum_{n=1}^N \frac{1}{\sqrt{\pi(n)}} |\alpha(n)|\right)^2 d\mu(\pi)\right)^{\frac{1}{2}}.$$

Now we define several functions of N, and show relationships between them. Our first function, $C_1(N)$, is the function that we are investigating.

Definition. *Let $C_1(N) = G^2(l_\infty^N \hookrightarrow L_{2,1}^N)$, that is, $C_1(N)$ is the least number C such that for all $x_1, x_2, \dots,$ $x_S \in l_\infty^N$ we have*

$$\mathbf{E}\left\|\sum_{s=1}^S \gamma_s x_s\right\|_\infty \ge \frac{1}{C}\left(\sum_{s=1}^S \|x_s\|_{L_{2,1}^N}^2\right)^{\frac{1}{2}}.$$

Definition. *Let $m_{S,N}$ be the bilinear map*

$$\begin{aligned} A \times Y &\to X_{2,1} \\ (\alpha_t), (y_t) &\mapsto (x_s), \end{aligned}$$

where

$$x_s(n) = \sum_{t=1}^\infty \alpha_t(n) y_t(s).$$

Let $C_2(N) = \sup_{S\in\mathbf{N}} \|m_{S,N}\|$.

Proposition 2.1. *For all $N \in \mathbf{N}$ we have $C_1(N) \approx C_2(N)$.*

Proof: This follows straight away from Corollary 0.2a. □

Definition. *Let $C_3(N)$ be the least number C such that for all probability measures μ on S_N and for all $\alpha_1, \alpha_2, \ldots, \alpha_T \in l_\infty^N$, we have*

$$\sum_{t=1}^{T} \frac{1}{\sqrt{\operatorname{lm} t}} \|\alpha_t\|_{Z_\mu} \le C \left\| \sum_{t=1}^{T} |\alpha_t| \right\|_\infty .$$

Definition. *Let $C_4(N)$ be the least number C such that for any probability measure μ on S_N, there is a probability measure λ on $[N]$ such that for all $\alpha \in l_\infty^N$ we have*

$$\|\alpha\|_{Z_\mu} \le C \|\alpha\|_{L_{T\sqrt{\operatorname{lm} T}}(\lambda)} .$$

Definition. *Let $C_5(T)$ be the least constant C such that the following holds. For any $T \in \mathbf{N}$, let $\bar{\mu}$ be a probability measure on S_T. Then for any positive natural numbers $i_1, i_2, \ldots, i_T$ such that $\sum_{t=1}^T i_t = N$ we have*

$$\frac{1}{\sqrt{N}} \sum_{t=1}^{T} \frac{1}{\sqrt{\operatorname{lm} t}} \left(\int_{S_T} \frac{i_t^2}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}} \, d\bar{\mu}(\sigma) \right)^{\frac{1}{2}} \le C.$$

Proposition 2.2. *For all $N \in \mathbf{N}$ we have*
i) $C_2(N) = C_3(N)$;
ii) $C_3(N) \approx C_4(N)$;
iii) $C_3(N) \approx C_5(N)$.

Proof of i): We use two elementary duality results.

a) $$\|(x_s)\|_{X_{2,1}} = \sup \left\{ \frac{1}{\sqrt{N}} \sum_{s=1}^{S} \sum_{n=1}^{N} u_s \frac{1}{\sqrt{\pi_s(n)}} |x_s(n)| \right\},$$

where the supremum is over all $\pi_1, \pi_2, \ldots, \pi_S \in S_N$ and all $u_1, u_2, \ldots, u_S \ge 0$ such that $\sum_{s=1}^S u_s^2 = 1$.

b) $$\|(y_t)\|_Y = \sup \left\{ \sum_{t=1}^{\infty} \sum_{s=1}^{S} \xi_t(s) y_t(s) : (\xi_t) \in Y' \right\}.$$

Now $C_2(N)$ is the least number C such that for all $S \in \mathbf{N}$, $(\alpha_t) \in A$ with $\alpha_t \ge 0$, and $(y_t) \in Y$ with $y_t \ge 0$, we have

$$\left\| \left(\sum_{t=1}^{\infty} \alpha_t y_t(s) \right) \right\|_{X_{2,1}} \le K \|(y_t)\|_Y \|(\alpha_t)\|_A .$$

So by (a), $C_2(N)$ is the least number C such that for all $S \in \mathbf{N}$, $(\alpha_t) \in A$ with $\alpha_t \ge 0$, $(y_t) \in Y$ with $y_t \ge 0$, $\pi_1, \pi_2, \ldots, \pi_S \in S_N$, and $u_1, u_2, \ldots, u_S \ge 0$ with $\sum_{s=1}^S u_s^2 = 1$, we have

$$\frac{1}{\sqrt{N}} \sum_{t=1}^{\infty} \sum_{s=1}^{S} \sum_{n=1}^{N} u_s \frac{1}{\sqrt{\pi_s(n)}} \alpha_t(n) y_t(s) \le K \|(y_t)\|_Y \|(\alpha_t)\|_A .$$

Thus by (b) we see that $C_2(N)$ is the least number C such that for all $S \in \mathbf{N}$, $(\alpha_t) \in A$ with $\alpha_t \ge 0$, $\pi_1, \pi_2, \ldots, \pi_S \in S_N$, and $u_1, u_2, \ldots, u_S \ge 0$ with $\sum_{s=1}^S u_s^2 = 1$, we have

$$\frac{1}{\sqrt{N}} \sum_{t=1}^{\infty} \frac{1}{\sqrt{\operatorname{lm} t}} \left\| \left(\sum_{n=1}^{N} u_s \frac{1}{\sqrt{\pi_s(n)}} \alpha_t(n) \right)_{s \in [S]} \right\|_{l_2^S} \le K \|(\alpha_t)\|_A .$$

So if we put the measure μ defined by $\mu(\{\pi\}) = \sum_{s : \pi_s = \pi} u_s^2$ on S_N, we have that

$$\left\| \left(\sum_{n=1}^{N} u_s \frac{1}{\sqrt{\pi_s(n)}} \alpha_t(n) \right)_{s \in [S]} \right\|_{l_2^S} = \left(\int_{S_N} \left(\sum_{n=1}^{N} \frac{1}{\sqrt{\pi_s(n)}} \alpha_t(n) \right)^2 d\mu(\pi) \right)^{\frac{1}{2}},$$

and the result follows. □

Proof of ii): By Theorem 1A:2.7(i) we have

$$C_3(N) \approx \inf\{\, \pi_{T\sqrt{\operatorname{lm} T},1}(l_\infty^N \hookrightarrow Z_\mu) : \mu \text{ is a probability measure on } S_N\}.$$

The result follows immediately from Theorem 1D:4.1. □

Proof of iii): Here we make use of the approximation

$$\sum_{n=n_1}^{n_2} \frac{1}{\sqrt{n}} \approx \frac{n_2 - n_1}{\sqrt{n_2}}.$$

First, we show that $C_5(N) \geq c^{-1}C_2(N)$. Let $\epsilon > 0$. From Proposition 1A:2.10, we see that there are $(y_t) \in Y$ and $\alpha_1, \alpha_2, \ldots, \alpha_T \in l_\infty^N$, with $|\alpha_t| \wedge |\alpha_u| = 0$ for $t \neq u \in [T]$ and $\|\alpha_t\|_\infty \leq 1$ for $t \in [T]$, such that

$$\|(x_s)\|_{X_{2,1}} \geq \|m_{S,T}\| \, \|(y_t)\|_Y - \epsilon,$$

where

$$x_s(n) = \sum_{t=1}^{T} \alpha_t(n) y_t(s).$$

Without loss of generality, we suppose that $y_t \geq 0$ and that $\alpha_t = \chi_{I_t}$ where $I_1, I_2, \ldots, I_T$ are disjoint sets whose union is $[T]$. We note that $x_s(n) = y_t(s)$ whenever $n \in I_t$. Thus x_s is constant on each I_t. Hence

$$\|(x_s)\|_{X_{2,1}} \approx \frac{1}{\sqrt{N}} \sum_{s=1}^{S} \sum_{n=1}^{N} u_s \frac{1}{\sqrt{\pi_s(n)}} x_s(n)$$

for some $u_1, u_2, \ldots, u_S \geq 0$ with $\sum_{s=1}^{S} u_s^2 = 1$, and some $\pi_1, \pi_2, \ldots, \pi_S \in S_N$ with the property that $\pi_s(I_t)$ is an interval in $[N]$. Write i_t for $|I_t|$, and let $\sigma_1, \sigma_2, \ldots, \sigma_S \in S_T$ be such that

$$\pi_s(I_t) = \left[\sum_{u=1}^{\sigma_s(t)-1} i_{\sigma_s^{-1}(u)} + 1, \sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)} \right].$$

To save space, write

$$M_{s,t} = \sum_{n \in I_t} \frac{1}{\sqrt{\pi_s(n)}}$$

so that by the observation above we have

$$M_{s,t} \approx \frac{i_t}{\left(\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)} \right)^{\frac{1}{2}}}.$$

Then

$$\begin{aligned}
\|(x_s)\|_{X_{2,1}} &= \frac{1}{\sqrt{N}} \sum_{s=1}^{S} \sum_{t=1}^{T} u_s M_{s,t} y_t(s) \\
&\leq \frac{1}{\sqrt{N}} \sum_{t=1}^{T} \frac{1}{\sqrt{\operatorname{lm} t}} \left(\sum_{s=1}^{S} u_s^2 M_{s,t}^2 \right)^{\frac{1}{2}} \|(y_t)\|_Y .
\end{aligned}$$

Now let $\epsilon \to 0$. We deduce that

$$\|m_{S,T}\| \leq \frac{1}{\sqrt{N}} \sum_{t=1}^{T} \frac{1}{\sqrt{\operatorname{lm} t}} \left(\sum^{S} u_s^2 M_{s,t}^2 \right)^{\frac{1}{2}} .$$

If we now define a probability measure $\bar\mu$ on S_T by $\bar\mu(\{\sigma\}) = \sum_{s:\sigma_s=\sigma} u_s^2$, we see that the right hand side of the above equation is dominated by

$$\frac{1}{2}\frac{1}{\sqrt{N}}\sum_{t=1}^{T}\frac{1}{\sqrt{\operatorname{lm} t}}\left(\int_{S_T}\frac{i_t^2}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}}\,d\bar\mu(\sigma)\right)^{\frac{1}{2}}.$$

Therefore $\|m_{S,T}\| \le cC_5(N)$.

Now we show that $C_5(N) \le cC_3(N)$. Suppose we are given $T \in \mathbf{N}$, a probability measure $\bar\mu$ on S_T, and positive natural numbers $i_1, i_2, \ldots, i_T$ such that $\sum_{t=1}^{T} i_t = N$. For each $t \in [N]$, let

$$I_t = \left[\sum_{u=1}^{t-1} i_u + 1, \sum_{u=1}^{t} i_u\right],$$

and $\alpha_t = \chi_{I_t}$. For each $\sigma \in S_T$, choose $\pi_\sigma \in S_N$ such that

$$\pi_\sigma(I_t) = \left[\sum_{u=1}^{\sigma(t)-1} i_{\sigma^{-1}(u)} + 1, \sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}\right],$$

and define a probability measure μ on S_N by setting $\mu(\{\pi_\sigma\}) = \bar\mu(\{\sigma\})$ and $\mu(\{\pi\}) = 0$ if $\pi \ne \pi_\sigma$ for any $\sigma \in S_T$. It is easily seen that

$$\frac{i_n}{\left(\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}\right)^{\frac{1}{2}}} \le c\sum_{n\in I_t}\frac{1}{\sqrt{\pi_\sigma(n)}}\alpha_t(n).$$

Therefore

$$\left(\int_{S_T}\frac{i_t^2}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}}\,d\bar\mu(\sigma)\right)^{\frac{1}{2}} \le c\,\|\alpha_t\|_{Z_\mu}.$$

The result follows. □

Now we consider trying to find good bounds for $C_1(N)$, and we concentrate our attention on the quantity $C_5(N)$. First, we will look for lower bounds.

Question 2.3. *Is $C_5(N) \ge c^{-1}\sqrt{\operatorname{lm}\operatorname{lm} N}$, or at least, is it the case that $C_5(N)$ is not uniformly bounded in N. Put another way, for each $N \in \mathbf{N}$, can we find a $T \in \mathbf{N}$, a probability measure $\bar\mu$ on $[T]$, and positive natural numbers $i_1, i_2, \ldots, i_T$ with $\sum_{t=1}^{T} i_t = N$, such that*

$$\frac{1}{\sqrt{N}}\sum_{t=1}^{T}\frac{1}{\sqrt{\operatorname{lm} t}}\left(\int_{S_T}\frac{i_t^2}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}}\,d\bar\mu(\sigma)\right)^{\frac{1}{2}}$$

is large?

I have not been able to answer this question.

If we are looking for lower bounds for $C_5(N)$, the following may help.

Definition. *Let $C_6(N)$ be the least constant N such that the following holds. For any positive natural numbers $i_1, i_2, \ldots, i_T$ such that $\sum_{t=1}^{T} i_t = N$ there are positive real numbers $\nu_1, \nu_2, \ldots, \nu_T$ such that*

$$\frac{1}{N}\left(\sum_{t=1}^{T}\frac{\nu_t}{\operatorname{lm} t}\right)\left(\sup_{\sigma\in S_T}\sum_{t=1}^{T}\frac{1}{\nu_t}\frac{i_t^2}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}}\right) \le C^2.$$

Proposition 2.4. *For all $N \in \mathbf{N}$ we have $C_5(N) \le C_6(N)$.*

Proof: Suppose we are given $T \in \mathbf{N}$, a probability measure $\bar\mu$ on S_T, and positive natural numbers $i_1, i_2, \dots, i_T$ such that $\sum_{t=1}^T i_t = N$. Then for any positive real numbers $\nu_1, \nu_2, \dots, \nu_T$ we have by the Cauchy-Schwartz inequality that

$$\begin{aligned}
&\frac{1}{N}\left(\sum_{t=1}^T \frac{1}{\sqrt{\operatorname{lm} t}}\left(\int_{S_T} \frac{i_t^2}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}}\, d\bar\mu(\sigma)\right)^{\frac12}\right)^2 \\
\le&\frac{1}{N}\left(\sum_{t=1}^T \frac{\nu_t}{\operatorname{lm} t}\right)\left(\sum_{t=1}^T \int_{S_T} \frac{1}{\nu_t}\frac{i_t^2}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}}\, d\bar\mu(\sigma)\right) \\
=&\frac{1}{N}\left(\sum_{t=1}^T \frac{\nu_t}{\operatorname{lm} t}\right)\left(\int_{S_T} \sum_{t=1}^T \frac{1}{\nu_t}\frac{i_t^2}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}}\, d\bar\mu(\sigma)\right) \\
\le&\frac{1}{N}\left(\sum_{t=1}^T \frac{\nu_t}{\operatorname{lm} t}\right)\left(\sup_{\sigma\in S_T} \sum_{t=1}^T \frac{1}{\nu_t}\frac{i_t^2}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}}\right).
\end{aligned}$$

The result follows. □

Thus we can formulate the following question.

Question 2.5. *Is $C_6(N)$ uniformly bounded in N, or at least, is $C_6(N) = o\big(\sqrt{\operatorname{lm}\operatorname{lm} N}\big)$?*

Now all choices of $i_1, i_2, \dots, i_T$ that I have tried indicate a positive answer. For example, if $i_1 = i_2 = \dots = i_T = \frac{N}{T}$, then choose $\nu_1 = \nu_2 = \dots = \nu_T = 1$, and see that

$$\begin{aligned}
\frac{1}{N}\left(\sum_{t=1}^T \frac{\nu_t}{\operatorname{lm} t}\right)\left(\sup_{\sigma\in S_T} \sum_{t=1}^T \frac{1}{\nu_t}\frac{i_t^2}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}}\right) &\approx \frac{1}{N}\frac{T}{\operatorname{lm} T}\frac{N}{T}\sum_{t=1}^T \frac{1}{t} \\
&\approx 1.
\end{aligned}$$

This example has further applications. For we have shown that if $I_1, I_2, \dots, I_T$ are equal sized disjoint subsets of $[N]$ and if μ is any probability measure on S_N, then

$$\sum_{t=1}^T \frac{1}{\sqrt{\operatorname{lm} t}} \left\|\chi_{I_t}\right\|_{Z_\mu} \le c. \tag{2.1}$$

Now we can prove the following result, which is similar to Corollary 2B:1.5a.

Proposition 2.6. *Let $a \in l_\infty^N$, and enumerate S_N as $\{\pi_1, \pi_2, \dots, \pi_S\}$. Let $x_s(n) = a(\pi_s(n))$. Then*

$$\mathbf{E}\left\|\sum_{s=1}^S \gamma_s x_s\right\|_\infty \ge c^{-1}\left(\sum_{s=1}^S \|x_s\|_{L_{2,1}^N}^2\right)^{\frac12}.$$

Proof: If x has the above form, then following through all the previous definitions and proofs, we see that in the definition of $C_3(N)$ that the only μ on S_N that we need to consider is the counting measure on S_N. But then $\|\cdot\|_{Z_\mu}$ is a 1-symmetric norm on $\mathbf{C}^N$. Thus we deduce from (2.1) that

$$\|\chi_I\|_{Z_\mu} \le c\frac{\sqrt{\operatorname{lm} T}}{T}$$

whenever $|I| = N/T$. But then by Lemma 1D:4.2, this is sufficient to show that for all $\alpha \in l_\infty^N$ we have

$$\|\alpha\|_{Z_\mu} \le c\,\|\alpha\|_{T\sqrt{\operatorname{lm} T},1},$$

and the result follows by Proposition 2.2(ii). □

But at least I can show the next result, although not easily. (This gives another proof of Theorem 1C:5.2(ii).)

Proposition 2.7. *For all $N \in \mathbf{N}$, we have $C_6(N) \le c\sqrt{\operatorname{lm}\operatorname{lm} N}$.*

Proof: First, without loss of generality, we may assume that $N \ge 3$. Suppose we are given $T \in \mathbf{N}$, and positive natural numbers $i_1, i_2, \dots, i_T$ such that $\sum_{t=1}^T i_t = N$. Let

$$\nu_t = i_t(\operatorname{lm}(N/i_t)).$$

Then, since $\operatorname{lm} x \le 2\sqrt{x}$ for all $x > 0$, we see that $\nu_t \le 2\sqrt{N i_t}$. Hence

$$\begin{aligned}\sum_{t=1}^T \frac{\nu_t}{2N(\operatorname{lm}(2N/\nu_t))} &\le \sum_{t=1}^T \frac{\nu_t}{2N(\operatorname{lm}\sqrt{N/i_t})} \\ &\le \sum_{t=1}^T \frac{\nu_t}{N(\operatorname{lm}(N/i_t))} \\ &= 1.\end{aligned}$$

Therefore $\|(\nu_t)\|_{T \operatorname{lm} T} \le 2N$. So by Theorem 1A:2.7(i), we have

$$\sum_{t=1}^T \frac{\nu_t}{\operatorname{lm} t} \le cN.$$

Now fix $\sigma \in S_T$. Split $[T]$ into the following $\lceil \log\log N\rceil + 1$ subsets:

$$\begin{aligned} A_k &= \left\{ t : N^{1-e^{1-k}} \le i_t < N^{1-e^{-k}} \right\} && \text{for } 1 \le k \le \lceil \log\log N\rceil; \\ A_k &= \left\{ t : N^{1-e^{1-k}} \le i_t \le N \right\} && \text{for } k = \lceil \log\log N\rceil + 1.\end{aligned}$$

(Note that $N < eN^{1-e^{-k}}$ for $k = \lceil \log\log N\rceil + 1$.) Now, for each k, let i'_k be the element of A_k such that $\sigma(i'_k)$ is smallest, and let $B_k = A_k \setminus \{i'_k\}$. Then since $\frac{x-y}{x} \le \log x - \log y$ for $x \ge y > 0$, we have for each k that

$$\begin{aligned} &\sum_{t \in A_k} \frac{1}{\nu_t} \frac{i_t^2}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}} \\ =& \sum_{t \in A_k} \frac{1}{\operatorname{lm}(N/i_t)} \frac{i_t}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}} \\ \le& \frac{1}{1 + \log(e^{-1} N^{e^{-k}})} \sum_{t \in A_k} \left(i_t \Big/ \sum_{\substack{u=1 \\ \sigma^{-1}(u) \in A_k}}^{\sigma(t)} i_{\sigma^{-1}(u)} \right) \\ \le& \frac{e^k}{\log N} \left(1 + \sum_{t \in B_k} \log \left(\sum_{\substack{u=1 \\ \sigma^{-1}(u) \in A_k}}^{\sigma(n)} i_{\sigma^{-1}(u)} \Big/ \sum_{\substack{u=1 \\ \sigma^{-1}(u) \in A_k}}^{\sigma(n)-1} i_{\sigma^{-1}(u)} \right) \right) \\ \le& \frac{e^k}{\log N} \left(1 + \log \left(\frac{\sum_{t \in A_k} i_t}{\inf_{t \in A_k} i_t} \right) \right) \\ \le& \frac{e^k}{\log N} \left(1 + \log \left(\frac{N}{N^{1-e^{1-k}}} \right) \right) \\ \le& e^2 + e. \end{aligned}$$

Therefore

$$\sum_{t=1}^T \frac{1}{\nu_t} \frac{i_t^2}{\sum_{u=1}^{\sigma(t)} i_{\sigma^{-1}(u)}} \le (e^2 + e)(\lceil \log\log N\rceil + 1).$$

The result follows. ∎

Chapter 2D — Comparison of Gaussian and Rademacher Cotype

In this chapter we prove Theorems 1C:5.3 and 1C:5.4.

Theorem 1. *Let $2 \le p < q < \infty$. Then for sufficiently large N (depending on p and q) we have*

$$G^p(l_\infty^N \hookrightarrow l_{q,p}^N) \le c\sqrt{p}\frac{1}{\sqrt{\operatorname{lm} N}} N^{\frac{1}{p}}.$$

Corollary 1a. *Let $2 \le p < q < \infty$. Then for sufficiently large N (depending on p and q) there is a bounded linear operator T from l_∞^N to L_q such that*

$$R^p(T) \ge c^{-1}\frac{1}{\sqrt{p}}\sqrt{\operatorname{lm} N}\, G^p(T).$$

Corollary 1b. *Let $2 < q < \infty$. Then for sufficiently large N (depending on p and q) there is a bounded linear operator T from l_∞^N to L_q such that*

$$\pi_{2,1}(T) \ge c^{-1}\sqrt{\operatorname{lm} N}\, G^2(T).$$

Proof of Corollaries 1a and 1b: The obvious example is $T\colon l_\infty^N \hookrightarrow l_q^N$, as it factors through both $l_{q,p}^N$ and $l_{q,2}^N$. By considering the unit vectors we immediately see that $R^p(T) \ge N^{\frac{1}{p}}$, and $\pi_{2,1}(T) \ge N^{\frac{1}{2}}$.

Proof of Theorem 1: We follow the proof of Proposition 2A:4.6 almost identically. First we take the L_p average of systems of strips of size k.

$$\mathbf{E}\left\|\sum_{s=1}^{S}\gamma_s x_s\right\|_\infty \ge \tfrac{1}{4}\left(\frac{1}{\left|[N]^{(k)}\right|}\sum_{\nu\in[N]^{(k)}}\left(\mathbf{E}\sup_{n\in[N]}|\Gamma_n^{\sigma(\nu)}|\right)^p\right)^{\frac{1}{p}},$$

which, by Proposition 2A:3.1, is approximately greater than

$$\frac{1}{\sqrt{p}}\left(\frac{1}{\left|[N]^{(k)}\right|}\sum_{\nu\in[N]^{(k)}}(\operatorname{lm} k)^{\frac{p}{2}}\frac{1}{k}\sum_{n=1}^{N}\|x_{\sigma_n}(n)\|_2^p\right)^{\frac{1}{p}},$$

which, since $\|\cdot\|_2 \ge \|\cdot\|_p$ for $p \ge 2$, is greater than

$$\frac{1}{\sqrt{p}}\left(\frac{1}{\left|[N]^{(k)}\right|}\sum_{\nu\in[N]^{(k)}}(\operatorname{lm} k)^{\frac{p}{2}}\frac{1}{k}\sum_{n=1}^{N}\|x_{\sigma_n}(n)\|_p^p\right)^{\frac{1}{p}},$$

$$\approx \frac{1}{\sqrt{p}}\left(\frac{(\operatorname{lm} k)^{\frac{p}{2}}}{k}\sum_{s=1}^{S}\sum_{n=1}^{N}|x_s(n)|^p\,\frac{\left|\{\,\nu\in[N]^{(k)} : s\in\sigma_n(\nu)\}\right|}{\left|[N]^{(k)}\right|}\right)^{\frac{1}{p}},$$

which, by Proposition 2A:2.2, is approximately greater than

$$\frac{1}{\sqrt{p}}\left(\frac{(\operatorname{lm} k)^{\frac{p}{2}}}{N}\sum_{n=1}^{N}\sum_{\{s:\pi_s(n)\le N/k\}}|x_s(n)|^p\right)^{\frac{1}{p}}.$$

Now we average over k with weighting G_k, to be chosen later, and write $G = \sum_{k=1}^{N/3} G_k$.

$$\begin{aligned}\mathbf{E}\left\|\sum_{s=1}^{S}\gamma_s x_s\right\|_\infty &\ge c^{-1}\frac{1}{\sqrt{p}}\left(\frac{1}{G}\sum_{k=1}^{N/3}\frac{G_k(\operatorname{lm} k)^{\frac{p}{2}}}{N}\sum_{n=1}^{N}\sum_{\{s:\pi_s(n)\le N/k\}}|x_s(n)|^p\right)^{\frac{1}{p}}\\ &= c^{-1}\frac{1}{\sqrt{p}}\frac{1}{(GN)^{\frac{1}{p}}}\left(\sum_{n=1}^{N}\sum_{k=1}^{N/\pi_s(n)}G_k(\operatorname{lm} k)^{\frac{p}{2}}\,|x_s(n)|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Now let

$$G_k = \frac{1}{(\operatorname{lm} k)^{\frac{p}{2}}} \frac{k^{1-\frac{p}{q}} - (k-1)^{1-\frac{p}{q}}}{N^{1-\frac{p}{q}}},$$

and observe that $G \approx (\operatorname{lm} N)^{-\frac{p}{2}}$ for sufficiently large N (depending on p and q). Then we see that the above is bounded above by

$$c^{-1} \frac{1}{\sqrt{p}} \frac{\sqrt{\operatorname{lm} N}}{N^{\frac{1}{p}}} \left(\sum_{n=1}^{N} \pi_s(n)^{\frac{p}{q}-1} |x_s(n)|^p \right)^{\frac{1}{p}}$$

which, by Proposition 1A:2.6(i), is at least

$$c^{-1} \frac{1}{\sqrt{p}} \frac{\sqrt{\operatorname{lm} N}}{N^{\frac{1}{p}}} \left(\sum_{n=1}^{N} \|x\|^p_{l^N_{q,p}} \right)^{\frac{1}{p}}$$

as desired. $\square$

Part 3 — Generalized Lorentz Spaces

Contents

Introduction to Part 3

This part is really an appendix to the rest of the thesis. Its main purpose is to prove some of the results of Sections 1A:2.3 and 1A:2.4, but it also discusses some new problems in Chapter 3C.

We start in Chapter 3A by giving the basic definitions, that is, of the rearrangement invariant positive homogeneous functional Banach spaces (r.i. spaces) (in Section 1), and the admissible functions (in Section 2). The r.i. spaces are an extension of the well known idea of normed rearrangement invariant spaces, and the admissible functions generalize the notion of Orlicz functions. For both of these, I define many convexity constants (and a few concavity constants). Then Section 3 describes Orlicz spaces and weak Orlicz spaces, and Section 4 describes the technical notion of normal spaces and functions. Finally, in Section 5, we define the main objects of our interest in this part, the generalized Lorentz spaces.

Chapter 3B gives some inequalities. Section 1 looks at the generalized Hölder's inequality, giving a different approach to that usually taken. These results are then used to prove the results of Section 2, which looks at changes in the second index of the generalized Lorentz functional. This section culminates in good formulae for calculating certain Orlicz functionals. (This extends results of Bennett and Rudnick.) Section 3 gives a single proposition that is required in Parts 1 and 2.

Chapter 3C looks at the surprisingly non-trivial problem of whether the generalized Lorentz spaces are quasi-normed. It also considers the Boyd indices of these spaces, and relates these to complicated conditions on the second index.

Chapter 3A — Definitions and Elementary Properties

1) Rearrangement Invariant Spaces

We start off by defining rearrangement invariant positive homogeneous functional Banach spaces. Although these definitions may seem rather long, they are only extensions of the well known notion of normed rearrangement invariant spaces (see [L–T2 Ch.2] or [B–S Ch.4 §2]).

Definition 1.1. *A measure space I is a standard measure space if I is one of the following.*
i) $I = I_{0,\infty} = [0,\infty)$ *with Lebesgue measure* λ.
ii) $I = I_{0,B} = [0,B]$ *with Lebesgue measure* λ, *where* $0 < B < \infty$.
iii) $I = I_{A,\infty} = A\mathbf{N} = \{\, An : n \in \mathbf{N}\}$ *with measure* $\lambda\{An\} = A$, *where* $0 < A < \infty$.
iv) $I = I_{A,B} = A\mathbf{N} \cap [0,B]$ *with measure* $\lambda\{An\} = A$, *where* $0 < A \le B < \infty$.

Definition 1.2. *Let I be a measure space. Define $L_0 = L_0(I)$ be the set of equivalence classes of measurable functions $f\colon I \to \mathbf{C}$, where the equivalence relation is equality almost everywhere. Give $L_0(I)$ the* measure topology, *that is, the neighbourhoods of zero are sets of the form $\{\, f : \lambda\{\, t \in I : |f(t)| > \epsilon\} < \epsilon\}$, where $\epsilon > 0$.*

Definition 1.3. *Let I be a standard measure space. A* rearrangement invariant positive homogeneous functional *on I is a function $\|\cdot\|\colon L_0(I) \to [0,\infty]$ satisfying*
i) *if $f \in L_0(I)$, then $\|f\| = 0 \Leftrightarrow f = 0$;*
ii) *if $f \in L_0(I)$ and $\alpha \in \mathbf{C}$, then $\|\alpha f\| = |\alpha|\,\|f\|$;*
iii) *if $f, g \in L_0(I)$, then $|f| \le |g| \Rightarrow \|f\| \le \|g\|$;*
iv) *if $f_n, f \in L_0(I)$ for $n \in \mathbf{N}$, then $|f_n| \nearrow |f| \Rightarrow \|f_n\| \to \|f\|$, as $n \to \infty$;*
v) *if $f_n \in L_0(I)$ for $n \in \mathbf{N}$, then $\|f_n\| \to 0 \Rightarrow f_n \to 0$ in the measure topology, as $n \to \infty$;*
vi) *if $\tau\colon I \to I$ is a measure preserving map and $f \in L_0(I)$, then $\|f\| = \|f \circ \tau\|$.*
A rearrangement invariant positive homogeneous functional Banach space *on I is a pair $(X, \|\cdot\|)$, where $\|\cdot\|$ is a rearrangement invariant positive homogeneous functional on I, and $X = \{f \in L_0(I) : \|f\| < \infty\}$. We abbreviate the phrase 'rearrangement invariant positive homogeneous functional' to* r.i. functional, *and 'rearrangement invariant positive homogeneous functional Banach space' to* r.i. space. *We always refer to a r.i. space by a single letter, X say, which also denotes its subset of $L_0(I)$, and we denote its norm by $\|\cdot\|_X$.*

Definition 1.4. *Let I be a standard measure space, and $f \in L_0(I)$. Define the* decreasing rearrangement *of f to be the function $I_{0,\infty} \to \mathbf{R}_+$ given by*

$$f^*(x) = \sup\{\, t : \lambda\{|f| > t\} > x\}.$$

Sometimes, with an abuse of notation, we will write f^* for $f^*\big|_I$. If $0 < A \le B < \infty$, then $(\, f^*(x) : x \in I_{A,B})$ is the sequence $(\,|f(x)| : x \in I_{A,B})$ rearranged in decreasing order. If X is a r.i. space, then $\|f\|_X = \|f^*\|_X$.

Definition 1.5. *Let X and Y be r.i. spaces, and $C < \infty$. We say that X and Y are* approximately equal, *with* constant of approximation *C (in symbols $X \overset{C}{\approx} Y$), if $\|\cdot\|_X \overset{C}{\approx} \|\cdot\|_Y$. If $X \overset{c}{\approx} Y$, then we simply say that X and Y are* approximately equal *(in symbols $X \approx Y$).*

1.1) Quasi-Norms

Here we give various convexity constants.

Definition 1.6. *Let I be a standard measure space, and X be a r.i. space on I.*

i) We say that X is quasi-normed if there is a $C < \infty$ such that for all $f, g \in L_0(I)$ we have

$$\|f+g\|_X \le C(\|f\|_X + \|g\|_X),$$

and we define the quasi factor of X to be

$$Q(X) = \inf\{\, C : \text{the above holds}\}.$$

ii) Let $0 < p < \infty$, $p \le q \le \infty$. We say that X is (p,q)-convex if there is a $C < \infty$ such that for all f_1, $f_2, \ldots, f_N \in L_0(I)$ we have

$$\left\| \left(\sum_{n=1}^{N} |f_n|^q \right)^{\frac{1}{q}} \right\|_X \le C \left(\sum_{n=1}^{N} \|f_n\|_X^p \right)^{\frac{1}{p}} \quad \text{for } q < \infty,$$

$$\left\| \sup_{n \in [N]} |f_n| \right\|_X \le C \left(\sum_{n=1}^{N} \|f_n\|_X^p \right)^{\frac{1}{p}} \quad \text{for } q = \infty,$$

and we define the (p,q)-convexity constant of X to be

$$C_{p,q}(X) = \inf\{\, C : \text{the above holds}\}.$$

If $p = q$, we say that X is p-convex if the above holds, and call the (p,q)-convexity constant the p-convexity constant and denote it by $C_p(X)$.

We have the following result.

Proposition 1.7. *Let X be a r.i. space. Then there is a $0 < p \le 1$ such that X is $(p,1)$-convex if and only if X is quasi-normed.*

Proof: See [K–P–R Thm.1.2]. □

However I do not know the answer to the following question.

Question. *Let X be a quasi-normed r.i. space. Is there a $0 < p < \infty$ such that X is p-convex?*

Proposition 1.8. *Let $0 < p < \infty$, $p \le q \le \infty$, and X be a (p,q)-convex r.i. space. Then there is a r.i. space X_1 such that $C_{p,q}(X_1) = 1$ and*

$$\|\cdot\|_{X_1} \le \|\cdot\|_X \le C_{p,q}(X)\, \|\cdot\|_{X_1} .$$

Proof: Suppose that $q < \infty$ (the argument is the same if $q = \infty$). Define $\|\cdot\|_{X_1}$ to be

$$\|f\|_{X_1} = \inf \left\{ \left(\sum_{n=1}^{N} \|f_n\|_X^p \right)^{\frac{1}{p}} : |f| = \left(\sum_{n=1}^{N} |f_n|^q \right)^{\frac{1}{q}} \right\}.$$

Clearly $\|\cdot\|_{X_1} \le \|\cdot\|_X \le C_{p,q}(X)\, \|\cdot\|_{X_1}$, and clearly X_1 satisfies (i) to (iii), (v) and (vi) of Definition 1.3. We check (iv) of Definition 1.3. Suppose $f_n \nearrow f$. Let $\epsilon > 0$, and choose g_1, $g_2, \ldots, g_M$ such that

$$|f| = \left(\sum_{m=1}^{M} |g_m|^q \right)^{\frac{1}{q}},$$

and

$$\left(\sum_{m=1}^{M}\|g_m\|_X^p\right)^{\frac{1}{p}} \geq \|f\|_{X_1} - \epsilon.$$

Let $g_m^{(n)} = \dfrac{f_n}{f} g_m$. Then $g_m^{(n)} \nearrow g_m$ as $n \to \infty$, and

$$|f_n| = \left(\sum_{m=1}^{M}\left|g_m^{(n)}\right|^q\right)^{\frac{1}{q}}.$$

Therefore, as $n \to \infty$ we have

$$\begin{aligned}
\|f_m\|_{X_1} &\geq \left(\sum_{m=1}^{M}\left\|g_m^{(n)}\right\|_X^p\right)^{\frac{1}{p}} \\
&\to \left(\sum_{m=1}^{M}\|g_m\|_X^p\right)^{\frac{1}{p}} \\
&\geq \|f\|_{X_1} - \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, we are done, and X_1 is a r.i. space.

Now we check that $C_{p,q}(X_1) = 1$. Let $\epsilon > 0$, and suppose that

$$|f| = \left(\sum_{n=1}^{N}|f_n|^q\right)^{\frac{1}{q}}.$$

For each $n \in [N]$, choose $g_1^{(n)}, g_2^{(n)}, \ldots, g_M^{(n)}$ such that

$$|f_n| = \left(\sum_{m=1}^{M}\left|g_m^{(n)}\right|^q\right)^{\frac{1}{q}}$$

and

$$(1+\epsilon)\|f_n\|_{X_1} \geq \left(\sum_{m=1}^{M}\|g_m(n)\|_X\right)^{\frac{1}{p}}.$$

Then

$$|f| = \left(\sum_{m=1}^{M}\sum_{n=1}^{N}\left|g_m^{(n)}\right|^q\right)^{\frac{1}{q}},$$

and so

$$\begin{aligned}
\|f\|_{X_1} &\leq \left(\sum_{m=1}^{M}\sum_{n=1}^{N}\left\|g_m^{(n)}\right\|_X^p\right)^{\frac{1}{p}} \\
&\leq (1+\epsilon)\left(\sum_{n=1}^{N}\|f_n\|_{X_1}^p\right)^{\frac{1}{p}}.
\end{aligned}$$

□

1.2) Dilatory Factors

Here we give some other numbers that describe r.i. spaces.

Definition 1.9. *Let I be a standard measure space, $0 < A \le B < \infty$, and $0 < \rho < \infty$. Define the map $d_\rho\colon I \to I$ by*

i) $x \mapsto \rho x$ *for* $I = I_{0,\infty}$;

ii) $x \mapsto \begin{cases} \rho x & \text{if } \rho x \le B \\ B & \text{if } \rho x > B \end{cases}$ *for* $I = I_{0,B}$;

iii) $x \mapsto A\lfloor \rho x / A \rfloor$ *for* $I = I_{A,\infty}$;

iv) $x \mapsto \begin{cases} A\lfloor \rho x / A \rfloor & \text{if } \rho x \le B \\ A\lfloor B \rfloor & \text{if } \rho x > B \end{cases}$ *for* $I = I_{A,B}$.

Let $D_\rho\colon L_0(I) \to L_0(I)$ be the map $f \mapsto f \circ d_\rho$.

Definition 1.10. *Let X be a r.i. space.*

i) If $0 < \rho < \infty$, then the ρ-dilatory factor of X is the number

$$D_\rho(X) = \sup\left\{ \frac{\|D_\rho(f)\|_X}{\|f\|_X} : 0 < \|f\|_X < \infty \right\}.$$

ii) We say that X is dilatory if $D_\rho(X) < \infty$ for all $0 < \rho < \infty$.

iii) Let $0 < p < \infty$. We say that X is p-lower Boyd if there is a number $C \le \infty$ such that

$$D_\rho(X) \le C\rho^{-\frac{1}{p}} \quad \text{for all } 0 < \rho \le 1.$$

The p-lower Boyd constant of X is

$$B_p(X) = \inf\{\, C : \text{the above holds}\}.$$

The lower Boyd index of X is

$$p_X = \sup\{\, p : B_p(X) < \infty\}.$$

iv) Let $0 < q < \infty$. We say that X is q-upper Boyd if there is a number $C \le \infty$ such that

$$D_\rho(X) \le C\rho^{-\frac{1}{q}} \quad \text{for all } 1 \le \rho < \infty.$$

The q-upper Boyd constant of X is

$$B^q(X) = \inf\{\, C : \text{the above holds}\}.$$

The upper Boyd index of X is

$$q_X = \inf\{\, q : B^q(X) < \infty\}.$$

We have the following simple results.

Proposition 1.11. *Let X be a r.i. space. The following are equivalent.*

i) X is dilatory.

ii) There is a $\rho < 1$ such that $D_\rho(X) < \infty$.

iii) $p_X > 0$.

Proof: (iii)⇒(i)⇒(ii) is clear. We show (ii)⇒(iii). Let X be a r.i. space on the standard measure space I. Let $\rho < 1$ be such that $D_\rho(X) < \infty$. Notice that for all $x \in I$ and $n \in \mathbf{N}$ we have $(d_\rho)^n(x) \le d_{\rho^n}(x)$, and so $D_{\rho^n}(f^*) \le (D_\rho)^n(f^*)$ for all $f \in L_0(I)$. Hence $D_{\rho^n}(X) \le (D_\rho(X))^n$, and therefore $D_\sigma(X) \le D_\rho(X)\sigma^q$ for all $\sigma < 1$, where $q = \log D_\rho(X) \big/ \log \rho$. ☐

Proposition 1.12. *Let $0 < p \leq q \leq \infty$, and X be a r.i. space. If X is quasi-normed, then it is dilatory. If X is (p,q)-convex, then $p_X \geq p$.*

Proof: Let X be a r.i. space on $I_{A,B}$. Given $f \in L_0(I_{A,B})$, and $\rho = \frac{1}{N}$, where $N \in \mathbf{N}$, let $f_1, f_2 \ldots, f_N \in L_0(I_{A,B})$ be disjoint functions each of which has the same distribution as $f\big|_{[0,B/N]}$. Then for any $0 < q < \infty$, we have

$$\begin{aligned} D_\rho(f^*) &= \left(\left(\sum_{n=1}^{N} |f_n|^q\right)^{\frac{1}{q}}\right)^* \\ &= \left(\sup_{n \in [N]} |f_n|\right)^* . \end{aligned}$$

Thus $\left\|D_{\frac{1}{2}}(f)\right\|_X \leq Q(X)\left\|f\right\|_X$, and $\left\|D_{\frac{1}{N}}(f)\right\|_X \leq C_{p,q}(X) N^{-\frac{1}{p}}\left\|f\right\|_X$. The results follow. $\square$

2) Admissible Functions

In this section we define the notion of admissible functions. Again, the long definitions disguise simple ideas, which are only meant to provide elementary extensions to the idea of Orlicz functions (see [B–S Ch.4 §8], [K–R], L–T1 Ch.4] or [Mu]).

Definition 2.1. *A function $F\colon [0,\infty] \to [0,\infty]$ is called admissible if*
i) F *is increasing;*
ii) $F(0) = 0$*;*
iii) F *is left continuous;*
iv) F *is continuous at* 0*;*
v) $F(t) \to \infty$ *as* $t \to \infty$*.*
If F satisfies only (i), (ii) and (iii), we say that F is almost admissible.

Definition 2.2. *Let F be an almost admissible function. Then we define F^{-1} and $\tilde{F}\colon [0,\infty] \to [0,\infty]$ to be*

$$F^{-1}(t) = \sup\{\, u : F(u) < t\}$$
$$\tilde{F}(u) = \sup\{\, 1/F(\tfrac{1}{t}) : t < u\}.$$

(Here $\sup \emptyset = 0$.)

Proposition 2.3. *If F is an almost admissible function, then so are $\tilde{F}$ and F^{-1}. If F and G are admissible functions, then $F \circ G$ is almost admissible. Furthermore, we have the following identities.*
i) $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$.
ii) $(F^{-1})^{-1} = F$.
iii) $\widetilde{F \circ G} = \tilde{F} \circ \tilde{G}$.
iv) $\tilde{\tilde{F}} = F$.
v) $\widetilde{F^{-1}} = \tilde{F}^{-1}$.

For comparing admissible functions we introduce the following notation.

Definition 2.4. *Let F and G be admissible functions. For $C < \infty$ we write $F \overset{C}{\prec} G$ if for all $t \in [0,\infty]$ we have $F(t) \le G(Ct)$, and we write $F \overset{C}{\succ} G$ if $G \overset{C}{\prec} F$, and $F \overset{C}{\asymp} G$ if $F \overset{C}{\prec} G$ and $G \overset{C}{\prec} F$. We write $F \prec G$ ($F \succ G$, $F \asymp G$) if there is a $C < \infty$ such that $F \overset{C}{\prec} G$ ($F \overset{C}{\succ} G$, $F \overset{C}{\asymp} G$). If $F \asymp G$, we say that F is equivalent to G.*

Proposition 2.5. *Let F and G be admissible functions, and $0 < C < \infty$. Then*
i) $F \le CG \Leftrightarrow \tilde{G} \le C\tilde{F}$,
ii) $F \overset{C}{\prec} G \Leftrightarrow \tilde{G} \overset{C}{\prec} \tilde{F}$,
iii) $F \le CG \Leftrightarrow G^{-1} \overset{C}{\prec} F^{-1}$.

2.1) Quasi-Orlicz Functions

Definition 2.6. *Let F be an admissible function.*
i) Let $0 < \rho < \infty$. The ρ-dilatory factor of F is defined to be

$$D_\rho(F) = \sup\left\{ \frac{F^{-1}(\rho^{-1}t)}{F^{-1}(t)} : 0 < F^{-1}(t) < \infty \right\}.$$

ii) We say that F is dilatory if $D_\rho(F) < \infty$ for all $0 < \rho < \infty$.
iii) Let $0 < p < \infty$. We say that F is p-quasi-Orlicz if there is a number $C < \infty$ such that $D_\rho(F) \le C\rho^{-\frac{1}{p}}$ for all $0 < \rho \le 1$.
iv) Let $0 < p < \infty$. We say that F is p-convex, or p-Orlicz if $F \circ T^{\frac{1}{p}}$ is a convex function.

Proposition 2.7. *Let F be an admissible function. The following are equivalent.*
i) F is dilatory.
ii) There is a $0 < \rho < 1$ such that $D_\rho(F) < \infty$.
iii) There are C_1, $C_2 > 1$ such that for all $t \in [0,\infty]$ we have $F(C_1 t) \ge C_2 F(t)$.
iv) There is a $0 < p < \infty$ such that F is p-quasi-Orlicz.
v) There is a $0 < p < \infty$ such that F is equivalent to a p-convex admissible function.
Furthermore the p that satisfy (iv) are the same as the p that satisfy (v).

Proof: (iv)⇒(i) and (i)⇒(ii) are clear. (ii)⇔(iii) is easy. (ii)⇒(iv) is a simpler version of the argument given for Proposition 1.11. (v)⇒(iv) is trivial. We show (iv)⇒(v).

Let $0 < p < \infty$ and $C < \infty$ be such that $D_\rho(F) \le C\rho^{\frac{1}{p}}$, that is, for all $u \ge 1$ and $0 \le t \le \infty$, we have

$$u^p F(t) \le F(Cut). \tag{2.1}$$

Define the function $G\colon [0,\infty] \to [0,\infty]$ by

$$\begin{aligned} G(t) &= \sup_{u \le t} \left\{ \left(\left(\frac{t}{u}\right)^p - 1 \right) F(u) \right\} \\ &= \sup_{u \in \mathbf{R}} \left\{ 0 \vee \left(\left(\frac{t}{u}\right)^p - 1 \right) F(u) \right\}. \end{aligned}$$

We see that $G \circ T^{\frac{1}{p}}$ is the supremum of convex functions, and so is itself convex. We also see that $G(t) \ge ((t/2^{-\frac{1}{p}}t)^p - 1)F(2^{-\frac{1}{p}}t) = F(2^{-\frac{1}{p}}t)$. Further, by equation (2.1), we have that $((t/u)^p - 1)F(u) \le F(Ct)$. Hence $G \asymp F$.

That G is admissible is now trivial to check. $\square$

2.2) Admissible Functions that Satisfy the Δ_2-Condition

Definition 2.8. *Let F be an admissible function. We say that F satisfies the Δ_2-condition if F^{-1} is a dilatory admissible function.*

We see from Proposition 2.7 that this definition is the same as that given in Section 1A:2.3.

Proposition 2.9. *Let $0 < p < \infty$, and F be an admissible function that is p-quasi Orlicz and that satisfies the Δ_2-condition. Then there is a $p \le q < \infty$, $C < \infty$, and an admissible function G such that*

i) $F \overset{C}{\asymp} G$;

ii) $G \circ T^{\frac{1}{p}}$ *is convex;*

iii) $G \circ T^{\frac{1}{q}}$ *is concave.*

Proof: By replacing F by $F \circ T^{\frac{1}{p}}$, we may assume that F is 1-quasi Orlicz, and by Proposition 2.7, we may assume that F is a convex function. In particular, we may assume that $\frac{F(t)}{t}$ increases as t increases.

None of this changes the fact that F satisfies the Δ_2-condition. By Proposition 2.7, there is a $0 < q < \infty$ such that F^{-1} is $\frac{1}{q}$-quasi Orlicz, that is, there is a number $C < \infty$ such that for all $0 < t < \infty$ and $u \ge 1$ we have

$$F(ut) \le C u^q F(t).$$

(Thus it is clear that we must have $q \ge 1$ if F is to be convex.) Define the function H to be

$$H(t) = \sup_{u \ge 1} \frac{F(ut)}{u^q}.$$

Clearly $F(t) \le H(t) \le CF(t)$, and as F is convex, this implies that $F \overset{C}{\asymp} H$. Furthermore, $\frac{H(t)}{t}$ increases with t, and $\frac{H(t)}{t^q}$ decreases with t.

Now define the function G to be

$$G(t) = \int_0^t \frac{H(u)}{u}\, du.$$

Since $\frac{H(u)}{u}$ increases with u, we have

$$G(t) \le t\frac{H(t)}{t} = H(t),$$

$$G(2t) \ge \int_t^{2t} \frac{H(u)}{u}\, du \ge H(t),$$

so that $G \overset{2}{\asymp} H$. We also have that G is convex. Furthermore,

$$\begin{aligned} G(t^{\frac{1}{q}}) &= \int_0^{t^{\frac{1}{q}}} \frac{H(u)}{u}\, du \\ &= \tfrac{1}{q} \int_0^t \frac{H(u^{\frac{1}{q}})}{u}\, du, \end{aligned}$$

which, being an integral of a decreasing function, is concave. It is now easy to show that H is admissible $\square$

2.3) Families of Admissible Functions

Finally in this section, we define the following for later use.

Definition 2.10. *Let I be a standard measure space. A family of admissible functions on I is a sequence $(F_t : t \in I)$, where for each $t \in I$, F_t is an admissible function.*

3) Functions from Spaces and Spaces from Functions

3.1) The Fundamental Function

Definition 3.1. *Let X be a r.i. space on $I_{0,\infty}$. Define the fundamental function $\Phi_X\colon[0,\infty]\to[0,\infty]$ to be such that*

$$\tilde{\Phi}_X^{-1}(t)=\left\|\chi_{[0,t]}\right\|_x .$$

Proposition 3.2. *Let X be a r.i. space. Then Φ_X is an admissible function. If X is dilatory, then so is Φ_X. If $p_X>0$, then Φ_X is p_X-convex.*

Proof: This is trivial. □

We could also define Φ_X when X is a r.i. space on $I_{A,B}$ where $(A,B)\neq(0,\infty)$, by 'joining the dots'. However we never have any need to do this.

3.2) The Orlicz Functional

Some of these ideas may also be found in [B–S Ch.4 §8], [K–R], [L–T1 Ch.4], [L–T2 p120] and [Mu].

Definition 3.3. *Let I be a standard measure space, and F be an admissible function. Define the L_F-functional (also called the F-Orlicz functional) $\|\cdot\|_F$ on $L_0(I)$ to be*

$$\|f\|_F=\sup\left\{x:\int_I F\left(\tfrac{|f(t)|}{x}\right)\,d\lambda(t)>1\right\}.$$

Let the F-Orlicz space be $L_F=L_F(I)=\{f\in L_0(I):\|f\|_F<\infty\}$.

Proposition 3.4. *Let I be a standard measure space, F be an admissible function, and $f\in L_0(I)$. Then*

i) $\int_I F(|f|)\,d\lambda>1\Rightarrow\|f\|_F\geq 1$;

ii) $\int_I F(|f|)\,d\lambda\leq 1\Rightarrow\|f\|_F\leq 1$;

iii) $\|f\|_F>1\Rightarrow\int_I F(|f|)\,d\lambda>1$;

iv) $\|f\|_F<1\Rightarrow\int_I F(|f|)\,d\lambda\leq 1$.

Furthermore, if F satisfies the Δ_2-condition, then

$$\|f\|_F=1\Leftrightarrow\int_I F(|f|)\,d\lambda=1.$$

Proof: These are straightforward to check. □

Proposition 3.5. *Let I be a standard measure space, and F be an admissible function. Then $L_F(I)$ is a r.i. space. If F is dilatory then $L_F(I)$ is quasi-normed, and if F is p-Orlicz then $L_F(I)$ is p-convex. If $I=I_{0,\infty}$ then $\Phi_{L_F}=F$.*

Proof: First we show that $L_F(I)$ is a r.i. space. The only problems in checking Definition 1.3 are properties (iv) and (v). To check (iv), suppose that $f_n\nearrow f$, with $\|f\|_F>1$. Then we have

$$\int_I F(|f|)\,d\lambda>1.$$

Since F is left continuous, $F(|f_n|)\nearrow F(|f|)$, and so by the monotone convergence theorem, there is a $N\in\mathbf{N}$ such that for all $n\geq N$ we have

$$\int_I F(|f_n|)\,d\lambda>1.$$

But this implies that $\|f_n\|_F \geq 1$, and we are done.

To check (v), let $\epsilon > 0$, and set $\delta = \epsilon / F^{-1}(\frac{1}{\epsilon})$ (notice that by property (v) of Definition 2.1 we have $\delta > 0$). If $\|f_n\|_F \to 0$ as $n \to \infty$, then there is a $N \in \mathbf{N}$ such that for all $n \geq N$ we have $\|f_n\|_F < \delta$, that is,

$$\int_I F\left(\tfrac{|f_n|}{\delta}\right) d\lambda \leq 1.$$

Let $A = \{ t \in I : F(|f_n|/\delta) \geq \frac{1}{\epsilon}\}$. Then $\lambda(A) \leq \epsilon$, and if $t \notin A$, then $|f_n| \leq \delta F^{-1}(\frac{1}{\epsilon}) = \epsilon$. Hence $\lambda\{ t \in I : |f_n(t)| > \epsilon\} \leq \epsilon$. Thus $f_n \to 0$ in the measure topology.

Now suppose that F is p-Orlicz. We show that L_F is p-convex. It is sufficient to show the following. If $\|f_1\|_F$, $\|f_2\|_F, \ldots, \|f_N\|_F < 1$, and $u_1, u_2, \ldots, u_N \in [0,1]$ with $\sum_{n=1}^N u_n^p = 1$, then $\left\|\sum_{n=1}^N u_n |f_n|\right\| \leq 1$. But then

$$\begin{aligned}\int_I F\left(\textstyle\sum_{n=1}^N u_n |f_n|\right) d\lambda &\leq \sum_{n=1}^N u_n^p \int_I F(|f_n|)\, d\lambda \\ &\leq 1,\end{aligned}$$

and the result follows. □

Proposition 3.6. *Let F and G be admissible functions, and $0 < C < \infty$. If $F \overset{C}{\prec} G$, then $\|\cdot\|_F \leq C \|\cdot\|_G$.*

Proof: This is trivial to check. □

Definition 3.7. *Let I be a standard measure space, X be a r.i. space on I, and F be an admissible function. Define the $X \circ F$-functional $\|\cdot\|_{X\circ F}$ on $L_0(I)$ to be*

$$\|f\|_{X\circ F} = \sup\left\{ x : \left\|F\left(\tfrac{|f|}{x}\right)\right\|_X > 1\right\}.$$

Proposition 3.8. *Let I be a standard measure space, X be a r.i. space on I, and F be an admissible function. Then $X \circ F$ is a r.i. space on I.*

Proof: As Proposition 3.5. □

3.3) The Weak Orlicz Functional

Definition 3.9. *Let I be a standard measure space, and F be an admissible function. Define the $L_{F,\infty}$-functional (also called the F-weak Orlicz functional) $\|\cdot\|_{F,\infty}$ on $L_0(I)$ to be*

$$\|f\|_{F,\infty} = \sup\{f^*(t)\tilde{F}^{-1}(t) : 0 < t < \infty\}.$$

Let the F-weak Orlicz space be $L_{F,\infty} = L_{F,\infty}(I) = \{f \in L_0(I) : \|f\|_{F,\infty} < \infty\}$.

Proposition 3.10. *Let I be a standard measure space, and F be an admissible function. Then $L_{F,\infty}(I)$ is a r.i. space. If F is dilatory then $L_F(I)$ is quasi-normed, and if F is p-quasi-Orlicz then $L_F(I)$ is (p,∞)-convex. If $I = I_{0,\infty}$ then $\Phi_{L_F} = F$.*

Proof: These are all elementary. □

Proposition 3.11. *Let F and G be admissible functions, and $0 < C < \infty$. If $F \overset{C}{\prec} G$, then $\|\cdot\|_F \leq C \|\cdot\|_G$.*

Proof: This is trivial. □

Proposition 3.12. *Let X be a r.i. space on $I_{0,\infty}$. Then $\|\cdot\|_{\Phi_X,\infty} \leq \|\cdot\|_X$.*

Proof: Let $f \in L_0(I)$, and $t \in I$. Then

$$f^*(t)\chi_{[0,t]} \leq f^*,$$

and the result follows by taking $\|\cdot\|_\Phi$ of both sides. □

3.4) Families of Admissible Functions.

This definition is similar to what may be found in [Mu Ch.2 §7].

Definition 3.13. *Let I be a standard measure space, and $(F_t : t \in I)$ be a family of admissible functions. Define the $L_{(F_t)}$-functional $\|\cdot\|_{(F_t)}$ on $L_0(I)$ to be*

$$\|f\|_{(F_t)} = \sup\left\{x : \int_I F_t\left(\tfrac{|f(t)|}{x}\right)\, d\lambda(t) > 1\right\}.$$

Note that $\|\cdot\|_{(F_t)}$ will not necessarily be a r.i. functional.

4) Un-L_∞ and Normal Spaces and Functions

As far as I know, these results are new.

Definition 4.1. *An admissible function F is said to be un-L_∞ if $F(t) = 0$ only if $t = 0$, and $F(t) = \infty$ only if $t = \infty$. We say that F is normal if in addition, F is continuous and strictly increasing.*

If X is a r.i. space on $I_{0,\infty}$ then X is said to be un-L_∞ (normal) if Φ_X is un-L_∞ (normal).

Proposition 4.2. *Let X be an un-L_∞, (p,q)-convex, r.i. space on $I_{0,\infty}$, where $0 < p < \infty$ and $p \le q \le \infty$. Then there is a normal, quasi-normed, r.i. space on $I_{0,\infty}$, Y, such that*

$$\tfrac{1}{2} \left\|\cdot\right\|_Y \le \left\|\cdot\right\|_X \le C_{p,q}(X) \left\|\cdot\right\|_Y .$$

Proof: By Proposition 1.8, there is a r.i. space X_1 with $C_{p,q}(X_1) = 1$ and $\left\|\cdot\right\|_{X_1} \le \left\|\cdot\right\|_X \le C_{p,q}(X) \left\|\cdot\right\|_{X_1}$. As $C_{p,q}(X_1) = 1$, we have $\left\|\chi_{[0,s+t]}\right\|_{X_1}^p \le \left\|\chi_{[0,s]}\right\|_{X_1}^p + \left\|\chi_{[0,t]}\right\|_{X_1}^p$, that is

$$\left(\tilde{\Phi}_{X_1}^{-1}(s+t)\right)^p \le \left(\tilde{\Phi}_{X_1}^{-1}(s)\right)^p + \left(\tilde{\Phi}_{X_1}^{-1}(t)\right)^p.$$

Since X is un-L_∞, $\tilde{\Phi}_{X_1}^{-1}(t) \to 0$ as $t \to 0$, and therefore $\tilde{\Phi}_{X_1}^{-1}$ is continuous.

Also, by Proposition 2.7 there is a p-convex admissible function $G \asymp \Phi_{X_1}$. Since G is un-L_∞, it is continuous and strictly increasing. So let

$$\left\|\cdot\right\|_Y = \left\|\cdot\right\|_{X_1} + \epsilon \left\|\cdot\right\|_{G,\infty} ,$$

where $\epsilon > 0$ is such that $\epsilon \left\|\cdot\right\|_{G,\infty} \le \left\|\cdot\right\|_X$. Then

$$\tilde{\Phi}_Y^{-1} = \tilde{\Phi}_{X_1}^{-1} + \epsilon \tilde{G}^{-1},$$

which is continuous and strictly increasing. The result follows. $\square$

Corollary 4.2a. *Let F be a un-L_∞, dilatory, admissible function. Then there is an equivalent normal, dilatory, admissible function.*

Proof: Let $X = L_F(I_{0,\infty})$, and choose Y as in Proposition 4.2. Then the desired function is Φ_Y. $\square$

Normal admissible functions have nice properties: if F and G are normal admissible functions, then so is $F \circ G$; and if F is a normal admissible function, then so are $\tilde{F}$ and F^{-1}, with $F \circ F^{-1} = F^{-1} \circ F = \mathrm{Id}_{[0,\infty]}$.

5) Generalized Lorentz Spaces

Now we come to define the main object of interest of Part 3. The motivation for their definition arises as follows. Let F be an admissible function, and X be a r.i. space on $I_{0,\infty}$. We desire to define a r.i. space on $I_{0,\infty}$, called $L_{F,X}$ such that L_{T^p,L_q} is the Lorentz space $L_{p,q}$. Thus we would like $\Phi_{L_{X,F}} = F$, and for $L_{X,F}$ to 'glue' together like X. Two definitions that immediately suggest themselves are the following.

i) Let $\omega\colon I_{0,\infty} \to \mathbf{R}_+$ be such that $\left\|\omega\chi_{[0,t]}\right\|_X = \tilde{F}^{-1}(t)$. Define
$$\|f\|_{X,F} = \|\omega f^*\|_X .$$

ii) Let $\Omega\colon I_{0,\infty} \to I_{0,\infty}$ be such that $\left\|\chi_{[0,t]} \circ \Omega\right\|_X = \tilde{F}^{-1}(t)$. Define
$$\|f\|_{X,F} = \|f^* \circ \Omega\|_X .$$

If $X = L_q$, then both definitions give the same space, (with $\omega(t) = ((\Omega^{-1})'(t))^{\frac{1}{q}}$). We also have the same space if $X = L_F$, then both definitions give $L_{X,F} = L_F$.

We will make use of the second definition. This is because it is easy to see that Ω should be $\tilde{F} \circ \tilde{\Phi}_X^{-1}$; it is not at all clear what ω should be.

Definition 5.1. *Let I be a standard measure space, X be a normal r.i. space on $I_{0,\infty}$, and F be a normal admissible function. Define the $L_{F,X}$-functional (also called the generalized F-Lorentz functional) $\|\cdot\|_{F,X}$ on $L_0(I)$ to be*
$$\|f\|_{F,X} = \left\|f^* \circ \tilde{F} \circ \tilde{\Phi}_X^{-1}\right\|_X .$$
Let the generalized F-Lorentz space be
$$L_{F,X} = L_{F,X}(I) = \{f \in L_0(I) : \|f\|_{F,X} < \infty\}.$$

Proposition 5.2. *Let I be a standard measure space, X be a normal r.i. space on $I_{0,\infty}$, and F be a normal admissible function. Then $L_{X,F}(I)$ is a r.i. space. Furthermore we have the following properties.*

i) If $I = I_{0,\infty}$, then $\Phi_{L_{F,X}} = F$.

ii) $L_{\Phi_X,X} = X$.

iii) $L_{F,L_F} = L_F$.

iv) If G is a normal admissible function, then $L_{F,L_{G,X}} = L_{F,X}$.

Proof: These are all straightforward to check. □

Proposition 5.3. *Let X be a normal, quasi-normed r.i. space on $I_{0,\infty}$, and F be a normal admissible function. Then $L_{F,X}$ is quasi-normed if and only if it is dilatory.*

Proof: If $f, g \in L_0$, then $(f+g)^* = f^* \circ d_{\frac{1}{2}} + g^* \circ d_{\frac{1}{2}}$. Thus, if $L_{F,X}$ is dilatory, it follows straight away from Definition 5.1 that $Q(L_{F,X}) \le D_{\frac{1}{2}}(L_{F,X})Q(X)$. The converse implication follows by Propositions 1.12 and 3.2. □

Proposition 5.4. *Let X and Y be normal r.i. spaces on $I_{0,\infty}$, and F and G be normal admissible functions. Suppose that X and Y are approximately equal, that F and G are equivalent, and that $L_{1,X}$ is dilatory. Then $L_{F,X}$ and $L_{G,Y}$ are approximately equal.*

Proof: Let $C < \infty$ be such that $X \overset{C}{\approx} Y$, and $F \approx CG$. Then for all $f \in L_0$, we have
$$\begin{aligned}\|f\|_{G,Y} &= \left\|f^* \circ \tilde{G} \circ \tilde{\Phi}_Y^{-1}\right\|_Y \\ &\le C \left\|f^* \circ \tilde{F} \circ d_{C^{-2}} \circ \tilde{\Phi}_X^{-1}\right\|_X \\ &\le C\, D_{C^{-2}}(L_{1,X})\ \|f\|_{F,X} ,\end{aligned}$$
and
$$\begin{aligned}\left(C D_{C^{-2}}(L_{1,X})\right)^{-1} \|f\|_{F,X} &\le C^{-1} \left\|f^* \circ \tilde{F} \circ d_{C^{-2}} \circ \tilde{\Phi}_X^{-1}\right\|_X \\ &\le \left\|f^* \circ \tilde{G} \circ \Phi_Y^{-1}\right\|_Y \\ &= \|f\|_{G,Y} .\end{aligned}$$
□

We consider the problem of whether $L_{F,X}$ is dilatory in Chapter 3C. However, we present the following simple result.

Proposition 5.5. *Let X be a normal, quasi-normed r.i. space on $I_{0,\infty}$, and F be a normal admissible function. If F satisfies the Δ_2-condition, then $L_{F,X}$ is dilatory.*

Proof: This is trivial. □

Proposition 5.6. *Let X be a normal r.i. space on $I_{0,\infty}$, and F and H be normal admissible functions. Then $L_{F,X} \circ H = L_{F\circ H, X\circ H}$.*

Proof: This is easy to check. □

Chapter 3B — Inequalities

1) Hölder's Inequality Generalized

In this section we investigate a well known generalization of Hölder's inequality. However, our approach is quite different to that usually presented (see [B–S Ch.4], [K–R Ch.4 §9], [L–T1 4.b] or [Mu Ch.2 §13]).

1.1) Complementary Functions and Dual Spaces

Definition 1.1. *Let F, G and H be admissible functions. We say that F and G are complementary with respect to H if there is a number $C < \infty$ such that*

$$H^{-1} \overset{C}{\approx} F^{-1}G^{-1}.$$

If $H = \mathrm{Id}_{[0,\infty]}$, we simply say that F and G are complementary.

If F is an admissible function such that $\frac{F(t)}{t}$ is increasing with t, we define the complementary function F^* to be an admissible function such that

$$F^{-1}F^{*-1} = \mathrm{Id}_{[0,\infty]}.$$

Definition 1.2. *Let X, Y and Z be r.i. spaces. We say that X and Y are sub-dual with respect to Z if there is a number $C < \infty$ such that for all f, $g \in L_0$ we have*

$$\|fg\|_Z \leq C \|f\|_X \|f\|_Y .$$

We say that X and Y are dual with respect to Z if in addition we have

i) given $f \in X \setminus \{0\}$ there is a $g \in Y \setminus \{0\}$ such that $\|fg\|_Z \geq C^{-1} \|f\|_X \|f\|_Y$;

ii) given $g \in Y \setminus \{0\}$ there is a $f \in X \setminus \{0\}$ such that $\|fg\|_Z \geq C^{-1} \|f\|_X \|f\|_Y$.

1.2) Hölder's Inequality Generalized

Theorem 1.3. *Let X be a quasi-normed r.i. space, F and G be normal admissible functions and H be a normal dilatory admissible function. If F and G are complementary with respect to H, then $X \circ F$ and $X \circ G$ are sub-dual with respect to $X \circ H$. If, in addition, F and G satisfy the Δ_2-condition, then $X \circ F$ and $X \circ G$ are dual with respect to $X \circ H$.*

Note that we do not need F, G and H to be normal, but it does make the proof simpler to assume that they are.

Proof: Let $C < \infty$ be such that $H^{-1} \overset{C}{\approx} F^{-1}G^{-1}$. First we show that there is a constant $C_1 < \infty$ such that for all $u, v \in [0, \infty]$ we have
$$H(C_1^{-1}uv) \leq F(u) + G(v).$$
Let $s = F(u)$ and $t = G(v)$. Then $uv = F^{-1}(s)G^{-1}(t) \leq F^{-1}(s \vee t)G^{-1}(s \vee t) \leq CH^{-1}(s \vee t) \leq CH^{-1}(s+t)$, and we have the result.

Now we show that there is a constant $C_2 < \infty$ such that for all $f, g \in L_0$ we have
$$\|fg\|_{X\circ H} \leq C_2 \|f\|_{X\circ F} \|g\|_{X\circ G},$$
for suppose that $\|f\|_{X\circ F}$, $\|g\|_{X\circ G} < 1$. Then
$$\begin{aligned}\left\|H(C_1^{-1}|fg|)\right\|_X &\leq \|F(|f|) + G(|g|)\|_X \\ &\leq Q(X)(\|F(|f|)\|_X + \|G(|g|)\|_X) \\ &\leq 2Q(X).\end{aligned}$$
Hence
$$\left\|H(\tfrac{1}{C_1 D_{(2Q(X))^{-1}}(H)}|fg|)\right\|_X \leq 1,$$
and so $\|fg\|_{X\circ H} \leq C_1 D_{(2Q(X))^{-1}}(H)$, as desired.

Now suppose that F and G satisfy the Δ_2-condition (so that H also satisfies the Δ_2-condition). We show that there is a number $C_3 < \infty$ such that if $f \in X \circ F \setminus \{0\}$, then there is a $g \in X \circ G \setminus \{0\}$ with $\|fg\|_{X\circ H} \geq C_3^{-1} \|f\|_{X\circ F} \|g\|_{X\circ G}$. For suppose that $\|f\|_{X\circ F} = 1$. Let $k = F(|f|)$ (so that $|f| = F^{-1}(k)$) and $g = G^{-1}(k)$. By Proposition 3A:3.4, we have that $\|k\|_X = 1$, $\|g\|_{X\circ G} = 1$, and $\left\|H^{-1}(k)\right\|_{X\circ H} = 1$. Therefore,
$$\|fg\|_{X\circ H} = \left\|F^{-1}(k)G^{-1}(k)\right\|_{X\circ H} \geq C^{-1}\left\|H^{-1}(k)\right\|_{X\circ H} = C^{-1}.$$

Similarly, there is a number $C_4 < \infty$ such that if $g \in X \circ G \setminus \{0\}$, then there is a $f \in X \circ F \setminus \{0\}$ with $\|fg\|_{X\circ H} \geq C_4^{-1} \|f\|_{X\circ F} \|g\|_{X\circ G}$. □

1.3) Families of Admissible Functions

Theorem 1.4. *Let I be a standard measure space, and $(F_t : t \in I)$ be a family of admissible functions on I such that for each $t \in I$ we have that F_t is normal and that $\frac{F_t(u)}{u}$ is increasing with u. Let $(F_t^* : t \in I)$ be a family of admissible functions, where for each $t \in I$, F_t^* is the complementary function of F_t. Then for all $f, g \in L_0(I)$ we have*
$$\|fg\|_1 \leq c\|f\|_{(F_t)} \|g\|_{(F_t^*)}.$$

Proof: As Theorem 1.3. □

2) The Second Index of Generalized Lorentz Functions

In this section we find inequalities between L_{F,G_1} and L_{F,G_2}, and between $L_{F,G}$ and $L_{F,\infty}$, where F, G, G_1 and G_2 are normal admissible functions. This work extends results due to Bennett and Rudnick (see [B–R] or [B–S Ch.4 §6]).

2.1) Inequalities Between L_{F,G_1} and L_{F,G_2}

In this section we are going to compare $L_{F,G}$ with $L_{F,H\circ G}$, where F, G and H are normal admissible functions, and H is 1-quasi Orlicz.

Theorem 2.1. *Let F, G and H be normal admissible functions such that G is dilatory, and H is 1-quasi Orlicz. Then there is a number $C < \infty$, depending on H and $D_{\frac{1}{2}}(G)$ only, such that*

$$\|\cdot\|_{F,H\circ G} \le C \|\cdot\|_{F,G} .$$

Proof: First we show that for some number $C_1 < \infty$ that$\|\cdot\|_H \le C_1 \|\cdot\|_{H,1}$.

For all $f \in L_0$ we have

$$\begin{aligned} \|f\|_{H,1} &= \int_0^\infty f^* \circ \tilde{H}(t)\, dt \\ &= \int_0^\infty f^*(t)\, d\tilde{H}^{-1}(t) \\ &= \int_0^\infty \tilde{H}^{-1}(t)\, d(-f^*(t)) \\ &= \int_0^\infty \left\|\chi_{[0,t]}\right\|_H \, d(-f^*(t)) \end{aligned}$$

and as $\|\cdot\|_H$ is 1-convex, there is a number $C_1 < \infty$, depending on H only, such that the above is

$$\begin{aligned} &\le C_1 \left\| \int_0^\infty \chi_{[0,t]}\, d(-f^*(t)) \right\|_H \\ &= C_1 \|f^*\|_H . \end{aligned}$$

Now, as G is dilatory, we have for some number $C_2 < \infty$ that

$$\|\cdot\|_{H\circ G, H\circ G} = \|\cdot\|_{L_{H,G}} \le C_2 \|\cdot\|_{L_{H,1}\circ G} = C_2 \|\cdot\|_{H\circ G, G} ,$$

and hence for all $f \in L_0$ we have

$$\begin{aligned} \|f\|_{F,H\circ G} &= \left\| f\circ \tilde{F} \circ (\tilde{H}\circ \tilde{G})^{-1} \right\|_{H\circ G, H\circ G} \\ &\le C_2 \left\| f\circ \tilde{F} \circ (\tilde{H}\circ \tilde{G})^{-1} \right\|_{H\circ G, G} \\ &= C_2 \|f\|_{F,H\circ G} . \end{aligned}$$

□

For the next theorem, we need an elementary definition and lemma.

Definition 2.2. *Suppose F is a normal admissible function. We denote by F' the unique left continuous function $(0,\infty) \to [0,\infty]$ such that*

$$F(t) = \int_0^t F'(u)\, du.$$

Lemma 2.3. *Suppose F is a normal p-convex admissible function, where $0 < p < \infty$. Then there is a number $C < \infty$, depending on p only, such that for all $0 < t < \infty$, we have*

$$p\frac{F(t)}{t} \le F'(t) \le C\frac{F(Ct)}{t}.$$

Proof: Let $G = F \circ T^{\frac{1}{p}}$, so that G is convex. It is well known that G' is increasing, and that $F'(t) = pt^{p-1}G'(t^p)$.

First we prove the left hand inequality. For all $0 < t < \infty$, we have

$$\begin{aligned} F(t) &= p \int_0^t u^{p-1} G'(u^p)\, du \\ &\le p \int_0^t u^{p-1} G'(t^p)\, du \\ &= t^p\, G'(t^p) \\ &= \tfrac{1}{p}\, t\, F'(t). \end{aligned}$$

For the right hand inequality, for all $0 < t < \infty$, we have

$$\begin{aligned} (1 - 2^{-p})\, F(2t) &\ge F(2t) - F(t) \\ &= p \int_t^{2t} u^{p-1} G'(u^p)\, du \\ &\ge p \int_t^{2t} u^{p-1} G'(t^p)\, du \\ &= (2^p - 1)\, t^p\, G'(t^p) \\ &= (2^p - 1)\, \tfrac{1}{p}\, t\, F'(t). \end{aligned}$$

$\square$

Theorem 2.4. *Let F, G and H be normal admissible functions such that G is dilatory, and H is 1-convex. If $\left\| \frac{1}{\tilde{H}^{*-1}} \right\|_{H^*} < \infty$, then there is a number $C < \infty$ such that*

$$\|\cdot\|_{F,G} \le C \, \|\cdot\|_{F,H\circ G} .$$

Proof: First we show that $\|\cdot\|_1 \le c \left\| \frac{1}{\tilde{H}^{*-1}} \right\|_{H^*} \|\cdot\|_{1,H}$. Suppose that $\|f\|_{1,H} < 1$. Then

$$\int_0^\infty H(f^*(t))\tilde{H}'(t)\, dt = \int_0^\infty H \circ f^* \tilde{H}^{-1}(t)\, dt \le 1.$$

So, if we let $(\, H_t : t \in I_{0,\infty})$ be the family of admissible functions defined by $H_t(u) = \tilde{H}'(t)H(u)$, then the above says that $\|f\|_{(H_t)} \le 1$. Now it is easy to see that

$$H_t^*(u) = \tilde{H}'(t) H^* \left(\tfrac{u}{\tilde{H}'(t)} \right) .$$

So by Theorem 1.4 we have

$$\|f\|_1 \le c \, \|f\|_{(H_t)} \, \|1\|_{(H_t^*)} \le c \, \|1\|_{(H_t^*)} .$$

Thus we only need to show that $\|1\|_{(H_t^*)} \le \left\| \frac{1}{\tilde{H}^{*-1}} \right\|_{H^*}$. This follows, as if $\|1\|_{(H_t^*)} > x$, then by Proposition 3A:3.4, we have that

$$\begin{aligned} 1 &< \int_0^\infty H_t^* \left(\tfrac{1}{x} \right)\, dt \\ &= \int_0^\infty H^* \left(\tfrac{1}{x\tilde{H}'(t)} \right)\, d\tilde{H}(t). \end{aligned}$$

By Lemma 2.3, this is

$$\begin{aligned}
&\leq \int_0^\infty H^*\left(\tfrac{t}{x\tilde{H}(t)}\right)\, d\tilde{H}(t) \\
&= \int_0^\infty H^*\left(\tfrac{\tilde{H}^{-1}(t)}{xt}\right)\, dt \\
&= \int_0^\infty H^*\left(\tfrac{1}{x\tilde{H^*}^{-1}(t)}\right)\, dt.
\end{aligned}$$

This implies that $\left\|\frac{1}{\tilde{H^*}^{-1}}\right\|_{H^*} \geq x$, as desired.

Finally, the result follows as in the end of the proof of Theorem 2.1. □

2.2) Inequalities Between $L_{F,G}$ and $L_{F,\infty}$

By Proposition 3A:3.12, we already know that if F and G are normal admissible functions, then $\|\cdot\|_{F,\infty} \leq \|\cdot\|_{F,G}$. We also have the following result.

Theorem 2.5. *Let F and G be normal admissible functions. Then*

$$\|\cdot\|_{F,G} \leq \left\|\tfrac{1}{\tilde{G}^{-1}}\right\|_G \|\cdot\|_{F,\infty}.$$

Proof: Suppose that $\|f\|_{F,\infty} \leq 1$. Then $f^* \leq \frac{1}{\tilde{F}^{-1}}$, and so $\|f\|_{F,G} \leq \left\|\frac{1}{\tilde{G}^{-1}}\right\|_G$.

□

2.3) Applications of these Inequalities

Now we prove Theorem 1A:2.7.

Lemma 2.6. *Let $L \asymp \mathrm{em}(T^\beta)$, where $\beta > 0$. Then $\left\|\frac{1}{\tilde{L}^{-1}}\right\|_L$ is finite.*

Proof: It follows straight away from Definition 3A:3.3 that we need only show that for some $x > 0$ we have

$$\int_0^\infty \mathrm{em}\left(\left(\tfrac{1}{x(\mathrm{lm}\, t)^{\frac{1}{\beta}}}\right)^\beta\right) dt < 1.$$

Let us assume that $x \geq 1$. Then

$$\begin{aligned}
\int_0^\infty \mathrm{em}\left(\left(\tfrac{1}{x(\mathrm{lm}\, t)^{\frac{1}{\beta}}}\right)^\beta\right) dt &= \int_0^\infty \mathrm{em}\left(\tfrac{1}{x^\beta \,\mathrm{lm}\, t}\right) dt \\
&= \int_0^{e^{1-x^\beta}} \exp\left(\tfrac{1+\log\frac{1}{t}}{x^\beta} - 1\right) dt \\
&\quad + \int_{e^{1-x^\beta}}^1 \exp\left(1 - \tfrac{x^\beta}{1+\log\frac{1}{t}}\right) dt \\
&\quad + \int_1^\infty \exp\left(1 - x^\beta(1+\log t)\right) dt \\
&\leq e^{(1/x^\beta)-1} \int_0^{e^{1-x^\beta}} t^{-1/x^\beta}\, dt + (1-e^{1-x^\beta})e^{1-x^\beta} \\
&\quad + e\int_1^\infty (et)^{-x^\beta}\, dt \\
&\leq e^{(2/x^\beta)-1-x^\beta} \frac{x^\beta}{x^\beta-1} + e^{1-x^\beta} + e^{2-x^\beta}\frac{1}{x^\beta-1},
\end{aligned}$$

and this tends to 0 as $x \to \infty$. □

Theorem 2.7. *(Bennett and Rudnick) Let $p < 0 < \infty$, and $\alpha \in \mathbf{R}$.*

i) There is a number $C < \infty$, depending on p and α, such that

$$\|\cdot\|_{T^p(\operatorname{lm} T)^\alpha} \overset{C}{\approx} \|\cdot\|_{T^p(\operatorname{lm} T)^\alpha, p}\,.$$

ii) There is a number $C < \infty$, depending on p only, such that

$$\|\cdot\|_{\operatorname{em}(T^p)} \overset{C}{\approx} \|\cdot\|_{\operatorname{em}(T^p),\infty}\,.$$

Proof of i): First we consider the case when $\alpha > 0$. Let $H = T(\operatorname{lm} T^{\frac{1}{p}})^\alpha$. Then H is 1-quasi Orlicz, and so by Theorem 2.1, we have for some $C < \infty$ that

$$\|\cdot\|_{T^p(\operatorname{lm} T)^\alpha, H\circ T^p} \le C\,\|\cdot\|_{T^p(\operatorname{lm} T)^\alpha, T^p}\,.$$

Furthermore, $\frac{H(t)}{t}$ is increasing with t, and $H^* \equiv \operatorname{em}(T^{\frac{1}{\alpha}})$. Hence, by Theorem 2.4 and Lemma 2.6, we have for some $C < \infty$ that

$$\|\cdot\|_{T^p(\operatorname{lm} T)^\alpha, T^p} \le C\,\|\cdot\|_{T^p(\operatorname{lm} T)^\alpha, H\circ T^p}\,.$$

Finally, as $\|\cdot\|_{T^p(\operatorname{lm} T)^\alpha, H\circ T^p} = \|\cdot\|_{T^p(\operatorname{lm} T)^\alpha}$, we are done.

The case $\alpha < 0$ follows by a similar argument, this time using the function H such that $H \circ T^p(\operatorname{lm} T)^\alpha = T^p$. $\square$

Proof of ii): This follows immediately from Proposition 3A:3.12, Theorem 2.5 and Lemma 2.6. $\square$

Obviously, we could prove a great many more inequalities of this form than we have given here.

3) A Result about Orlicz Spaces

Here we prove Proposition 1A:2.3(ii).

Proposition 3.1. *(See also [K–R Thm.8.1] and [L–T1 4.a.5].) Let F and G be admissible functions such that F is dilatory, and $0 \le A < \infty$, $0 < B \le \infty$ with $A \le B$. Then there is a number $C < \infty$, depending on $D_{\frac{1}{2}}(F)$ only, such that if $\tilde{F}^{-1}(t) \le \tilde{G}^{-1}(t)$ for $A \le t \le B$, then for all $f \in L_0(I_{A,B})$ we have $\|f\|_F \le C\,\|f\|_G$.*

Proof: Let $f \in L_0(I_{A,B})$ be such that $\|f\|_G < 1$, so that

$$\int_{I_{A,B}} G(|f(t)|)\,d\lambda(t) \le 1.$$

First we show that $\sup_{t\in I_{A,B}} G(|f(t)|) \le \frac{1}{A}$, and so values of $\tilde{F}^{-1}(t)$ and $\tilde{G}^{-1}(t)$ for $t < A$ are immaterial. This is obvious if $A = 0$; otherwise the above integral becomes the sum

$$\sum_{I_{A,B}} AG(|f(t)|) \le 1,$$

and we are done.

Now, if $G(|f(t)|) \ge \frac{1}{B}$, then $F(|f(t)|) \le G(|f(t)|)$, and if $G(|f(t)|) \le \frac{1}{B}$, then $F(|f(t)|) \le \frac{1}{B}$. Let

$$J = \{\, t \in I_{A,B} : G(|f(t)|) \le \tfrac{1}{B}\}.$$

Then

$$\int_{I_{A,B}\setminus J} F(|f(t)|)\,d\lambda(t) \le \int_{I_{A,B}} G(|f(t)|)\,d\lambda(t) \le 1,$$

and

$$\int_J F(|f(t)|)\,d\lambda(t) \le \tfrac{1}{B}\lambda(I_{A,B}) \le 1.$$

Therefore

$$\int_{I_{A,B}} F(|f(t)|)\,d\lambda(t) \le 2,$$

and so $\|f\|_F \le D_{\frac{1}{2}}(F)$. □

Chapter 3C — Boyd Indices of Generalized Lorentz Spaces

1) Introduction

In this chapter, I present preliminary work on an important question about the generalized Lorentz spaces.

Question 1.1. *Let X be a normal quasi-normed r.i. space on $I_{0,\infty}$, and F be a normal dilatory admissible function. Is $L_{F,X}$ dilatory?*

(We know by Proposition 3A:5.5 that this is so if Φ_X satisfies the Δ_2-condition.)

We will concentrate most of our investigation on the case when $X = L_G$, where G is a normal dilatory admissible function. We will also consider the following questions.

Question 1.2. *Let X be a normal quasi-normed r.i. space on $I_{0,\infty}$, and F be a normal dilatory admissible function. Is it true that $p_{L_{F,X}} = \sup\{\, p : F$ is p-quasi Orlicz$\}$?*

Question 1.3. *Let X be a normal quasi-normed r.i. space on $I_{0,\infty}$, and F be a normal dilatory admissible function. Is it true that $q_{L_{F,X}} = \inf\{\, q : F \circ T^{\frac{1}{q}}$ is equivalent to a concave function$\}$?*

Positive answers to the last two questions have many applications. We do not go into this, but the reader is referred to [B–S Ch.3] and [L–T2 2.b]. For general X, the answer to the last two questions is negative, for we have the following result due to T. Shimogaki.

Theorem 1.4. *(See [Sh].) There is a 1-convex r.i. space X such that $\Phi_X = T^2$, but $p_X = 1$ and $q_X = \infty$.*

In considering all these questions, it is sufficient to look only at $L_{1,X}$, for we have the following result.

Proposition 1.5. *Let X be a normal quasi-normed r.i. space on $I_{0,\infty}$, and F be a normal dilatory admissible function.*

i) If $L_{1,X}$ is dilatory, then so is $L_{F,X}$.
ii) If $p_{L_{1,X}} = 1$, then $p_{L_{F,X}} = \sup\{\, p : F$ is p-quasi Orlicz$\}$.
iii) If $q_{L_{1,X}} = 1$, then $q_{L_{F,X}} = \inf\{\, q : F \circ T^{\frac{1}{q}}$ is equivalent to a concave function$\}$.

Proof: First we show that $p_{L_{F,X}} \le \sup\{\, p : F$ is p-quasi Orlicz$\}$. For if $0 < \rho \le 1$, then

$$\begin{aligned} D_\rho(L_{L,F}) &= \sup\left\{ \frac{\|D_\rho(f)\|_{F,X}}{\|f\|_{F,X}} : 0 < \|f\|_{F,X} < \infty \right\} \\ &\ge \sup\left\{ \frac{\left\|D_\rho(\chi_{[0,t]})\right\|_{F,X}}{\left\|\chi_{[0,t]}\right\|_{F,X}} : 0 < t < \infty \right\} \\ &= \sup\left\{ \frac{\tilde F^{-1}(\rho t)}{\tilde F^{-1}(t)} : 0 < t < \infty \right\} \\ &= \sup\left\{ \frac{F^{-1}(\rho t)}{F^{-1}(t)} : 0 < t < \infty \right\} \\ &= D_\rho(F). \end{aligned}$$

The result follows.

Now we note that

$$\begin{aligned} D_\rho(L_{F,X}) &= \sup\left\{ \frac{\|D_\rho(f)\|_{F,X}}{\|f\|_{F,X}} : 0 < \|f\|_{F,X} < \infty \right\} \\ &= \sup\left\{ \frac{\left\|f^* \circ d_\rho \circ \tilde F\right\|_{1,X}}{\left\|f^* \circ \tilde F\right\|_{1,X}} : 0 < \|f\|_{F,X} < \infty \right\}. \end{aligned}$$

However $d_\rho \circ \tilde{F} \le \tilde{F} \circ d_{D_\rho(F)}$, and hence the above is bounded by

$$= \sup \left\{ \frac{\left\| f^* \circ \tilde{F} \circ d_\rho \right\|_{1,X}}{\left\| f^* \circ \tilde{F} \right\|_{1,X}} : 0 < \|f\|_{F,X} < \infty \right\},$$

and this is bounded by $D_{D_\rho(F)}(L_{1,X})$.

Hence we have (i). Furthermore, it is now easy to deduce (ii), for suppose that F is p-quasi Orlicz, so that there is a $C < \infty$ such that $D_\rho(F) < C\rho^{-\frac{1}{p}}$ for $0 < \rho \le 1$. If $p_{L_{F,X}} = 1$, then for each $\epsilon > 0$ there is a $C_\epsilon < \infty$ such that $D_\rho(L_{F,X}) \le C_\epsilon \rho^{1+\epsilon}$ for $0 < \rho \le 1$. Hence

$$D_\rho(L_{F,X}) \le D_{D_\rho(F)}(L_{1,X}) \le C_\epsilon (C\rho^{-\frac{1}{p}})^{1+\epsilon},$$

and the result follows.

The proof of (iii) is entirely similar (note that, by a similar argument to that presented in the proof of Proposition 3A:2.7, $F \circ T^{\frac{1}{q}}$ is equivalent to a concave function if and only if there is a number $C < \infty$ such that $D_\rho(F) \le C\rho^{-\frac{1}{q}}$ for all $1 \le \rho < \infty$.) $\square$

2) The Boyd Indices of $L_{1,G}$

In this section we attempt to find the Boyd indices of $L_{1,G}$ where G is a normal dilatory admissible function. We do not solve this problem, but we do indicate that this problem is difficult. We try to tackle this problem by finding sufficient conditions on G to calculate the Boyd indices. But first we give an example to show that we cannot rely on $D_\rho(L_{1,G})$ being proportional to ρ^{-1}.

Proposition 2.1. *Suppose that $0 < p, q < \infty$. Let G be the normal dilatory admissible function given by*

$$G(t) = \begin{cases} t^p & \text{if } t \leq 1 \\ t^q & \text{if } t \geq 1. \end{cases}$$

Then there is a number $C < \infty$, depending on p and q only, such that $D_\rho(L_{1,G}) \geq \rho^{-1}(\operatorname{lm}\rho^{-1})^{\frac{1}{p}-\frac{1}{q}}$ for $0 < \rho < \infty$.

Sketch Proof: If $0 < \rho \leq 1$, then let $f \in L_0(I_{0,\infty})$ be defined by

$$f(t) = \begin{cases} \rho^{-1} & \text{if } t \leq \rho \\ \frac{1}{t} & \text{if } \rho < t \leq 1 \\ 0 & \text{if } 1 < t. \end{cases}$$

Then

$$\|f\|_{1,G} \approx \|f\|_{1,q} \approx \left(\frac{1}{q}\operatorname{lm}\rho^{-1}\right)^{\frac{1}{q}},$$

whereas

$$\|D_\rho(f)\|_{1,G} \approx \rho^{-1}\left(1 \vee \left(\frac{1}{q}\operatorname{lm}\rho^{-1}\right)^{\frac{1}{q}}\right),$$

and hence $D_\rho(L_{1,G}) \geq \frac{q^{\frac{1}{q}}}{p^{\frac{1}{p}}}\rho^{-1}(\operatorname{lm}\rho)^{\frac{1}{p}-\frac{1}{q}}$, as desired.

The argument for $1 \leq \rho < \infty$ is very similar. $\square$

2.1) Equivalent Quantities Describing the Boyd Indices

Now we define another quantity that is approximately greater than $D_\rho(L_{1,G})$.

Definition 2.2. *Let G be a normal admissible function, and $0 < \rho < \infty$. Define $E_\rho(G)$ to be the infimum over all E such that there is a function $u\colon I_{0,\infty} \to \mathbf{R}_+$ with $\int_0^\infty u(t)\,dt \le 1$, and*

$$\frac{G^{-1}\left(\frac{u(t)}{\tilde{G}'(t)}\right)}{G^{-1}\left(\frac{u(t)}{\rho\tilde{G}'(\rho t)}\right)} \le E \quad \text{for all } 0 < t < \infty.$$

Theorem 2.3. *Let G be a normal dilatory admissible function. Then there is a number $C < \infty$, depending on $D_{\frac{1}{2}}(G)$ only, such that for all $0 < \rho < \infty$ we have that $D_\rho(L_{1,G}) \le C\,E_\rho(G)$.*

Proof: First we note that

$$\|D_\rho f\|_{1,G} = \|f^*\|_{(G_t^\rho)}\,,$$

where $G_t^\rho(u) = \rho\tilde{G}'(\rho t)G(u)$. If we could find a family of admissible functions $(\,H_t : t \in I_{0,\infty})$ such that for some $C_1 < \infty$ we had $(G_t^\rho)^{-1} \overset{C_1}{\approx} H_t^{-1} \cdot (G_t^1)^{-1}$, then we could say by something like Theorem 3B:1.3 that there was a number $C_2 < \infty$ such that $\|D_\rho f\|_{1,G} \le C_2\,\|1\|_{(H_t)}\,\|f\|_{1,G}$. We produce a rigorous version of this argument, and show that $D_\rho(L_{1,G}) \le 2\big(D_{\frac{1}{2}}(G)\big)^2 E_\rho(G)$, as follows.

Define the functions $(\,L_t : t \in I_{0,\infty})$ by

$$L_t(u) = \frac{(G_t^\rho)^{-1}(u)}{(G_t^1)^{-1}(u)} \quad \text{for } u \in [0,\infty].$$

It is easy to see that $L_t(u) \cdot (G_t^1)^{-1}(v) \le (G_t^\rho)^{-1}(u+v)$ for $u, v \in [0,\infty]$. Now, by the definition of $E_\rho(G)$, there is a function $u\colon I_{0,\infty} \to \mathbf{R}_+$ such that $\int_0^\infty u(t)\,dt \le 1$, and

$$L_t(u(t)) \ge \frac{1}{2E_\rho(G)}.$$

Suppose that $\|f\|_{1,G} < 1$, so that if $v(t) = G_t^1\big(f^*(t)\big)$ then $\int_0^\infty v(t)\,dt \le 1$. Then

$$\begin{aligned}
\int_0^\infty G_t^\rho\left(\frac{f^*(t)}{2\big(D_{\frac{1}{2}}(G)\big)^2 \cdot E_\rho(G)}\right) dt &\le \tfrac{1}{4}\int_0^\infty G_t^\rho\Big(L\big(u(t)\big) \cdot (G_t^1)^{-1}\big(v(t)\big)\Big)\,dt \\
&\le \tfrac{1}{4}\int_0^\infty \big(u(t) + v(t)\big)\,dt \\
&\le \tfrac{1}{2},
\end{aligned}$$

and the result follows. □

Thus Questions 1.1, 1.2 and 1.3 have positive answers if the following do.

Question 2.4. *Let G be a normal dilatory admissible function. Is there a $0 < \rho < 1$ such that that $E_\rho(G)$ is finite?*

Question 2.5. *Let G be a normal dilatory admissible function. Is it true that for all $p < 1$ there is a number $C < \infty$ such that for all $0 < \rho < 1$ we have $E_\rho(G) \le C\rho^{-\frac{1}{p}}$?*

Question 2.6. *Let G be a normal dilatory admissible function. Is it true that for all $q > 1$ there is a number $C < \infty$ such that for all $1 < \rho < \infty$ we have $E_\rho(G) \le C\rho^{-\frac{1}{q}}$?*

These are somewhat difficult conjectures to check. However, if G satisfies the Δ_2-condition, then we can write these conjectures in terms of sequences rather than functions. Obviously Question 1.1 then has a positive answer, and it is not hard to see that Question 2.4 also has a positive answer. However we still have the Boyd indices to contend with.

The technique is to see that the problem is simplified by considering only the values that G takes at 2^n $(n \in \mathbf{Z})$.

Definition 2.7. *Suppose G is a normal admissible function. Define the function $\alpha_G\colon \mathbf{Z} \to \mathbf{Z}$ to be such that $2^{\alpha_G(n)} \le G(2^n) < 2^{\alpha_G(n)+1}$.*

Definition 2.8. *If $\alpha\colon \mathbf{Z} \to \mathbf{Z}$ is an increasing function, define $\alpha^{-1}\colon \mathbf{Z} \to \mathbf{Z}$ to be*

$$\alpha^{-1}(n) = \sup\{\, m : m \le \alpha(n)\}.$$

Proposition 2.9. *Let G be a normal dilatory admissible function that satisfies the Δ_2-condition. Then there is a number $C < \infty$ such that*

$$\left|\mathrm{Id}_{\mathbf{Z}} - \alpha \circ \alpha^{-1}\right| \text{ and } \left|\mathrm{Id}_{\mathbf{Z}} - \alpha^{-1} \circ \alpha\right| \le C.$$

Proof: As G is dilatory and satisfies the Δ_2-condition, there are numbers $C_1 < \infty$ and $0 < p \le q < \infty$ such that for all $0 < t < \infty$ and $u \ge 1$ we have

$$C_1^{-1}\, u^p\, G(t) \le G(ut) \le C_1\, u^q\, G(t).$$

Hence there is a number $C_2 < \infty$ such that for all $n \in \mathbf{Z}$ and $m \in \mathbf{N}$ we have

$$-C_2 + \alpha_G(n) + pm \le \alpha_G(n+m) \le C_2 + \alpha_G(n) + qm.$$

It is now easy to see the result. ☐

Definition 2.10. *Let G be a normal admissible function, and $r \in \mathbf{Z}$. Define $F_r(G)$ to be the infimum over all F such that there is a function $\upsilon\colon \mathbf{Z} \to \mathbf{Z}$, with $\sum_{n=-\infty}^{\infty} 2^{\upsilon(n)} \le 1$, and*

$$\alpha_G^{-1}\big(\upsilon(n) + \alpha_G(-n)\big) - \alpha_G^{-1}\big(\upsilon(n) + \alpha_G(-n-r)\big) \le F \quad \text{for all } n \in \mathbf{Z}.$$

Theorem 2.11. *Let G be a normal dilatory admissible functionthat satisfies the Δ_2-condition. Then there is a number $C < \infty$, depending on G only, such that for all $r \in \mathbf{Z}$ we have*

$$E_{2^r}(G) \le C\, 2^{F_r(G)}.$$

Proof: Choose $\upsilon\colon \mathbf{Z} \to \mathbf{Z}$ with $\sum_{n=-\infty}^{\infty} 2^{\upsilon(n)} \le 1$, and

$$\alpha_G^{-1}\big(\upsilon(n) + \alpha_G(-n)\big) - \alpha_G^{-1}\big(\upsilon(n) + \alpha_G(-n-r)\big) \le 1 + F_r(G).$$

Let $u\colon I_{0,\infty} \to \mathbf{R}_+$ be defined to be

$$u\big|_{[2^n,2^{n+1})} = 2^{\upsilon(n)-n},$$

so that $\int_0^\infty u(t)\,dt = \sum_{n=-\infty}^{\infty} 2^{\upsilon(n)} \le 1$. By Lemma 3B:2.3, there is a number $C_1 < \infty$ such that

$$\frac{G^{-1}\left(\frac{u(t)}{\tilde G'(t)}\right)}{G^{-1}\left(\frac{u(t)}{\tilde G'(2^r t)}\right)} \le \frac{G^{-1}\left(C_1 \frac{t u(t)}{\tilde G(t)}\right)}{G^{-1}\left(C_1^{-1} \frac{t u(t)}{\tilde G(C_1 2^r t)}\right)},$$

and as G satisfies the Δ_2-condition, we have for some $C_2 < \infty$ that this is

$$\le \frac{G^{-1}\left(C_1 \frac{t u(t)}{\tilde G(t)}\right)}{G^{-1}\left(C_2^{-1} \frac{t u(t)}{\tilde G(2^r t)}\right)}.$$

Now, if $t \in [2^n, 2^{n+1})$, then $u(t) = 2^{\upsilon(n)-n}$, and $\tilde G(t) \overset{2}{\approx} 2^{-\alpha_G(-n)}$. Hence the above is

$$\le \frac{G^{-1}\left(2C_1\, 2^{1+\upsilon(n)-\alpha_G(-n)}\right)}{G^{-1}\left(\frac{1}{2} C_2^{-1}\, 2^{\upsilon(n)-\alpha_G(-n-r)}\right)}.$$

As G is dilatory and satisfies the Δ_2-condition, we have for some $C_3 < \infty$ that the above is

$$\le C_3 \frac{G^{-1}\left(2^{\upsilon(n)-\alpha_G(-n)}\right)}{G^{-1}\left(2^{\upsilon(n)-\alpha_G(-n-r)}\right)}.$$

Now, by Proposition 2.9, we have for some $C_4 < \infty$ that $G^{-1}(2^n) \overset{C_4}{\approx} 2^{\alpha_G^{-1}(n)}$, and hence the above is

$$\begin{aligned} &\le C_3\, C_4^2\, 2^{\left(\alpha_G^{-1}\left(\upsilon(n)-\alpha_G(-n)\right)\alpha_G^{-1}\left(\upsilon(n)-\alpha_G(-n-r)\right)\right)} \\ &\le C_3\, C_4^2\, 2^{1+F_r(G)}, \end{aligned}$$

as desired. ☐

Thus Questions 2.5 and 2.6 become the following.

Question 2.12. *Let G be a normal dilatory function satisfying the Δ_2-condition. Is it true that for all $p < 1$ there is a number $C < \infty$ such that for all $r \in \mathbf{Z}$ with $r < 0$, we have $F_r(G) \leq C - \frac{1}{p} r$.*

Question 2.13. *Let G be a normal dilatory function satisfying the Δ_2-condition. Is it true that for all $q > 1$ there is a number $C < \infty$ such that for all $r \in \mathbf{Z}$ with $r > 0$, we have $F_r(G) \leq C - \frac{1}{q} r$.*

I have not made much headway in answering these questions. Part of the difficulty lies in that, as a consequence of the work of Section 3B:2, simpler normal dilatory admissible functions, G, that one might consider, satisfy $L_{1,G} = L_{1,p}$ for some $0 < p \leq \infty$.

To recap, we are interested in characterizing the Boyd indices of the $L_{F,G}$ spaces purely in terms of F, where F and G are normal dilatory admissible functions. We reduce this to showing that the Boyd indices of $L_{1,G}$ are 1, and this is tackled by considering new quantities that depend on G, which we call $E_\rho(G)$ and $F_r(G)$ $(0 < \rho < \infty,\ r \in \mathbf{Z})$.

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