Tangent Sequences in Orlicz and Rearrangement Invariant Spaces

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Abstract

Let (f_n) and (g_n) be two sequences of random variables adapted to an increasing sequence of σ-algebras $(\mathcal{F}_n)$ such that the conditional distributions of f_n and g_n given $\mathcal{F}_{n-1}$ coincide. Suppose further that the sequence (g_n) is conditionally independent. Then it is known that $\|\sum f_k\|_p \le C\,\|\sum g_k\|_p$, $1 \le p \le \infty$, where the number C is a universal constant. The aim of this paper is to extend this result to certain classes of Orlicz and rearrangement invariant spaces. This paper includes fairly general techniques for obtaining rearrangement invariant inequalities from Orlicz norm inequalities.

1. Introduction

Let $(\mathcal{F}_n)$ be an increasing sequence of σ-algebras on some probability space $(\Omega, \mathcal{F}, P)$. We will assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$. A sequence (f_n) of random variables is called $(\mathcal{F}_n)$-*adapted* if f_n is $\mathcal{F}_n$-measurable for each $n \ge 1$. In the sequel we will simply write 'adapted' if there is no risk of confusion. For any sequence (f_n) of random variables, we will write $f^* = \sup_n |f_n|$ and $f_n^* = \max_{1\le k\le n} |f_k|$. Throughout the paper all equalities or inequalities between random variables are assumed to hold almost surely.

Given a σ-algebra $\mathcal{A} \subset \mathcal{F}$ and an integrable random variable f, we will denote the conditional expectation of f given $\mathcal{A}$ by $E_{\mathcal{A}} f$. If $\mathcal{A} = \mathcal{F}_k$ then we will simply write $E_k f$ for $E_{\mathcal{F}_k} f$.

The *conditional distribution* of a random variable f given $\mathcal{A}$ is denoted by $\mathcal{L}(f|\mathcal{A})$. Thus, $\mathcal{L}(f|\mathcal{A}) = \mathcal{L}(g|\mathcal{A})$ means that for each real number t we have that $P(f > t|\mathcal{A}) = P(g > t|\mathcal{A})$.

The following definition was introduced by Kwapień and Woyczyński in a preprint of their paper [**10**], which was distributed as early as 1986. We refer the reader to their book [**11**] for more information on tangent sequences.

Definition. Let $(\mathcal{F}_n)$ be an increasing sequence of σ-algebras on $(\Omega, \mathcal{F}, P)$.

(a) Two adapted sequences (f_n) and (g_n) of random variables are *tangent* if for each $n \ge 1$ we have

$$\mathcal{L}(f_n|\mathcal{F}_{n-1}) = \mathcal{L}(g_n|\mathcal{F}_{n-1}).$$

(b) An adapted sequence (g_n) of random variables satisfies *condition (CI)* if there exists a σ-algebra $\mathcal{G} \subset \mathcal{F}$ such that for each $n \ge 1$

$$\mathcal{L}(g_n|\mathcal{F}_{n-1}) = \mathcal{L}(g_n|\mathcal{G}),$$

and (g_n) is a sequence of $\mathcal{G}$-conditionally independent random variables.

A sequence (f_n) is *conditionally symmetric* if (f_n) and $(-f_n)$ are tangent sequences of random variables.

Every sequence of random variables (f_n) admits (possibly on an enlarged probability space) a tangent sequence which satisfies condition (CI) (cf. e.g. [**11** p. 104]). Throughout this paper, such a sequence will be denoted by $(\overline{f}_n)$, and will be called a *decoupled version* of (f_n). It is useful to note that the σ-algebra $\mathcal{G}$ can be chosen so that the random variables f_n, $n \ge 1$, are $\mathcal{G}$-measurable.

In this paper, we will be interested in comparing Orlicz and rearrangement invariant norms of sums of tangent sequences. A *rearrangement invariant space* X is a space of random variables f equipped with a complete quasi-norm $\|\cdot\|_X$ such that either the conditions (i), (ii) and (iii) or (i), (ii) and (iii′) below hold:

(i) if $g^\# \le f^\#$ and $f \in X$, then $g \in X$ with $\|g\|_X \le \|f\|_X$;
(ii) if f is simple with finite support then $f \in X$;
(iii) $f_n \in X$ and $f_n \searrow 0$ implies $\|f_n\|_X \searrow 0$
(iii′) $f_n \in X$ and $0 \le f_n \nearrow f$ and $\sup_n \|f_n\|_X < \infty$ imply $f \in X$ with $\|f\|_X = \sup_n \|f_n\|_X$.

Here $f^\#$ denotes the decreasing rearrangement of $|f|$, that is, $f^\#(s) = \sup\{\, t : P(|f| > t) > s\,\}$.

Examples of rearrangement invariant spaces include Orlicz spaces and Lorentz spaces. Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\Phi(0) = 0$, and such that there is a constant $c > 0$ such that $\Phi(ct) \ge 2\Phi(t)$ for all $t \ge 0$. (Functions satisfying the latter condition have been called *dilatory* in [**14**]; let us note that if Φ is convex, and $\Phi(0) = 0$ then this condition is satisfied with $c = 2$.) Given such a function Φ we define the *Orlicz norm* of a random variable f to be

$$\|f\|_\Phi = \inf\{\lambda > 0 : E(\Phi(|f|/\lambda)) \le 1\}\,.$$

We let $L_\Phi = \{f : \|f\|_\Phi < \infty\}$. Note that if Φ is convex, then L_Φ is a normed space. However, we do not wish to restrict ourselves to normed spaces.

Other examples are the Lorentz spaces. Given $0 < p, q \le \infty$, we define the space $L_{p,q}$ to be those random variables f for which the following quantity is finite:

$$\|f\|_{p,q} = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^1 s^{(q/p)-1} f^\#(s)^q\,ds\right)^{1/q} & \text{if } q < \infty \\ \displaystyle\sup_{s>0} s^{1/p} f^\#(s) & \text{if } q = \infty\ . \end{cases}$$

Note that $L_{p,q}$ is not a normed space unless $1 \le q \le p \le \infty$.

Note that the L_p spaces are special cases: $L_p = L_\Phi = L_{p,p}$, where $\Phi(t) = t^p$. We refer the reader to [**13**] for more details about these spaces.

In the present paper we will be interested in the domination of a rearrangement invariant norm of a sum of an arbitrary sequence of adapted random variables by the rearrangement invariant norm of a sum of its decoupled version. It is already known (see [**4**]) that if Φ satisfies the Δ_2-condition, that is, there is a constant $c > 0$ such that

$\Phi(2t) \leq c\,\Phi(t)$ for all $t \geq 0$, then there is a constant C_Φ such that for every adapted sequence (f_n) of random variables one has:

$$\|\sum f_i\|_\Phi \leq C_\Phi \|\sum \overline{f}_i\|_\Phi. \tag{1.1}$$

Building on some special situations considered by Klass [**4**, Theorem 3.1] and Kwapień [**9**], Hitczenko [**6**] began to investigate how the constant C_Φ depends upon Φ. He showed that there is a universal constant $C > 0$ such that

$$\|\sum f_i\|_p \leq C\,\|\sum \overline{f}_i\|_p \qquad 1 \leq p \leq \infty. \tag{1.2}$$

In this present paper, we will show among other things, that inequality (1.1) holds with C_Φ uniformly bounded, at least for certain classes of Orlicz functions. Our first theorem extends a result of Klass who proved (1.1) for randomly stopped sums of independent random variables.

Let us define classes of Orlicz functions. Following Klass [**8**], for $q > 0$ we define the class F_q as the class of all functions Φ such that:
(i) $\Phi : [0,\infty) \to [0,\infty)$, $\Phi(0) = 0$,
(ii) Φ is nondecreasing and continuous, and
(iii) Φ satisfies the growth condition: $\Phi(cx) \leq c^q\Phi(x)$ for all $x \geq 0$, $c \geq 2$.
For $p > 0$, we define the class G_p as the class of all functions Φ such that
(i) $\Phi : [0,\infty) \to [0,\infty)$, $\Phi(0) = 0$,
(ii) Φ is nondecreasing and continuous, and
(iii) Φ satisfies the growth condition: $\Phi(cx) \geq c^p\Phi(x)$ for all $x \geq 0$, $c \geq 2$.

Then we obtain the following results.

Theorem 1.1. *There is a universal constant $C > 0$ such that if $\Phi \in F_q$ for some $q > 0$, then for every adapted sequence (f_n) of random variables one has:*

$$E\Phi\left(\left|\sum f_i\right|\right) \leq C^{1+q} E\Phi\left(\left|\sum \overline{f}_i\right|\right).$$

This inequality had already been obtained by Klass [**8**] in the special case that $f_k = I(\tau \geq k)\xi_k$, where (ξ_k) is a sequence of independent random variables and τ is a stopping time. More precisely, Klass proved his result for Banach space valued random variables (ξ_k) (with absolute value replaced by norm). To discuss Banach space valued random variables one needs to adjust notation; for a random variable Y and a σ-algebra $\mathcal{A}$, we use $\mathcal{L}(Y|\mathcal{A})$ to denote the regular version of the conditional distribution of Y given $\mathcal{A}$, that is, $\mathcal{L}(Y|\mathcal{A}) = \mathcal{L}(Z|\mathcal{A})$ means that for every Borel subset A of the Banach space, we have $P(Y \in A|\mathcal{A}) = P(Z \in A|\mathcal{A})$. Recall that the existence of the regular versions of the conditional distributions is guaranteed, as long as our random variables take values in a separable Banach space. As it turns out, in our generality, the inequality of Theorem 1.1 need not hold (with *any* constant), unless some extra conditions are imposed on the geometry of the underlying Banach space (see e.g. [**3**]). Since it is unclear at this time for which Banach spaces the inequality

$$\left(E\|\sum f_k\|^p\right)^{1/p} \leq c_p\left(E\|\sum \overline{f}_k\|^p\right)^{1/p}$$

holds (even if the constant c_p is allowed to depend on p), we confine our discussion to real valued random variables.

Corollary 1.2. *Given numbers $p_0 > 0$ and $r \geq 1$, there is a constant $c_{p_0,r}$ such that if $p \geq p_0$, and if $\Phi \in G_p \cap F_{rp}$, then*

$$\| \sum f_i \|_\Phi \leq c_{p_0,r} \| \sum \overline{f}_i \|_\Phi .$$

The next step is to extend these results to rearrangement invariant spaces. This will be accomplished through a rather general method of obtaining rearrangement invariant norm inequalities from Orlicz norm inequalities. We believe that this technique will prove useful in other contexts as well. In particular, we would like to mention that this method could be used to deduce martingale inequalities obtained by Johnson and Schechtman [**7**] from the corresponding inequalities for Orlicz functions.

Corresponding to the notions of Orlicz spaces lying in $G_p \cap F_q$, we have the following notion. We say that a rearrangement invariant space is an *interpolation space for* (L_p, L_q) (in short, a (p,q)-*interpolation space*) if there is a constant $c > 0$ such that for every operator $T : L_p \cap L_q \to L_p \cap L_q$ for which $\|T\|_{L_p \to L_p} \leq 1$ and $\|T\|_{L_q \to L_q} \leq 1$ we have that $\|T\|_{X \to X} \leq c$.

However, this notion is not quite what we need. Define

$$K_{p,q}(f,t) = \inf\{ \|f'\|_p + t\,\|f''\|_q : f' + f'' = f^\# \}.$$

We will say that a rearrangement invariant space X is a (p,q)-K-*interpolation space* if there is a constant c such that whenever f and g are such that $K_{p,q}(f,t) \leq K_{p,q}(g,t)$ $(t > 0)$, and $g \in X$, then $f \in X$ and $\|f\|_X \leq c\,\|g\|_X$. The (p,q)-K-*interpolation constant* of X, denoted by $C_{p,q}(X)$, is the infimum of c that work for all functions f and g.

It is quite easily seen that every (p,q)-K-interpolation space is a (p,q)-interpolation space. It is also known that if $1 \leq p, q \leq \infty$, then every normed (p,q)-interpolation space is a (p,q)-K-interpolation space (see [**1**]). We are able to establish the following method for obtaining rearrangement invariant inequalities from Orlicz inequalities.

Theorem 1.3. *Suppose that $\|f\|_\Phi \leq \|g\|_\Phi$ for all $\Phi \in F_q \cap G_p$, where $0 < p < q < \infty$. If X is a (p,q)-K-interpolation space, then $\|f\|_X \leq 2^{2+1/p} C_{p,q}(X)\,\|g\|_X$.*

Corollary 1.4. *Given numbers $p_0 > 0$ and $r \geq 1$, there is a constant $c_{p_0,r}$ such that if $p \geq p_0$, and if X is a (p,pr)-K-interpolation space, then*

$$\| \sum f_i \|_X \leq c_{p_0,r} C_{p,pr}(X) \| \sum \overline{f}_i \|_X .$$

In particular, we are able to obtain the following result for Lorentz spaces. Note that if one is interested in normed Lorentz spaces, then p_0 below can be taken to be 1, and the resulting inequality extends (1.2).

Corollary 1.5. *Given a number $p_0 > 0$, there is a constant c_{p_0} such that if $p, q \geq p_0$, then*

$$\| \sum f_i \|_{p,q} \leq c_{p_0} \| \sum \overline{f}_i \|_{p,q} .$$

2. Inequalities for Orlicz functions

We begin with a proof of Theorem 1.1. Since our proof is based on well understood techniques we will be somewhat sketchy and we refer the reader to [**6**] for details that are not explained here. Throughout this section we let (M_n) be a martingale with difference sequence (Δ_k). Since $F_{q_1} \subset F_{q_2}$ whenever $q_1 \le q_2$, we can assume without loss of generality that $q \ge 1$. Our departing point is the following result which can be found in the just mentioned paper (Theorem 5.1 and the beginning of the proof of Lemma 2.3).

Lemma 2.1. *Let $1 \le q < \infty$, and let (Δ_k) be a conditionally symmetric martingale difference sequence, and $(\overline{\Delta}_k)$ its decoupled version. Set $T_{n,q}(M) = (E|\sum_{k=1}^n \overline{\Delta}_k|^q \big|\mathcal{G})^{1/q}$. Then there exist $\delta_1 > 0$, $\beta > 1 + \delta_1$ and ϵ with $0 < \epsilon \le 1/2$ such that for every $\lambda > 0$ we have*

$$P(M^* \ge \beta\lambda, (T_q^*(M) \vee \Delta^*) < \delta_1\lambda) \le \epsilon^q P(M^* \ge \lambda).$$

From this, we obtain

Lemma 2.2. *Let (Δ_k) be as above, and assume that w_n is a $\mathcal{F}_{n-1}$-measurable random variable such that $|\Delta_n| \le w_n$ for each $n \ge 1$. Set $N_n = \sum_{k=1}^n \overline{\Delta}_k$ $(n \ge 1)$. Suppose that δ_1, β, and ϵ are as in Lemma 2.1. Then, there exist $\delta > 0$, $\delta_2 > 0$ and $0 < \alpha \le 1/2$ such that for every $\lambda > 0$ we have*

$$P(M^* \ge \beta\lambda, N^* < \delta\lambda) \le \epsilon^q P(M^* \ge \lambda) + P(w^* \ge \delta_2\lambda) + (1 - \alpha^q)P(M^* \ge \beta\lambda).$$

Proof: We have that

$$\begin{aligned} P(M^* \ge \beta\lambda, N^* < \delta\lambda) \le & P(M^* \ge \beta\lambda, T_q^*(M) < \delta_1\lambda, w^* < \delta_2\lambda) + P(w^* \ge \delta_2\lambda) \\ & + P(M^* \ge \beta\lambda, T_q^*(M) \ge \delta_1\lambda, w^* < \delta_2\lambda, N^* < \delta\lambda). \end{aligned} \tag{2.1}$$

Suppose $\delta_2 \le \delta_1$. Then, in view of Lemma 2.1, for the first probability on the right-hand side of (2.1) we have that

$$\begin{aligned} P(M^* \ge \beta\lambda, T_q^*(M) < \delta_1\lambda, w^* < \delta_2\lambda) \le & P(M^* \ge \beta\lambda, T_q^*(M) \vee w^* < \delta_1\lambda) \\ \le & \epsilon^q P(M^* \ge \lambda). \end{aligned}$$

It remains to estimate the last probability in (2.1). Since M^*, w^* and $T_q^*(M)$ are $\mathcal{G}$-measurable, by conditioning on $\mathcal{G}$, we see that the last probability in (2.1) is equal to:

$$E\Big\{I(M^* \ge \beta\lambda, T_q^*(M) \ge \delta_1\lambda, w^* < \delta_2\lambda)P(N^* < \delta\lambda|\mathcal{G})\Big\}.$$

By Kolmogorov's converse inequality (see e.g. [**12**, Remark 6.15, p. 161]) for all sequences of independent and symmetric random variables (ξ_k) and for all $t > 0$ we have that

$$P(S^* \ge t) \ge \frac{1}{2^q}\Big(1 - \frac{2^{2q}(t^q + E(\xi^*)^q)}{E(S^*)^q}\Big),$$

where $S_n = \sum_{k=1}^n \xi_k$. Applying this result conditionally on $\mathcal{G}$, we obtain that

$$P(N^* < \delta\lambda|\mathcal{G}) \le 1 - \frac{1}{2^q}\Big(1 - \frac{2^{2q}((\delta\lambda)^q + E_{\mathcal{G}}(\overline{\Delta}^*)^q)}{E_{\mathcal{G}}(N^*)^q}\Big). \tag{2.2}$$

Also, if $w_n < \delta_2\lambda$, then $|\Delta_n| < \delta_2\lambda$, and since w_n is $\mathcal{F}_{n-1}$-measurable, and the conditional distributions of Δ_n and $\overline{\Delta}_n$ coincide, it follows that $|\overline{\Delta}_n| < \delta_2\lambda$. Therefore, on the set

$$\{T_q^*(M) \ge \delta_1\lambda, w^* < \delta_2\lambda\},$$

we have

$$\frac{(\delta\lambda)^q + E_{\mathcal{G}}(\overline{\Delta}^*)^q}{E_{\mathcal{G}}(N^*)^q} \le \frac{(\delta\lambda)^q + (\delta_2\lambda)^q}{(\delta_1\lambda)^q},$$

so that the conditional probability in (2.2) does not exceed

$$1 - \frac{1}{2^q}\Big(1 - \frac{4^q(\delta^q + \delta_2^q)}{\delta_1^q}\Big).$$

Choosing $\delta = \delta_2 = \delta_1/12$, we obtain that

$$1 - \frac{1}{2^q}\Big(1 - \frac{4^q(\delta^q + \delta_2^q)}{\delta_1^q}\Big) = 1 - \frac{1}{2^q}\Big(1 - \frac{2}{3^q}\Big) \le 1 - \alpha^q,$$

whenever $\alpha \le 1/6$. Therefore,

$$\begin{aligned} E\Big\{&I(M^* \ge \beta\lambda, T_q(M) \ge \delta_1\lambda, w^* < \delta_2\lambda)P(N^* < \delta\lambda|\mathcal{G})\Big\} \\ &\le (1-\alpha^q)EI(M^* \ge \beta\lambda, T_q(M) \ge \delta_1\lambda, w^* < \delta_2\lambda) \le (1-\alpha^q)P(M^* \ge \beta\lambda). \end{aligned}$$

This completes the proof of Lemma 2.2.

Now we are ready to complete the proof of Theorem 1.1. By an argument similar to one used in [**6**, proof of Lemma 2.1], it follows that in order to prove

$$E\Phi(|\sum f_k|) \le c^q E\Phi(|\sum \overline{f}_k|), \tag{2.3}$$

it suffices to establish (2.3) for $(f_k) = (\Delta_k)$, a conditionally symmetric martingale difference sequence. By a routine application of Davis' decomposition (cf. e.g. [**2**] and references therein), we may also assume that $|\Delta_n| \le w_n$, where w_n is a $\mathcal{F}_{n-1}$-measurable random variable, and that $w^* \le 2\Delta^*$. The latter inequality, together with the inequality $P(f^* \ge t) \le 2P(g^* \ge t)$ valid for all tangent sequences (f_k) and (g_k) (cf. [**4**] or [**11**, Theorem 5.2.1 (i)]), implies that

$$P(w^* \ge t) \le P(\Delta^* \ge t/2) \le 2P(\overline{\Delta}^* \ge t/2) \le 2P(N^* \ge t/4). \tag{2.4}$$

By Lemma 2.2, we have that

$$P(M^* \geq \beta\lambda) \leq P(N^* \geq \delta\lambda) + P(w^* \geq \delta_2\lambda) + (1-\alpha^q)P(M^* \geq \beta\lambda),$$

so that

$$\alpha^q P(M^* \geq \beta\lambda) \leq P(N^* \geq \delta\lambda) + P(w^* \geq \delta_2\lambda).$$

Consequently,

$$\alpha^q E\Phi(M^*/\beta) \leq E\Phi(N^*/\delta) + E\Phi(w^*/\delta_2).$$

Therefore,

$$\begin{aligned}\Big(\frac{\alpha}{\beta}\Big)^q E\Phi(M^*) =& \Big(\frac{\alpha}{\beta}\Big)^q E\Phi(\beta M^*/\beta) \\ \leq& \Big(\frac{\alpha}{\beta}\Big)^q \beta^p E\Phi(M^*/\beta) \leq E\Phi(N^*/\delta) + E\Phi(w^*/\delta_2) \\ \leq& \delta^{-q} E\Phi(N^*) + \delta_2^{-q} E\Phi(w^*).\end{aligned}$$

By (2.4) we have that

$$E\Phi(w^*) \leq 2E\Phi(4N^*) \leq 2\cdot 4^q E\Phi(N^*),$$

so that

$$E\Phi(M^*) \leq \Big(\frac{\beta}{\alpha}\Big)^q \Big\{\frac{1}{\delta^q} + \frac{2\cdot 4^q}{\delta_2^q}\Big\} E\Phi(N^*).$$

Conditionally given $\mathcal{G}$, $(\overline{\Delta}_k)$ is a sequence of independent and symmetric random variables. Therefore, by Levy's inequality,

$$P(N^* \geq t\big|\mathcal{G}) \leq 2P(N \geq t\big|\mathcal{G}).$$

This implies (see e.g. [**11**, Proposition 0.2.1]) that for every increasing function $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ we have $E\phi(N^*) \leq 2E\phi(N)$. This completes the proof of Theorem 1.1.

As we mentioned in the introduction, inequality (2.3) extends a result of Klass, who considered sequences (f_k) of special form. On the other hand, it follows from a result of Kwapień [**9**] that if $f_k = (\sum_{j=1}^{k-1} a_{j,k}\xi_j)\xi_k$, where (ξ_j) is a sequence of independent zero mean random variables, then (2.3) holds in the stronger form:

$$E\Phi(|\sum f_k|) \leq E\Phi(c|\sum \overline{f}_k|),$$

for some absolute constant c and for *every* convex function Φ. Thus one may wonder whether our restrictions on Φ can be relaxed. We wish to close this section with a negative result showing that (2.3) does not hold for all convex functions Φ. (However, it is still possible that (2.3) holds under weaker assumption than ours.) Our example is an easy adaptation of an example due to Talagrand concerning comparison of tail behavior for sums of tangent sequences. This example was included in [**5**], and we refer the reader to the latter paper for details that are not included here.

Proposition 2.3. *For every constant $c > 0$, there exists a convex function $\Phi : \mathbf{R} \to \mathbf{R}$ and a sequence (f_k) such that*

$$E\Phi(|\sum f_k|) \geq cE\Phi(c|\sum \overline{f}_k|).$$

Proof: We will show that for every $k \in \mathbf{N}$ there exists a convex function Φ and a sequence (f_k) for which

$$E\Phi(|\sum f_k|) \geq \frac{2^{2^{k+2}}}{k^2 2^{2k}} E\Phi(\frac{k}{4}|\sum \overline{f}_k|).$$

Let (r_n) denote the Rademacher random variables, that is, a sequence of independent random variables such that $P(r_n = \pm 1) = 1/2$. Fix $k \in \mathbf{N}$. Given an integer N_1 to be specified in a moment, define $N_2, \ldots N_k$ as follows:

$$N_i - N_{i-1} = 2^{-(i-1)} N_1, \quad i = 2, \ldots, k.$$

Put

$$\Omega_1 = \{r_1 = \ldots = r_{N_1} = 1\},$$

and then

$$\Omega_i = \Omega_{i-1} \cap \{r_{N_{i-1}+1} = \ldots = r_{N_i}\}, \quad i = 2, \ldots, k.$$

Define a sequence of random variables (v_i) by the formulas:

$$\begin{aligned} v_1 = \ldots = v_{N_1} &= 1 \\ v_{N_1+1} = \ldots = v_{N_2} &= 2I_{\Omega_1} \\ &\cdots\cdots\cdots\cdots \\ v_{N_{k-1}+1} = \ldots = v_{N_k} &= 2^{k-1} I_{\Omega_{k-1}}. \end{aligned}$$

We let $f_j = v_j r_j$ for $j = 1, \ldots N_k$. Then $\overline{f}_j = v_j r'_j$, where (r'_j) is an independent copy of (r_j) (cf. [**10**, Example 4.3.1]). For $0 < \delta < 1$, let Φ_δ be a convex function defined by $\Phi_\delta(x) = (x - \delta k N_1)^+$. Note that $\sum |v_j| = kN_1$, and therefore

$$E\Phi_\delta(|\sum v_j r_j|) \geq (1-\delta) k N_1 P(|\sum v_j r_j| \geq kN_1).$$

On the other hand, if $|\sum v_j r'_j| < 4N_1$, then $|\sum v_j r'_j| \leq 4N_1 - 1$, so that with $\delta = 1 - 1/(4N_1)$ we get

$$\Phi_\delta(\frac{k}{4}|\sum v_j r'_j|) \leq (kN_1 - \frac{k}{4} - \delta k N_1)^+ = 0.$$

Since

$$P(|\sum v_j r'_j| \geq 4N_1) \leq k 2^{-N_1/2^{k-2}} P(|\sum v_j r_j| \geq kN_1),$$

(cf. [**5**, top half of page 176]) we obtain

$$\begin{aligned}E\Phi_\delta(\frac{k}{4}|\sum v_j r_j'|) \leq &\Phi_\delta(\frac{k^2 N_1}{4}) P(|\sum v_j r_j'| \geq 4N_1) \\ \leq &k N_1 \frac{k}{4} k 2^{-N_1/2^{k-2}} P(|\sum v_j r_j| \geq k N_1),\end{aligned}$$

and it follows that

$$\frac{E\Phi_\delta(|\sum v_j r_j|)}{E\Phi_\delta((k/4)|\sum v_j r_j'|)} \geq \frac{4(1-\delta) k N_1 2^{N_1/2^{k-2}}}{k^3 N_1} = \frac{2^{N_1/2^{k-2}}}{k^2 N_1} \geq \frac{2^{2^{k+2}}}{k^2 2^{2k}},$$

for $N_1 \geq 2^k$. This completes the proof.

In the above example the sequence (f_k) may be constructed so that $\sum f_k$ is a randomly stopped sum of independent random variables (see Remark on p. 176 of [**5**]). Thus, the conclusion of this Remark applies here as well.

3. Rearrangement invariant norm inequalities

Lemma 3.1. *Suppose that* Φ *and* Ψ *are two Orlicz functions. Let* $\Theta = \Phi \wedge \Psi$, *and* $\Theta_1(x) = \frac{1}{2}\Theta(x)$. *Then*

$$\tfrac{1}{2}\left\|f\right\|_{\Theta_1} \leq \inf\{\left\|f'\right\|_\Phi + \left\|f''\right\|_\Psi : f' + f'' = f^\#\} \leq 2\left\|f\right\|_\Theta.$$

Proof: To show the left hand side, suppose that $f^\# = f' + f''$, and $\left\|f'\right\|_\Phi + \left\|f''\right\|_\Psi \leq 1$. Then $E\Phi(|f'|) \leq 1$ and $E\Psi(|f''|) \leq 1$, and so

$$\begin{aligned}E\Theta_1(\tfrac{1}{2}|f|) \leq &\tfrac{1}{2} E\Theta(\max\{|f'|, |f''|\}) = \tfrac{1}{2} E\max\{\Theta(|f'|), \Theta(|f''|)\} \\ \leq &\tfrac{1}{2}(E\Phi(|f'|) + E\Psi(|f''|)) \leq 1.\end{aligned}$$

To show the right hand side, suppose that $\left\|f\right\|_\Theta \leq 1$, that is, $E\Theta(|f|) \leq 1$. Let

$$f'(t) = \begin{cases} f^\#(t) & \text{if } \Phi(|f(t)|) \leq \Psi(|f(t)|) \\ 0 & \text{otherwise,} \end{cases}$$

and $f'' = f^\# - f'$. Then we see that $E\Phi(|f'|) \leq E\Theta(|f|) \leq 1$ and that $E\Psi(|f''|) \leq E\Theta(|f|) \leq 1$, and the result follows.

The next lemma follows immediately.

Lemma 3.2. *Let* $\Phi_t(x) = x^p \wedge (tx)^q$, *where* $0 < p < q < \infty$. *Then*

$$2^{-1-1/p}\left\|f\right\|_{\Phi_t} \leq K_{p,q}(f,t) \leq 2\left\|f\right\|_{\Phi_t}.$$

Now we will prove Theorem 1.3, using the above Lemma. From the hypothesis of Theorem 3.1, it follows that $\left\|f\right\|_{\Phi_t} \leq \left\|g\right\|_{\Phi_t}$. Hence by Lemma 3.2, it follows that $K_{p,q}(f,t) \leq 2^{2+1/p} K_{p,q}(g,t)$. Now the result follows by the definition of (p,q)-K-interpolation space.

Corollary 1.4 follows easily from Theorems 1.1 and 1.3. To show Corollary 1.5, we only need the following result. The methods below are all fairly standard in interpolation theory, and indeed if one is not concerned about uniform estimates, may be taken directly from the literature.

Lemma 3.3. *Given $p_0 > 0$, there is a constant $c_{p_0} > 0$ such that if $p, q \ge p_0$, then $L_{p,q}$ is a $(p/2, 2p)$-K-interpolation space with constant bounded by c_{p_0}.*

Proof: First let us define some norms. For $p \le q$, let

$$\|f\|_{a(t)} = \inf\{t^{-2/p} \|f'\|_{p/2} + t^{-1/2p} \|f''\|_{2p} : f' + f'' = f^\#\},$$

$$\|f\|_{b(t)} = \left(\frac{1}{t}\int_0^t f^\#(s)^{p/2}\,ds\right)^{2/p} + \left(\frac{1}{t}\int_t^\infty f^\#(s)^{2p}\,ds\right)^{1/2p}.$$

Clearly $\|f\|_{a(t)} \le \|f\|_{b(t)}$. Also $\|f\|_{a(t)} \ge \min\{1, 2^{1-2/p}\} f^\#(t)$. This is because if $f^\# = f' + f''$, then

$$\begin{aligned} t^{-2/p} \|f'\|_{p/2} + t^{-1/2p} \|f''\|_{2p} &\ge \left(\frac{1}{t}\int_0^t |f'(s)|^{p/2}\,ds\right)^{2/p} + \left(\frac{1}{t}\int_0^t |f''(s)|^{2p}\,ds\right)^{1/2p} \\ &\ge \left(\frac{1}{t}\int_0^t |f'(s)|^{p/2}\,ds\right)^{2/p} + \left(\frac{1}{t}\int_0^t |f''(s)|^{p/2}\,ds\right)^{2/p} \\ &\ge \min\{1, 2^{1-2/p}\} \left(\frac{1}{t}\int_0^t f^\#(s)^{p/2}\,ds\right)^{2/p} \\ &\ge \min\{1, 2^{1-2/p}\} f^\#(t). \end{aligned}$$

Next, given a function f, let us define the function $Hf(t) = \|f\|_{b(t)}$. Then it follows that $\|Hf\|_{p,q} \le 32^{1/\min\{p,q\}} \|f\|_{p,q}$. To see this, first note that $\|Hf\|_{p,q} \le (\|H_1 f\|_{p,q}^q + \|H_2 f\|_{p,q}^q)^{1/q}$, where

$$\begin{aligned} H_1 f(t) &= \left(\frac{1}{t}\int_0^t f^\#(s)^{p/2}\,ds,\right)^{2/p} \\ H_2 f(t) &= \left(\frac{1}{t}\int_t^\infty f^\#(s)^{2p}\,ds\right)^{1/2p} \end{aligned}$$

We will use two properties of $L_{p,q}$. First, if $v \le q$, and if $f_1, f_2, \ldots, f_n$ are functions, then

$$\left\| \left(\sum_{i=1}^n (f_i^\#)^v\right)^{1/v} \right\|_{p,q} \le \left(\sum_{i=1}^n \|f_i\|_{p,q}^v\right)^{1/v}.$$

Second, if we define the operators $D_a f(t) = f^\#(at)$ for $0 < a < \infty$, then $\|D_a f\|_{p,q} = a^{-1/p} \|f\|_{p,q}$. (The first property follows from Minkowski's inequality for $L_{q/v}$, the second is a simple change of variables argument.) Let $u = \max\{p/q, 2\}$, $v = \min\{q, p/2\}$. Then

$$\|H_1 f\|_{p,q} \le \left\| \left(\int_0^1 (D_a f^\#)^{p/2}\,da\right)^{2/p} \right\|_{p,q}$$

$$
\begin{aligned}
&\leq \left\| \left(\sum_{n=0}^{\infty} \int_{2^{-u(n+1)}}^{2^{-un}} (D_a f^{\#})^{p/2}\, da \right)^{2/p} \right\|_{p,q} \\
&\leq \left\| \left(\sum_{n=0}^{\infty} 2^{-un} (D_{2^{-u(n+1)}} f^{\#})^{p/2} \right)^{2/p} \right\|_{p,q} \\
&\leq \left\| \left(\sum_{n=0}^{\infty} 4^{-n} (D_{2^{-u(n+1)}} f^{\#})^{v} \right)^{1/v} \right\|_{p,q} \\
&\leq \left(\sum_{n=0}^{\infty} 4^{-n} \left\| D_{2^{-u(n+1)}} f^{\#} \right\|_{p,q}^{v} \right)^{1/v} \\
&\leq \left(\sum_{n=0}^{\infty} 4^{-n} 2^{u(n+1)v/p}) \right)^{1/v} \|f\|_{p,q} \\
&\leq \left(\sum_{n=0}^{\infty} 2^{1-n} \right)^{1/v} \|f\|_{p,q} \\
&\leq 4^{1/v} \|f\|_{p,q} .
\end{aligned}
$$

Now let $u = \max\{2p/q, 1\}$, $v = \min\{q, 2p\}$.

$$
\begin{aligned}
\|H_2 f\|_{p,q} &\leq \left\| \left(\int_{1}^{\infty} (D_a f^{\#})^{2p}\, da \right)^{1/2p} \right\|_{p,q} \\
&\leq \left\| \left(\sum_{n=0}^{\infty} \int_{2^{un}}^{2^{u(n+1)}} (D_a f^{\#})^{2p}\, da \right)^{1/2p} \right\|_{p,q} \\
&\leq \left\| \left(\sum_{n=0}^{\infty} 2^{u(n+1)} (D_{2^{un}} f^{\#})^{2p} \right)^{1/2p} \right\|_{p,q} \\
&\leq \left\| \left(\sum_{n=0}^{\infty} 2^{n+1} (D_{2^{un}} f^{\#})^{v} \right)^{1/v} \right\|_{p,q} \\
&\leq \left(\sum_{n=0}^{\infty} 2^{n+1} \left\| D_{2^{un}} f^{\#} \right\|_{p,q}^{v} \right)^{1/v} \\
&\leq \left(\sum_{n=0}^{\infty} 2^{n+1} 2^{-unv/p}) \right)^{1/v} \|f\|_{p,q} \\
&\leq \left(\sum_{n=0}^{\infty} 2^{1-n} \right)^{1/v} \|f\|_{p,q}
\end{aligned}
$$

$$\leq 4^{1/v} \|f\|_{p,q}.$$

Finally, to finish, suppose that $K_{p/2,2p}(f,t) \leq K_{p/2,2p}(g,t)$ for all $t > 0$. Then it follows that

$$\begin{aligned} f^{\#}(t) \leq& \max\{1, 2^{2/p-1}\} \|f\|_{a(t)} \leq \max\{1, 2^{2/p-1}\} \|g\|_{a(t)} \leq \max\{1, 2^{2/p-1}\} \|g\|_{b(t)} \\ =& 2^{2/p} Hg(t). \end{aligned}$$

Hence

$$\|f\|_{p,q} \leq 2^{2/p} \|Hg\|_{p,q} \leq 128^{1/\min\{p,q\}} \|g\|_{p,q}.$$

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