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GLOBAL REGULARITY OF THE NAVIER-STOKES EQUATION ON THIN THREE-DIMENSIONAL DOMAINS WITH PERIODIC BOUNDARY CONDITIONS

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Abstract

This paper gives another version of results due to Raugel and Sell, and similar results due to Moise, Temam and Ziane, that state the following: the solution of the Navier-Stokes equation on a thin three-dimensional domain with periodic boundary conditions has global regularity, as long as there is some control on the size of the initial data and the forcing term, where the control is larger than that obtainable via "small data" estimates. The approach taken is to consider the three-dimensional equation as a perturbation of the equation when the vector field does not depend upon the coordinate in the thin direction.

§1. Introduction

The celebrated Navier-Stokes equation is concerned with the velocity vector u on a domain Ω, describing the flow of an incompressible fluid. A famous unsolved problem is the following: if Ω is a nice enough 3-dimensional domain, and if the initial data is smooth, and the forcing term is uniformly smooth in time, then does it follow that the solution is smooth for all time? What is known is that a weak solution exists, although it is not known if that solution is unique. For details, we refer the reader to a number of books and papers, including [CF], [DG] and [T].

For the 2-dimensional problem, the solution is known. A heuristic approach to solving the 3-dimensional problem is as follows: if the solution becomes less smooth, then since we are dealing with an incompressible fluid, the complicated activity is going to get squashed into flat sheets, and one might hope that the solutions on the flat sheets can be somehow dealt with by the 2-dimensional case. Certainly, this "flattening" is observed in numerical and physical experiments.

For this reason, it would seem that in order to get some handle on the real problem, that it might be important to understand what happens to the solution to the Navier-Stokes equation on thin domains, that is, domains of the form $M \times [0, \epsilon]$, where M is some 2-dimensional manifold, and $\epsilon > 0$ is a small number. This is what Raugel and Sell did in a series of papers [RS1], [RS2], [RS3], as did Avrin [A], Temam and Ziane [TZ1], [TZ2], Moise, Temam and Ziane [MTZ], and Iftimie [I1], [I2].

In this paper, we consider the same situation considered by Raugel and Sell in [RS2], or by Moise, Temam and Ziane in [MTZ], or by Iftimie in [I1], [I2]. Let

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$\Omega_\epsilon = [0,l_1]\times[0,l_2]\times[0,\epsilon]$, where $l_1 \ge l_2$ are some positive numbers, and $\epsilon \in (0, l_2/4)$ is some small number. Let us consider vector fields $u : \Omega_\epsilon \to \mathbf{R}^3$ satisfying the periodic boundary conditions

$$u(0,y,z) = u(l_1,y,z), \quad u(x,0,z) = u(x,l_2,z), \quad u(x,y,0) = u(x,y,\epsilon). \tag{1}$$

Given any appropriately smooth (where being in L_2 is smooth enough) vector field u satisfying these boundary conditions, we may split it into its divergence-free part Lu and its gradient part. Thus L is the so called Leray projection.

The Navier-Stokes equation considered in this paper is the equation for a function $u(t)$, $t \ge 0$, taking values in the space of 3-dimensional divergence-free vector fields on Ω_ϵ satisfying the boundary conditions (1). The equation is

$$\partial_t u = \nu \Delta u - L(u\cdot\nabla u) + L(f), \tag{2}$$

where ν is a positive constant (the viscosity), and f is a function of t taking values in the 3-dimensional vector fields satisfying the boundary conditions (the forcing term). For simplicity let us suppose $f = L(f)$.

In fact, to simplify our presentation, it will make sense to assume that $\int_{\Omega_\epsilon} f\,dV = 0$, and that $\int_{\Omega_\epsilon} u(0)\,dV = 0$. In that case it is not hard to see that we have that $\int_{\Omega_\epsilon} u(t)\,dV = 0$ for all $t \ge 0$. This assumption does not really affect the solution very much, for suppose that we do not have this assumption. Given any function g on Ω_ϵ, let $\bar g$ denote its mean value $(l_1 l_2 \epsilon)^{-1}\int_{\Omega_\epsilon} g\,dV$. Let $(\xi_t,\eta_t,\zeta_t) = \int_0^t \bar u(0) + \bar f(s)\,ds$. Then replacing $u(x,y,z,t)$ by $u(x+\xi_t, y+\eta_t, z+\zeta_t, t) - \bar u$ gives us a solution to the Navier-Stokes equation in which f is replaced by $f - \bar f$, and in which $\bar u = 0$.

It is known that in order to show global regularity of u that it is sufficient to show that u stays within the Sobolev space H^1, that is, the space in which one derivative is in L_2. Furthermore, once this is established, it also follows that the solution is unique. (See [CF] or [DG].) We will also include results concerning the Sobolev space H^2, the space in which two derivatives are in L_2.

Throughout this paper, we will use the letter c to denote a positive constant (typically larger than one), whose value will change with each occurrence. Only in certain places (such as in Lemmas 3 and 5) will we use subscripts on the c's to identify them, so as to avoid confusion.

Theorem 1. *Let u satisfy the Navier-Stokes equation (2) with periodic boundary conditions (1), and $\int_{\Omega_\epsilon} u\,dV = 0$. Let*

$$U = \|u(0)\|_{H^1}, \quad F = \sup_t \|f(t)\|_2, \quad M = \max\left\{U, \frac{l_1}{\nu}F\right\}.$$

If $M \le c^{-1}\dfrac{\nu l_2^{1/2}}{l_1}$, then there exists a solution with the following properties. First, $u(t)$ is in H^1 for all $t \ge 0$, with

$$\|u(t)\|_{H^1} \le c \max\left\{M, \frac{l_1^{3/2}}{\nu l_2^{1/2}}\epsilon^{-1/2}M^2\right\}.$$

If $t \geq c\,\dfrac{l_1^2}{\nu}$, *then*

$$\|u(t)\|_{H^1} \leq c \max\left\{\frac{l_1}{\nu} F, \frac{l_1^{7/2}}{\nu^3 l_2^{1/2}}\,\epsilon^{-1/2} F^2\right\}.$$

Second, $u(t)$ *is in* H^2 *for almost all* $t \geq 0$, *and indeed*

$$\int_0^t \|u(s)\|_{H^2}^2\,ds < \infty$$

for all $0 \leq t < \infty$.

Using a rescaling argument, it may be shown that is is sufficient to show Theorem 1 in the case that l_1, l_2 and ν are of order one, say, that these quantities all lie between 1/2 and 2. We will demonstrate this at the end of the proof of Theorem 1. So for the remainder of this discussion, let us focus on this case. Then in effect, $M = \max\{U, F\}$, and Theorem 1 gives global regularity in the case that $M \leq c^{-1}$.

This result is not obtainable by the usual "small data" arguments, because these would only give regularity in the case that $M \leq c^{-1}\epsilon^{1/2}$. (The $\epsilon^{1/2}$ comes from the fact that we are calculating an L_2 norm on a domain whose total measure is of order ϵ.)

We will obtain Theorem 1 by considering it as a perturbation of the Navier-Stokes equation in which neither u nor f depend upon the third coordinate of Ω_ϵ (the 'z' coordinate). This approach was also taken by Iftimie [I1], [I2], and by Moise, Temam and Ziane [MTZ]. Note that the following result effectively does not depend upon ϵ, in that if the result is obtained for one value of ϵ, it is then automatically obtained for all others by a rescaling argument.

Theorem 2. *Let* u *satisfy the Navier-Stokes equation (2) with periodic boundary conditions (1), and* $\int_{\Omega_\epsilon} u\,dV = 0$, *and neither* u *nor* f *depend upon the third coordinate of* Ω_ϵ. *Let*

$$U = \|u(0)\|_{H^1}, \quad F = \sup_t \|f(t)\|_2, \quad M = \max\left\{U, \frac{l_1}{\nu}\,F\right\}.$$

Then there exists a solution with the following properties. First, $u(t)$ *is in* H^1 *for all* $t \geq 0$, *with*

$$\|u(t)\|_{H^1} \leq c \max\left\{M, \frac{l_1^{3/2}}{\nu l_2^{1/2}}\,\epsilon^{-1/2} M^2\right\}.$$

If $t \geq c\,\dfrac{l_1^2}{\nu}$, *then*

$$\|u(t)\|_{H^1} \leq c \max\left\{\frac{l_1}{\nu} F, \frac{l_1^{7/2}}{\nu^3 l_2^{1/2}}\,\epsilon^{-1/2} F^2\right\}.$$

Second, $u(t)$ *is in* H^2 *for almost all* $t \geq 0$, *and indeed*

$$\int_0^t \|u(s)\|_{H^2}^2\,ds < \infty$$

for all $0 \leq t < \infty$.

In order to compare Theorem 1 with the results in the literature, let us define the following projections. Let

$$Pu(x,y,z) = \frac{1}{\epsilon}\int_0^\epsilon u(x,y,\zeta)\,d\zeta,$$

and let $Qu = u - Pu$. As we stated above, we will discuss these results only in the case when l_1, l_2 and ν are of order one. Then the result of Raugel and Sell [RS2] gives global H^1 boundedness of the solution as long as

$$\begin{aligned}
\|Pu(0)\|_{H^1} &\leq \epsilon^{\frac{7}{24}+\delta_1}(\log(1/\epsilon))^{\delta_2} \\
\|Qu(0)\|_{H^1} &\leq \epsilon^{-\frac{5}{48}+\delta_3}(\log(1/\epsilon))^{\delta_4} \\
\sup_t \|Pf(t)\|_2 &\leq \epsilon^{\frac{7}{24}+\delta_5}(\log(1/\epsilon))^{\delta_6} \\
\sup_t \|Qf(t)\|_2 &\leq \epsilon^{-\frac{1}{2}+\delta_7}(\log(1/\epsilon))^{\delta_8},
\end{aligned}$$

where δ_i $(1 \leq i \leq 8)$ are positive numbers.

The result of Moise, Temam and Ziane [MTZ] gives H^1 boundedness for $t \in [0, T(\epsilon)]$, where $\lim_{\epsilon\to 0} T(\epsilon) = \infty$, and also an integral condition on the H^2 norm, as long as

$$\begin{aligned}
\|Pu(0)\|_{H^1} &\leq \alpha(\epsilon)\epsilon^{\frac{1}{6}+\delta} \\
\|Qu(0)\|_{H^1} &\leq \alpha(\epsilon)\epsilon^{-\frac{1}{6}+\delta} \\
\sup_t \|Pf(t)\|_2 &\leq \alpha(\epsilon)\epsilon^{\frac{1}{6}+\delta} \\
\sup_t \|Qf(t)\|_2 &\leq \alpha(\epsilon)\epsilon^{-\frac{1}{6}+\delta}
\end{aligned}$$

where δ is a positive number, and $\lim_{\epsilon\to 0} \alpha(\epsilon) = 0$.

Iftimie [I1], [I2] gets global existence results under conditions that use certain 'anisotropic' Sobolev spaces. For example, one case of his Theorem 4.1 gives global existence under the condition that the forcing term f is zero, and

$$\|Qu\|_{H^{1/2}} \exp(c\epsilon^{-1}\|Pu\|_2^2) \leq c^{-1}.$$

Even though his conclusions are slightly different, it is instructive to see how his hypothesis relates to that of this paper. Indeed, his condition is true if

$$\begin{aligned}
\|Pu\|_{H^1} &\leq c_\delta^{-1}\epsilon^{1/2}\sqrt{\log(1/\epsilon)} \\
\|Qu\|_{H^1} &\leq c^{-1}\epsilon^{-1/2+\delta},
\end{aligned}$$

where c_δ depends upon $\delta > 0$.

Before proceeding further, let us set our notation, and also state the "tools of the trade," that is, the standard results that many people in this area use. (See [CF] or [DG] — in particular [DG] considers the case of periodic boundary conditions, and many of the following calculations may be found there.)

As well as the operators P and Q given above, let us define the operators R and S: $R((u_1, u_2, u_3)) = (u_1, u_2, 0)$, and $Su = u - Ru = (0, 0, u_3)$. Split $u = v + w = r + s + w$, where $v = Pu$, $w = Qu$, $r = Rv$, $s = Sv$. Since r and s do not depend upon z (where we label the coordinates of Ω_ϵ by x, y and z), it is clear that r, s, v and w are all divergent free vector fields.

As a notational device, I will write

$$\|D^\alpha u\|_p$$

for the Sobolev space α derivatives in L_p. (Thus D might represent the operator $\sqrt{-\Delta}$.) We have the Sobolev inequalities: if f is a mean zero function on $[0, l_1] \times [0, l_2]$, $1 < p < q < \infty$, and $\alpha > 0$, then

$$\|f\|_q \le c\|D^\alpha f\|_p,$$

where $\frac{1}{q} + \frac{\alpha}{2} = \frac{1}{p}$, and c depends upon p and q, as well as l_1 and l_2. Thus, if f is a mean zero function on Ω_ϵ, then

$$\|Pf\|_q \le c\epsilon^{-\alpha/2}\|D^\alpha Pf\|_p.$$

In fact, the only condition under which we will use this inequality is in the case $p = 4$ and $q = 2$, when $\alpha = 1/2$. For this case, we will include an elementary proof in the Appendix.

We have the interpolation inequality: if f is a mean zero function, α_0, α_1 are real numbers, and $0 \le \theta \le 1$, then

$$\|D^{\alpha_\theta} f\|_2 \le c\|D^{\alpha_0} f\|_2^{1-\theta}\|D^{\alpha_1} f\|_2^{\theta},$$

where $\alpha_\theta = (1 - \theta)\alpha_0 + \theta\alpha_1$. This inequality is easy to show using Parseval's identity and Hölder's inequality. (See for example the proofs of Lemmas 4 or 6 for the statement of Parseval's identity.)

We also have the Poincaré inequalities: if f is a mean zero function on Ω_ϵ, with periodic boundary conditions, and $\alpha > 0$, then

$$\|f\|_2 \le c\|D^\alpha f\|_2,$$

where c depends upon α as well as l_1 and l_2. Again, this is easy to show using Parseval's identity.

If u is a divergence-free vector field on the domain Ω_ϵ with periodic boundary conditions, and if f and g are two other functions on Ω_ϵ with periodic boundary conditions, sufficiently smooth so that the following integrals make sense, then by integration by parts we get

$$\int_{\Omega_\epsilon} f(u \cdot \nabla g)\, dV = -\int_{\Omega_\epsilon} g(u \cdot \nabla f)\, dV,$$

and so

$$\int_{\Omega_\epsilon} f(u\cdot\nabla f)\,dV = 0.$$

If r is a two-dimensional, divergence-free vector field on the domain $[0,l_1]\times[0,l_2]$ with periodic boundary conditions that is sufficiently smooth, then we have the "enstrophy miracle:"

$$\int_{[0,l_1]\times[0,l_2]} \Delta r\cdot(r\cdot\nabla r)\,dA = 0.$$

This is obtained as follows. First, integrating by parts, we see that the left hand side is equal to

$$\begin{aligned} &-\int_{[0,l_1]\times[0,l_2]} \partial_x r\cdot(\partial_x r\cdot\nabla r)\,dA - \int_{[0,l_1]\times[0,l_2]} \partial_x r\cdot(r\cdot\nabla\partial_x r)\,dA \\ &\quad -\int_{[0,l_1]\times[0,l_2]} \partial_y r\cdot(\partial_y r\cdot\nabla r)\,dA - \int_{[0,l_1]\times[0,l_2]} \partial_y r\cdot(r\cdot\nabla\partial_y r)\,dA. \end{aligned}$$

(Here, as in the rest of the paper, ∂_x, ∂_y and ∂_z represent partial differentiation with respect to x, y and z respectively, that is, the first, second and third coordinates respectively.) We see that the second and fourth terms are zero. Expanding and collecting the first and third terms, and remembering that r is divergence-free, we see that they also total to zero.

Navier-Stokes for Flows Independent of z

Let us start with the proof of Theorem 2, the case when $w = Qu = 0$, that is, when $u = Pu = v$. We will prove Theorem 2 in the case that the quantities l_1, l_2 and ν all lie between $1/2$ and 2. The general result may be obtained as shown at the end of the proof of Theorem 1.

First we need to find a solution to the Navier-Stokes equation. This is done using so called Galerkin solutions. Let S_n denote the projection that takes a function f on Ω_ϵ onto the nth partial Fourier series. (Quite how we index this sum is not important, as long as S_n converges formally to the identity.) Then we consider the solution u_n to the problem

$$\partial_t u_n = \nu\Delta u_n - S_n L(u_n\cdot\nabla u_n) + S_n L(f),$$

with $u_n(0) = S_n u(0)$ for which $u = S_n u$. It is a well known argument to show that this equation, essentially an ODE, has a global solution, and that the solutions u_n converge weakly to some function u. This is the so called weak solution to the Navier-Stokes equation. In that case, for any appropriate norm $\|\cdot\|$, we will have that $\|u\| \le \liminf_{n\to\infty}\|u_n\|$.

Thus, in order to prove our theorem, it is sufficient to prove it for the Galerkin solutions. This is what we shall do. However, carrying the symbol S_n throughout the proof could be a little cumbersome, and so for this reason, we will replace L by $S_n L$, and suppose that both f and $u(0)$ lie in the range of S_n. (We will also suppose that f lies in the range of L.)

So let us proceed. Notice that the Navier-Stokes equation becomes

$$\partial_t v = \nu \Delta v - L(r \cdot \nabla v) + f,$$

since $v \cdot \nabla v = r \cdot \nabla v$, because $\partial_z v = 0$. If we apply the R and S operator to this, we get the following pair of equations:

$$\begin{aligned} \partial_t r &= \nu \Delta r - L(r \cdot \nabla r) + Rf \\ \partial_t s &= \nu \Delta s - L(r \cdot \nabla s) + Sf. \end{aligned} \tag{3}$$

The first equation is merely the 2-dimensional Navier-Stokes for the flow r. The second equation essentially says that the 1-dimensional quantity s is being pushed around by the 2-dimensional flow r (and indeed in the second equation, the operator L acts as the identity).

Let us write

$$\phi = \|Dr\|_2, \quad \psi = \|Ds\|_2, \quad \tilde{\phi} = \|D^2 r\|_2, \quad \tilde{\psi} = \|D^2 s\|_2, \quad \theta = \|u\|_2.$$

Poincaré's inequality tells us immediately that $\phi \le c\tilde{\phi}$, $\psi \le c\tilde{\psi}$, and $\theta^2 \le c(\phi^2 + \psi^2)$.

The process for comprehending ϕ and $\tilde{\phi}$ is well known. Start with the first equation in (3), dot product both sides with $-\Delta r$, integrate over Ω_ϵ, use the self-adjointness of L, apply some integration by parts, use the Cauchy-Schwartz inequality, and use the "enstrophy miracle," to get

$$\frac{1}{2}\partial_t \|Dr\|_2^2 \le -\nu \|D^2 r\|_2^2 + \|D^2 r\|_2 F.$$

Use the inequality $ab \le a^2 + b^2$ to get that $\|D^2 r\|_2 F \le (\nu/2)\|D^2 r\|_2^2 + (2/\nu)F^2$. Thus we have that

$$\partial_t \phi^2 \le -c^{-1}\tilde{\phi}^2 + cF^2.$$

This differential inequality is easy to solve, but before we do so, let us first understand ψ and $\tilde{\psi}$. Take the second equation from (3), dot product both sides with $-\Delta s$, integrate over Ω_ϵ, and work as before. But in this case, the "enstrophy miracle" does not work — there is a term:

$$\int_{\Omega_\epsilon} \Delta s \cdot (r \cdot \nabla s)\, dV.$$

To get a grip on this term, see that it splits into

$$\int_{\Omega_\epsilon} \partial_x^2 s \cdot (r \cdot \nabla s)\, dV$$

plus another term with ∂_y^2 in place of ∂_x^2. Integrate by parts to get

$$-\int_{\Omega_\epsilon} \partial_x s \cdot (\partial_x r \cdot \nabla s)\, dV - \int_{\Omega_\epsilon} \partial_x s \cdot (r \cdot \nabla \partial_x s)\, dV.$$

The second term is zero. For the first term, we may use Hölder's inequality and the Sobolev inequality to bound it by:

$$\begin{aligned}\|Dr\|_2\|Ds\|_4^2 &\le c\epsilon^{-1/2}\|Dr\|_2\|D^{3/2}s\|_2^2 \\ &\le c\epsilon^{-1/2}\|Dr\|_2\|Ds\|_2\|D^2s\|_2 \\ &\le \frac{1}{2}\|D^2s\|_2^2 + c\epsilon^{-1}\|Dr\|_2^2\|Ds\|_2^2,\end{aligned}$$

where in the last step we use the inequality $ab \le a^2 + b^2$.

Putting this all together, we get a differential inequality:

$$\partial_t\psi^2 \le -c^{-1}\tilde{\psi}^2 + c\epsilon^{-1}\phi^2\psi^2 + cF^2.$$

We will also require a differential equation for θ: take the Navier-Stokes equation, dot product both sides with u, integrate over Ω_ϵ, and do the usual stuff, to get

$$\frac{1}{2}\partial_t\|u\|_2^2 \le -\nu\|Du\|_2^2 + \|Du\|_2 F.$$

Since $\|Du\|_2 F \le (\nu/2)\|Du\|_2^2 + (2/\nu)F^2$, we get

$$\frac{1}{2}\partial_t\|u\|_2^2 \le -\frac{\nu}{2}\|Du\|_2^2 + \frac{2}{\nu}F^2,$$

that is

$$\partial_t\theta^2 \le -c^{-1}(\phi^2 + \psi^2) + cF^2.$$

Then Theorem 2 will be established once we have obtained the following result.

Lemma 3. *Let U, F and T be positive numbers, where T may be infinity. Let ϕ, ψ, $\tilde{\phi}$, $\tilde{\psi}$, θ be positive differentiable functions of t. Suppose that for some positive constants c_i $(1 \le i \le 10)$ we have*

$$\phi(0) \le U \tag{4.1}$$
$$\psi(0) \le U \tag{4.2}$$
$$\theta^2 \le c_1(\phi^2 + \psi^2) \tag{4.3}$$
$$\phi \le c_2\tilde{\phi} \tag{4.4}$$
$$\psi \le c_3\tilde{\psi} \tag{4.5}$$
$$\partial_t\phi^2 \le -c_4^{-1}\tilde{\phi}^2 + c_5F^2 \tag{4.6}$$
$$\partial_t\psi^2 \le -c_6^{-1}\tilde{\psi}^2 + c_7\epsilon^{-1}\phi^2\psi^2 + c_8F^2 \tag{4.7}$$
$$\partial_t\theta^2 \le -c_9^{-1}(\phi^2 + \psi^2) + c_{10}F^2 \tag{4.8}$$

for $0 \le t < T$. Let $M = \max\{U, F\}$. Then there exist positive constants c_i $(11 \le i \le 17)$, depending only upon c_i $(1 \le i \le 10)$ such that we have the inequalities

$$\theta^2 \le c_{11}(F^2 + (U^2 - F^2)e^{-c_{12}^{-1}t}) \tag{4.9}$$
$$\phi^2 \le c_{13}(F^2 + (U^2 - F^2)e^{-c_{14}^{-1}t}) \tag{4.10}$$
$$\psi \le c_{15}\max\{\epsilon^{-1/2}M^2, M\}, \tag{4.11}$$

and if $t \ge c_{16}$ then

$$\psi \le c_{17} \max\{\epsilon^{-1/2} F^2, F\}. \tag{4.12}$$

for $0 \le t < T$. *Furthermore, we have that*

$$\int_0^t (\phi(s)^2 + \psi(s)^2)\, ds < \infty \tag{4.13}$$

for $0 \le t < T$.

Proof: Inequalities (4.9) and (4.10) are easy to obtain from combining (4.1) and (4.4) with (4.6), and (4.1), (4.2) and (4.3) with (4.8), by using Gronwall's inequality.

Let us obtain (4.11). From (4.10), we see that there is a positive number c_{18} such that $\phi \le c_{18} M$. Combining this with inequalities (4.5), (4.7) and (4.8), we see that for some positive constants c_{19} and c_{20} that

$$\partial_t \psi^2 + c_{19} \psi^2 \le c_{20} \epsilon^{-1} M^2 (F^2 - \partial_t \theta^2) + c_{20} F^2.$$

Multiply both sides by $e^{c_{19} t}$ and integrate from 0 to t to get that

$$e^{c_{19} t} \psi^2 - \psi(0)^2 \le c_{20} \int_0^t \epsilon^{-1} M^2 e^{c_{19} s} (F^2 - \partial_s \theta(s)^2) + F^2 e^{c_{19} s}\, ds,$$

which, by evaluating the integrals, and integrating by parts, is less than or equal to

$$c_{20} c_{19}^{-1} e^{c_{19} t} (\epsilon^{-1} M^2 F^2 + F^2) + c_{20} \epsilon^{-1} M^2 (\theta(0)^2 - e^{c_{19} t} \theta^2) + c_{20} c_{19} \epsilon^{-1} M^2 \int_0^t e^{c_{19} s} \theta(s)^2\, ds.$$

Now, from (4.9), we see that there is a positive constant c_{21} such that $\theta \le c_{21} M$. Hence

$$\begin{aligned} e^{c_{19} t} \psi^2 - \psi(0)^2 &\le c_{20} c_{19}^{-1} e^{c_{19} t} (\epsilon^{-1} M^2 F^2 + F^2) + c_{20} c_{21}^2 \epsilon^{-1} M^4 + c_{20} c_{19} \epsilon^{-1} M^2 c_{21}^2 \int_0^t e^{c_{19} s}\, ds \\ &\le c_{20} c_{19}^{-1} e^{c_{19} t} (\epsilon^{-1} M^2 F^2 + F^2) + c_{20} c_{21}^2 \epsilon^{-1} M^4 + c_{20} \epsilon^{-1} M^2 c_{21}^2 e^{c_{19} t}. \end{aligned}$$

Hence

$$\psi^2 \le e^{-c_{19} t} U + c_{20} c_{19}^{-1} (\epsilon^{-1} M^2 F^2 + F^2) + c_{20} c_{21}^2 \epsilon^{-1} e^{-c_{19} t} M^4 + c_{20} c_{21}^2 \epsilon^{-1} M^2,$$

and from here it is easy to obtain (4.11).

To obtain (4.12) is similar. We see that there are positive numbers c_{22}, c_{23} and c_{24} such that if $t \ge c_{22}$ then $\phi \le c_{23} F$ and $\theta \le c_{24} F$. Apply the above argument, except integrate from c_{22} to t instead of from 0 to t.

Finally, (4.13) may be obtained by integrating (4.6) and (4.7), and using (4.9), (4.10) and (4.11). Q.E.D.

Proof of Theorem 1

The proof of Theorem 1 will follow the same lines as the proof of Theorem 2, with some additional work for dealing with the $w = Qu$ part. Let us start by assuming that the quantities l_1, l_2 and ν all lie between 1/2 and 2.

We need a couple of Poincaré/Sobolev type inequalities on Ω_ϵ.

Lemma 4. *Let $w = Qu$, then*

$$\|w\|_\infty \leq c\epsilon^{1/2}\|D^2 w\|_2$$
$$\|w\|_4 \leq c\epsilon^{1/4}\|Dw\|_2.$$

Proof: Let $\hat{w}$ denote the Fourier coefficients of w:

$$\hat{w}(m,n,p) = (l_1 l_2 \epsilon)^{-1} \int_{\Omega_\epsilon} w(x,y,z) \exp(-2\pi i(mx/l_1 + ny/l_2 + pz/\epsilon)\, dx\, dy\, dz,$$

where m, n and p are integers. Then the function w can be reconstructed using the Fourier series

$$w(x,y,z) = \sum_{m,n,p} \hat{w}(m,n,p) \exp(2\pi i(mx/l_1 + ny/l_2 + pz/\epsilon).$$

We recall Parseval's identity:

$$\|w\|_2^2 = l_1 l_2 \epsilon \sum_{m,n,p} |\hat{w}(m,n,p)|^2,$$

and the Hausdorff-Young inequality: if $2 \leq p \leq \infty$, and $p' = p/(p-1)$, then

$$\|w\|_p \leq (l_1 l_2 \epsilon)^{1/p} \left(\sum_{m,n,p} |\hat{w}(m,n,p)|^{p'} \right)^{1/p'}.$$

Since $w = Qu$, we have that $\hat{w}(m,n,p) = 0$ if $p = 0$.

Now for any real number α, we have that

$$\widehat{D^\alpha w}(m,n,p) = (-2\pi i)^\alpha (m^2/l_1^2 + n^2/l_2^2 + p^2/\epsilon^2)^{\alpha/2} \hat{w}(m,n,p),$$

and thus by Parseval's identity

$$\|D^\alpha w\|_2 = (2\pi)^\alpha (l_1 l_2 \epsilon)^{1/2} \left(\sum_{m,n,p} (m^2/l_1^2 + n^2/l_2^2 + p^2/\epsilon^2)^\alpha |\hat{w}(m,n,p)|^2 \right)^{1/2}.$$

Let us start with showing the first inequality. Apply Cauchy-Schwartz to get

$$\begin{aligned}\|w\|_\infty &\leq \sum_{m,n,p} |\hat{w}(m,n,p)| \\ &\leq \left(\sum_{m,n,p} I_{p\neq 0} (m^2/l_1^2 + n^2/l_2^2 + p^2/\epsilon^2)^{-2} \right)^{1/2} \times \\ &\quad \left(\sum_{m,n,p} (m^2/l_1^2 + n^2/l_2^2 + p^2/\epsilon^2)^2 |\hat{w}(m,n,p)|^2 \right)^{1/2}.\end{aligned}$$

By approximating sums by integrals, and using other elementary inequalities, we see that

$$\begin{aligned}\sum_{m,n,p} I_{p\neq 0}(m^2/l_1^2+n^2/l_2^2+p^2/\epsilon^2)^{-2} &\leq c\sum_{m,n,p} I_{p\neq 0}\frac{1}{\max\{m^4,n^4,p^4/\epsilon^4\}}\\ &\leq c\sum_{n,p} I_{p\neq 0}\frac{1}{\max\{n^3,p^3/\epsilon^3\}}\\ &\leq c\sum_{p} I_{p\neq 0}\frac{1}{p^2/\epsilon^2}\\ &\leq c\epsilon^2.\end{aligned}$$

Hence we obtain the first inequality.

The second inequality has a similar proof: start by using Hölder's inequality to get

$$\begin{aligned}\|w\|_4 &\leq (l_1l_2\epsilon)^{1/4}\left(\sum_{m,n,p}|\hat{w}(m,n,p)|^{4/3}\right)^{3/4}\\ &\leq (l_1l_2\epsilon)^{1/4}\left(\sum_{m,n,p} I_{p\neq 0}(m^2/l_1^2+n^2/l_2^2+p^2/\epsilon^2)^{-2}\right)^{1/4}\times\\ &\qquad\left(\sum_{m,n,p}(m^2/l_1^2+n^2/l_2^2+p^2/\epsilon^2)|\hat{w}(m,n,p)|^2\right)^{1/2},\end{aligned}$$

and proceed as with the proof of the first inequality. Q.E.D.

Proof of Theorem 1: As in the proof of Theorem 2, we argue that we work with the Galerkin approximations. We will obtain differential inequalities for the following quantities:

$$\phi=\sqrt{\|Dr\|_2^2+\|Dw\|_2^2},\quad \psi=\sqrt{\|Ds\|_2^2+\|Dw\|_2^2},$$
$$\tilde{\phi}=\sqrt{\|D^2r\|_2^2+\|D^2w\|_2^2},\quad \tilde{\psi}=\sqrt{\|D^2s\|_2^2+\|D^2w\|_2^2},$$
$$\chi=\|D^2w\|_2,\quad \theta=\|u\|_2.$$

Poincaré's inequality tells us immediately that $\phi\leq c\tilde{\phi}$, $\psi\leq c\tilde{\psi}$, and $\theta^2\leq c(\phi^2+\psi^2)$.

Let us start with the Navier-Stokes equation, and apply the operator P. Note that if f and g are functions on Ω_ϵ, then $P((Pf)(Qg))=0$. Thus, we obtain

$$\partial_t v=\nu\Delta v-LP(v\cdot\nabla v)-LP(w\cdot\nabla w)+Pf=\nu\Delta v-LP(r\cdot\nabla v)-LP(w\cdot\nabla w)+Pf.$$

Now apply R to both sides:

$$\partial_t r=\nu\Delta r-LP(r\cdot\nabla r)-LP(w\cdot\nabla Rw)+PRf.$$

Finally, take the dot product of both sides with $-\Delta r$, and integrate over Ω_ϵ, and do all the usual stuff. A lot of the terms work in exactly the same way that they did in the previous section. The only term that we did not deal with is the following:

$$\int_{\Omega_\epsilon} \Delta r \cdot (w \cdot \nabla R w)\, dV.$$

This splits into a term:

$$\int_{\Omega_\epsilon} \partial_x^2 r \cdot (w \cdot \nabla R w)\, dV,$$

and a similar one with ∂_y^2 in place of ∂_x^2. The bounds for the second term will be as for the first, so let us only deal with the first. Integrate by parts to get

$$-\int_{\Omega_\epsilon} \partial_x r \cdot (\partial_x w \cdot \nabla R w)\, dV - \int_{\Omega_\epsilon} \partial_x r \cdot (w \cdot \nabla \partial_x R w)\, dV.$$

The first term is bounded by

$$c\|Dv\|_2 \|Dw\|_4^2,$$

and the second by

$$c\|Dv\|_2 \|D^2 w\|_2 \|w\|_\infty,$$

Combining all this with Lemma 4, we get

$$\partial_t \|Dr\|_2^2 \le -c^{-1}\|D^2 r\|_2^2 + c\epsilon^{1/2}\|Dv\|_2 \|D^2 w\|_2^2 + cF^2. \tag{5}$$

The work for s is practically identical, and we get

$$\partial_t \|Ds\|_2^2 \le -c^{-1}\|D^2 s\|_2^2 + c\epsilon^{-1}\|Dr\|_2^2\|Ds\|_2^2 + c\epsilon^{1/2}\|Dv\|_2 \|D^2 w\|_2^2 + cF^2. \tag{6}$$

We also need to establish an equation for w. Take the Navier-Stokes equation and apply Q. Note that if f and g are two functions on Ω_ϵ, then $Q((Pf)(Pg)) = 0$. Thus we get

$$\partial_t w = \nu \Delta w - LQ(w \cdot \nabla v) - LQ(v \cdot \nabla w) - LQ(w \cdot \nabla w) + Qf.$$

Take the dot product with $-\Delta w$, and integrate over Ω_ϵ, doing all the stuff as before. Let us see what happens to the non-linear terms, only bothering with the $\partial_x^2 w$ part of Δw, knowing that the other parts will give the same estimates.

First we get

$$\int_{\Omega_\epsilon} \partial_x^2 w \cdot (w \cdot \nabla v)\, dV = -\int_{\Omega_\epsilon} \partial_x w \cdot (\partial_x w \cdot \nabla v)\, dV - \int_{\Omega_\epsilon} \partial_x w \cdot (w \cdot \nabla \partial_x v)\, dV.$$

The first term is bounded in absolute value by

$$\|Dw\|_4^2 \|Dv\|_2 \le c\epsilon^{1/2}\|D^2 w\|_2^2 \|Dv\|_2.$$

The second term is equal to

$$\int_{\Omega_\epsilon} \partial_x v \cdot (w \cdot \nabla \partial_x w)\, dV,$$

which is bounded in absolute value by

$$\|w\|_\infty \|D^2 w\|_2 \|Dv\|_2 \le c\epsilon^{1/2} \|D^2 w\|_2^2 \|Dv\|_2.$$

Next, we have

$$\int_{\Omega_\epsilon} \partial_x^2 w \cdot (v \cdot \nabla w)\, dV = -\int_{\Omega_\epsilon} \partial_x w \cdot (\partial_x v \cdot \nabla w)\, dV - \int_{\Omega_\epsilon} \partial_x w \cdot (v \cdot \nabla \partial_x w)\, dV.$$

The first term is bounded in absolute value by

$$\|Dw\|_4^2 \|Dv\|_2 \le c\epsilon^{1/2} \|D^2 w\|_2^2 \|Dv\|_2,$$

and the second term is zero.

Finally we have

$$\int_{\Omega_\epsilon} \partial_x^2 w \cdot (w \cdot \nabla w)\, dV = -\int_{\Omega_\epsilon} \partial_x w \cdot (\partial_x w \cdot \nabla w)\, dV - \int_{\Omega_\epsilon} \partial_x w \cdot (w \cdot \nabla \partial_x w)\, dV.$$

The first term is bounded in absolute value by

$$\|Dw\|_4^2 \|Dw\|_2 \le c\epsilon^{1/2} \|D^2 w\|_2^2 \|Dw\|_2,$$

and the second term is zero.

So, doing all the same stuff as above, we get

$$\partial_t \|Dw\|_2^2 \le -\|D^2 w\|_2^2 + c\epsilon^{1/2} \|Dv\|_2 \|D^2 w\|_2^2 + c\epsilon^{1/2} \|Dw\|_2 \|D^2 w\|_2^2 + cF^2. \tag{7}$$

If we add equations (5) and (7), and also (6) and (7), (and apply liberally the inequalities $\sqrt{a^2+b^2} \le a+b \le \sqrt{2}\sqrt{a^2+b^2}$ for positive a and b) we get the two differential inequalities:

$$\begin{aligned}
\partial_t \phi^2 &\le -c^{-1}\tilde{\phi}^2 - c^{-1}\chi^2 + c\epsilon^{1/2}(\phi+\psi)\chi^2 + cF^2 \\
\partial_t \psi^2 &\le -c^{-1}\tilde{\psi}^2 - c^{-1}\chi^2 + c\epsilon^{-1}\phi^2\psi^2 + c\epsilon^{1/2}(\phi+\psi)\chi^2 + cF^2.
\end{aligned}$$

In addition, arguing as in the previous section, we get the differential inequality

$$\partial_t \theta^2 \le -c^{-1}(\phi^2 + \psi^2) + cF^2$$

Thus the theorem will be established when we have proved the following lemma.

Lemma 5. *Let U and F be positive numbers. Let ϕ, ψ, $\tilde\phi$, $\tilde\psi$, θ be positive differentiable functions of t. Suppose that for some positive constants c_i ($1 \le i \le 10$, $18 \le i \le 19$) we have*

$$\phi(0) \le U \tag{8.1}$$
$$\psi(0) \le U \tag{8.2}$$
$$\theta^2 \le c_1(\phi^2 + \psi^2) \tag{8.3}$$
$$\phi \le c_2\tilde\phi \tag{8.4}$$
$$\psi \le c_3\tilde\psi \tag{8.5}$$
$$\partial_t\phi^2 \le (-c_{18}^{-1} + c_{19}\epsilon^{-1/2}(\phi+\psi))\chi - c_4^{-1}\tilde\phi^2 + c_5F^2 \tag{8.6}$$
$$\partial_t\psi^2 \le (-c_{18}^{-1} + c_{19}\epsilon^{-1/2}(\phi+\psi))\chi - c_6^{-1}\tilde\psi^2 + c_7\epsilon^{-1}\phi^2\psi^2 + c_8F^2 \tag{8.7}$$
$$\partial_t\theta^2 \le -c_9^{-1}(\phi^2+\psi^2) + c_{10}F^2 \tag{8.8}$$

for $0 \le t < \infty$. Let $M = \max\{U, F\}$. Then there exist positive constants c_i ($11 \le i \le 17$, $i = 20$), depending only upon c_i ($1 \le i \le 10$) such that if $M \le c_{20}^{-1}$, then we have the inequalities

$$\theta^2 \le c_{11}(F^2 + (U^2 - F^2)e^{-c_{12}^{-1}t}) \tag{8.9}$$
$$\phi^2 \le c_{13}(F^2 + (U^2 - F^2)e^{-c_{14}^{-1}t}) \tag{8.10}$$
$$\psi \le c_{15}\max\{\epsilon^{-1/2}M^2, M\}, \tag{8.11}$$

and if $t \ge c_{16}$ then

$$\psi \le c_{17}\max\{\epsilon^{-1/2}F^2, F\}. \tag{8.12}$$

for $0 \le t < \infty$. Furthermore, we have that

$$\int_0^t (\phi(s)^2 + \psi(s)^2)\, ds < \infty \tag{8.13}$$

for $0 \le t < \infty$.

Proof: Let $(c_i)_{11\le i\le 17}$ depend upon $(c_i)_{1\le i\le 10}$ as in Lemma 3. Let us suppose that $U, F \le c_{20}^{-1}$, where c_{20} will be chosen momentarily. Let

$$T = \inf\{t > 0 : c_{19}\epsilon^{1/2}(\phi+\psi) > c_{18}^{-1}\}.$$

Suppose for a contradiction that $T < \infty$. But then for $t \in [0, T]$, the quantities ϕ, ψ, $\tilde\phi$, $\tilde\psi$ and θ satisfy the hypothesis of Lemma 3. But then by the conclusion of Lemma 3, we know that for some constant $c_{21} > 0$ that

$$\phi + \psi \le c_{21}\epsilon^{-1/2}c_{20}^{-2}.$$

Setting c_{20} small enough, we see then that for $t \in [0, T]$ that we have

$$c_{19}\epsilon^{1/2}(\phi+\psi) \le c_{18}^{-1}/2,$$

and thus that there is a neighborhood of T such that

$$c_{19}\epsilon^{1/2}(\phi+\psi) \le c_{18}^{-1},$$

contradicting the definition of T.

Thus $T = \infty$, and thus the functions ϕ, ψ, $\tilde{\phi}$, $\tilde{\psi}$ and θ satisfy the hypothesis of Lemma 3, and the result follows.

Now let us relax the restriction that l_1, l_2 and ν all lie between 1/2 and 2. Let n be the integer part of $\frac{l_1}{l_2}$, and define new vector fields $\tilde{u}$ and $\tilde{f}$ on $[0,1] \times \left[0, \frac{nl_2}{l_1}\right] \times \left[0, \frac{\epsilon}{l_1}\right]$ according to the formulae

$$\begin{aligned}
\tilde{u}(x_1, x_2, x_3, t) &= \frac{l_1}{\nu} \quad u\left(l_1 x_1, (l_1 x_2 \bmod l_2), l_1 x_3, \frac{l_1^2}{\nu} t\right), \\
\tilde{f}(x_1, x_2, x_3, t) &= \frac{l_1^3}{\nu^2} \quad f\left(l_1 x_1, (l_1 x_2 \bmod l_2), l_1 x_3, \frac{l_1^2}{\nu} t\right).
\end{aligned}$$

Then it may be easily verified that these satisfy the equation

$$\partial_t \tilde{u} = \Delta \tilde{u} - L(\tilde{u} \cdot \nabla \tilde{u}) + L(\tilde{f}),$$

that is, one may apply the version of Theorem 1 that we already have to the functions $\tilde{u}$ and $\tilde{f}$. Obtaining the more general version of Theorem 1 is then merely a question of interpreting what it says about $\tilde{u}$ and $\tilde{f}$, taking into account the following identities:

$$\begin{aligned}
\|f\|_2 &= \frac{\nu^2}{n^{1/2} l_1^{3/2}} \|\tilde{f}\|_2, \\
\|u\|_{H^1} &= \frac{\nu}{n^{1/2} l_1^{1/2}} \|\tilde{u}\|_{H^1}.
\end{aligned}$$

Q.E.D.

Appendix: the Sobolev Inequality

The following result is essentially part of the literature. For example, in [S] this result is found for functions on Euclidean space. However, for our special case, we are able to provide an elementary proof (a proof motivated by Littlewood-Paley theory).

Lemma 6. *Let f be a function on $[0,l_1]\times[0,l_2]$ satisfying periodic boundary conditions, that is mean zero. Then there is a positive constant c, depending only upon l_1 and l_2, such that*

$$\|f\|_4 \le c\|D^{1/2}f\|_2.$$

Proof: For each $r=(r_1,r_2)$ a pair of integers, write $|r| = \sqrt{r_1^2/l_1^2 + r_2^2/l_2^2}$. Define the Fourier coefficients of f:

$$\hat f_r = (l_1 l_2)^{-1}\int_0^{l_1}\int_0^{l_2} f(x,y)\exp(-2\pi i(r_1x/l_1 + r_2y/l_2))\,dy\,dx.$$

The original function can be reconstructed using the Fourier series

$$f(x,y) = \sum_r \hat f_r \exp(2\pi i(r_1x/l_1 + r_2y/l_2)),$$

and we have Parseval's identity

$$\|f\|_2^2 = (l_1l_2)\sum_r |\hat f_r|^2.$$

We see that

$$\widehat{D^{1/2}f}_r = \sqrt{-2\pi i}(r_1^2/l_1^2 + r_2^2/l_2^2)^{1/4}\hat f_r,$$

and so by Parseval's equality we see that

$$\|D^{1/2}f\|_2^2 = 2\pi(l_1l_2)\sum_r (r_1^2/l_1^2 + r_2^2/l_2^2)^{1/2}|f_r|^2.$$

For m a non-negative integer, set

$$A_m = \left(\sum_{2^m\le|r|<2^{m+1}} |f_r|^2\right)^{1/2}.$$

Notice then that

$$\sum_{m=0}^\infty 2^m A_m^2 \le c\|D^{1/2}f\|_2^2.$$

Now,

$$\|f\|_4^4 = \int_0^{l_1}\int_0^{l_2} f(x,y)^2\overline{f(x,y)^2}\,dy\,dx,$$

and expanding this using the Fourier series, we obtain that

$$\begin{aligned}\|f\|_4^4 &= (l_1 l_2) \sum_{r^{(1)}+r^{(2)}-r^{(3)}-r^{(4)}=0} \hat{f}_{r^{(1)}} \hat{f}_{r^{(2)}} \overline{\hat{f}_{r^{(3)}} \hat{f}_{r^{(4)}}} \\ &\le 24(l_1 l_2) \sum_{\substack{r^{(1)}+r^{(2)}-r^{(3)}-r^{(4)}=0 \\ |r^{(1)}|\le|r^{(2)}|\le|r^{(3)}|\le|r^{(4)}|}} |\hat{f}_{r^{(1)}} \hat{f}_{r^{(2)}} \hat{f}_{r^{(3)}} \hat{f}_{r^{(4)}}|\end{aligned}$$

which in turn is bounded above by

$$\begin{aligned}24(l_1 l_2) \sum_{0\le m_1\le m_2\le m_3} \sum_{2^{m_1}\le|r^{(1)}|<2^{m_1+1}} |\hat{f}_{r^{(1)}}| \sum_{2^{m_2}\le|r^{(2)}|<2^{m_2+1}} |\hat{f}_{r^{(2)}}| \\ \sum_{2^{m_3}\le|r^{(3)}|<2^{m_3+1}} I_{|r^{(1)}+r^{(2)}-r^{(3)}|\ge|r^{(3)}|}\, |\hat{f}_{r^{(3)}} \hat{f}_{r^{(1)}+r^{(2)}-r^{(3)}}|.\end{aligned}$$

In bounding this quantity, let us start by looking at the inner sum:

$$\sum_{2^{m_3}\le|r^{(3)}|<2^{m_3+1}} I_{|r^{(1)}+r^{(2)}-r^{(3)}|\ge|r^{(3)}|}\, |\hat{f}_{r^{(3)}} \hat{f}_{r^{(1)}+r^{(2)}-r^{(3)}}|.$$

Using the Cauchy-Schwartz formula, this can be bounded above by

$$\Big(\sum_{2^{m_3}\le|r^{(3)}|<2^{m_3+1}} |\hat{f}_{r^{(3)}}|^2 \Big)^{1/2} \Big(\sum_{2^{m_3}\le|r^{(3)}|<2^{m_3+1}} I_{|r^{(1)}+r^{(2)}-r^{(3)}|\ge|r^{(3)}|}\, |\hat{f}_{r^{(1)}+r^{(2)}-r^{(3)}}|^2 \Big)^{1/2}.$$

The first term in this product is A_{m_3}. As for the second term, since $m_1 \le m_2 \le m_3$, it follows that $|r^{(1)} + r^{(2)} - r^{(3)}| \le 3 \cdot 2^{m_3}$, and hence the second term is bounded by $(A_{m_3}^2 + A_{m_3+1}^2 + A_{m_3+2}^2)^{1/2}$. Thus, the above quantity can be bounded above by $A_{m_3}^2 + A_{m_3+1}^2 + A_{m_3+2}^2$.

Furthermore

$$\sum_{2^m\le|r|<2^{m+1}} |\hat{f}_r| \le \left(\sum_{2^m\le|r|<2^{m+1}} 1 \right)^{1/2} \left(\sum_{2^m\le|r|<2^{m+1}} |\hat{f}_r|^2 \right)^{1/2} \le c 2^m A_m,$$

since the number of points r such that $2^m \le |r| < 2^{m+1}$ is 2^{2m} to within a constant factor.

Thus

$$\|f\|_4^4 \le c \sum_{m_3=0}^{\infty} (A_{m_3}^2 + A_{m_3+1}^2 + A_{m_3+2}^2) \sum_{m_2=0}^{m_3} 2^{m_2} A_{m_2} \sum_{m_1=0}^{m_2} 2^{m_1} A_{m_1}.$$

Now, applying Cauchy-Schwartz, we see that the inner sum obeys the inequalities

$$\sum_{m_1=0}^{m_2} 2^{m_1} A_{m_1} \le \left(\sum_{m_1=0}^{m_2} 2^{m_1} \right)^{1/2} \left(\sum_{m_1=0}^{m_2} 2^{m_1} A_{m_1}^2 \right)^{1/2} \le c 2^{m_2/2} \|D^{1/2} f\|_2.$$

Thus

$$\|f\|_4^4 \le c\|D^{1/2}f\|_2 \sum_{m_3=0}^{\infty} (A_{m_3}^2 + A_{m_3+1}^2 + A_{m_3+2}^2) \sum_{m_2=0}^{m_3} 2^{3m_2/3} A_{m_2}.$$

Applying Cauchy-Schwartz once more in a similar fashion, we get that

$$\sum_{m_2=0}^{m_3} 2^{3m_2/3} A_{m_2} \le c 2^{m_3} \|D^{1/2}f\|_2,$$

and so

$$\|f\|_4^4 \le c\|D^{1/2}f\|_2^2 \sum_{m_3=0}^{\infty} 2^{m_3} (A_{m_3}^2 + A_{m_3+1}^2 + A_{m_3+2}^2),$$

from which we see that $\|f\|_4^4 \le c\|D^{1/2}f\|_2^4$ as required.

Q.E.D.

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On page 3, in both Theorems 1 and 2, the sentence that begins "*If* $t \geq c\,\dfrac{l_1^2}{\nu}$ " should read

Furthermore

$$\limsup_{t\to\infty} \|u(t)\|_{H^1} \leq c \max\left\{ \frac{l_1}{\nu} F, \frac{l_1^{7/2}}{\nu^3 l_2^{1/2}}\, \epsilon^{-1/2} F^2 \right\}.$$

Similarly, for Lemma's 3 and 5 (pages 8/9 and 14 respectively), the phrase that begins "*and if* $t \geq c_{16}$ *then*" and ends "*for* $0 \leq t < T$" (respectively "*for* $0 \leq t < \infty$") should be replaced by

and

$$\limsup_{t\to\infty} \psi \leq c_{17} \max\{\epsilon^{-1/2} F^2, F\}. \tag{4.12}$$

For Lemma 5 the corresponding equation number is (8.12).

The second to last paragraph of the proof of Lemma 3 (page 9) should be changed to

> To obtain (4.12) is similar. We see that given $\epsilon > 0$ there are positive numbers τ, c_{23} and c_{24} (where τ depends upon ϵ) such that if $t \geq \tau$ then $\phi \leq c_{23} F + \epsilon$ and $\theta \leq c_{24} F + \epsilon$. Apply the above argument, except integrate from τ to t instead of from 0 to t.