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A Note on Sums of Independent Random Variables

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Abstract. In this note a two sided bound on the tail probability of sums of independent, and either symmetric or nonnegative, random variables is obtained. We utilize a recent result by Latała on bounds on moments of such sums. We also give a new proof of Latała's result for nonnegative random variables, and improve one of the constants in his inequality.

1. Introduction

Recently Latała (1997) obtained the following remarkable result: for a sequence of random variables (X_n) and $1 \le p < \infty$ define the following Orlicz norm

$$|||(X_k)|||_p = \inf\{\lambda > 0 : \prod_n \mathbb{E}|1 + X_n/\lambda|^p \le e^p\}. \tag{1.1}$$

Latała proved that

$$\frac{e-1}{2e^2}|||(X_k)|||_p \le \left(\mathbb{E}|\sum X_k|^p\right)^{1/p} \le e|||(X_k)|||_p, \tag{1.2}$$

provided (X_n) are either symmetric or positive, and in the first case $p \ge 2$, and in the second case $p \ge 1$. The main novelty here is the fact that, contrary to the classical inequalities, the constants here are independent of p. Certain particular cases of Latała's result had been known earlier (see e.g. Hitczenko (1993), Gluskin and Kwapień (1995) or Hitczenko, Montgomery-Smith and Oleszkiewicz (1997)), but they can be easily deduced from Latała's inequality.

Of course, the ultimate goal is to obtain bounds on the tail probabilities for sums of random variables. Latała's result prompted us to investigate that problem. This program has been completed; our methods, which are based on estimates for the decreasing rearrangement of a random variable, work in a rather general setting. As a result we were able to obtain extensions of Latała's result in various directions. The details of that approach will be presented elsewhere. The goal of this note is quite different; we will present a very simple argument that allows

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one to deduce tail bounds from Latała's result. As a matter of fact, this approach formally does not really depend on Latała's result, but it requires a knowledge of his bounds on moments in order to be employed successfully. We will also present a short proof (based on decoupling techniques) of Latała's result for non-negative random variables. Our proof gives a slightly better constant on the left-hand side of (1.2).

Our notation is standard; for a sequence (z_k) we let $z_n^* = \max_{1\le k\le n} |z_k|$. The letters c and C denote absolute constants whose values may change from one use to the next. We will write $S = \sum_{k=1}^\infty X_k$, and $S_n = \sum_{k=1}^n X_k$, and $\|S\|_p = (\mathbb{E}|S|^p)^{1/p}$.

2. Tail estimates via moment estimates

In this section we will to obtain two-sided estimates for tails of sums of independent random variables. For the sake of brevity we will concentrate on symmetric random variables, although it will be clear that our arguments work for nonnegative random variables as well. In certain special cases tail inequalities have been obtained from moment inequalities (see Gluskin and Kwapień (1995), Hitczenko and Kwapień (1994) or Hitczenko, Montgomery-Smith and Oleszkiewicz (1997)). Also, in the case of multiples of Rademacher random variables, two-sided estimates have been obtained by Montgomery and Odlyzko (1988), and Montgomery-Smith (1990).

Theorem 2.1. *There exist positive constants c, C, α and δ such that for all sequences of independent symmetric random variables (X_n), and for all t such that*

$$t \ge \frac{1}{2}\Big\|\sum X_i I(|X_i| \le t)\Big\|_2 = \frac{1}{2}\left(\sum_{i=1}^n \|X_i I(|X_i|\le t)\|_2^2\right)^{1/2},$$

the following holds: Let p_t be the least p such that

$$\Big\|\sum X_i I(|X_i| \le t)\Big\|_p \ge 2t.$$

Then we have the inequalities

$$\mathbb{P}(|S_n| > t) \ge c\big\{\mathbb{P}(X_n^* > t) + \exp(-\alpha p_t)\big\}, \tag{2.1}$$

and

$$\mathbb{P}(|S_n| > 4t) \le C\big\{\mathbb{P}(X_n^* > t) + \exp(-\delta p_t)\big\}. \tag{2.2}$$

If $t \le \frac{1}{2}\|\sum X_i I(|X_i| \le t)\|_2$, then

$$\mathbb{P}(|S_n| > t) \ge c.$$

Proof. For a given t, let $Y_i = X_i I(|X_i| \le t)$, and let $s_n = \sum_{j=1}^n Y_j$. Notice that $\|s_n\|_p$ is a continuous, increasing function of p, and that $\|s_n\|_2 \le 2t$. Hence either $2 \le p_t < \infty$ and $\|s_n\|_{p_t} = 2t$, or $p_t = \infty$ and $\|s_n\|_\infty \le 2t$.

Let us start by proving (2.1). It follows from Levy's inequality and contraction principle (see e.g. Kwapień Woyczyński (1992, Propositions 1.1.2 and 1.2.1)) that

$$\mathbb{P}(X_n^* > t) \le 2\mathbb{P}(|S_n| > t),$$

and,

$$\mathbb{P}(|s_n| > t) \le 2\mathbb{P}(|S_n| > t).$$

Hence

$$\mathbb{P}(|S_n| > t) \geq \tfrac{1}{4}\Big(\mathbb{P}(X_n^* > t) + \mathbb{P}(|s_n| > t)\Big). \tag{2.3}$$

Now we can see that if $p_t = \infty$, then the inequality is established. In the case that $p_t < \infty$, we need to obtain a lower estimate for the tail probability of a maximum of partial sums of uniformly bounded symmetric random variables (Y_i). But for such random variables, the following inequality is true (cf. Hitczenko (1994)): for all $q \geq p \geq 1$, we have

$$\begin{aligned} \|s_n\|_q \leq & C\frac{q}{p}\Big\{\|s_n\|_p + \|Y_n^*\|_q\Big\} \\ \leq & C\frac{q}{p}\Big\{\|s_n\|_p + t\Big\}. \end{aligned} \tag{2.4}$$

We also use the Paley-Zygmund inequality that states that for any non-negative random variable Z, and $0 < \lambda < 1$,

$$\mathbb{P}(Z > \lambda EZ) \geq (1-\lambda)^2 \frac{(EZ)^2}{EZ^2}.$$

Since $t = \frac{1}{2}\|s_n\|_{p_t}$ we have that

$$\begin{aligned} \mathbb{P}(|s_n| > t) &\geq \mathbb{P}(|s_n|^{p_t} > 2^{-p_t}\|s_n\|_{p_t}^{p_t}) \\ &\geq (1-2^{-p_t})^2 \frac{\|s_n\|_{p_t}^{2p_t}}{\|s_n\|_{2p_t}^{2p_t}}. \end{aligned}$$

It follows from (2.4) that the denominator is no more than

$$C^{2p_t}\{\|s_n\|_{p_t} + t\}^{2p_t} \leq (\tfrac{3}{2}C)^{2p_t}\|s_n\|_{p_t}^{2p_t}.$$

Therefore, we get the estimate

$$\mathbb{P}(|s_n| \geq t) \geq (1-2^{-p_t})^2(\tfrac{3}{2}C)^{-p_t} \geq \exp(-\alpha p_t),$$

which, together with (2.3) gives (2.1).

Inequality (2.2) is an easy consequence of Chebyshev's inequality. If $p_t < \infty$, then

$$\begin{aligned} \mathbb{P}(|S_n| > 4t) \leq & \mathbb{P}(X_n^* > t) + \mathbb{P}(|s_n| > 4t) \leq \mathbb{P}(X_n^* > t) + \frac{E|s_n|^{p_t}}{(4t)^{p_t}} \\ \leq & \mathbb{P}(X_n^* > t) + 2^{-p_t} = \mathbb{P}(X_n^* > t) + \exp(-\delta p_t). \end{aligned}$$

If $p_t = \infty$, we use the same ideas, noticing that $\mathbb{P}(|s_n| > 2t) = 0$.

Finally, if $t \leq \frac{1}{2}\|s_n\|_2$, we apply the contraction principle, the Paley-Zygmund inequality, and (2.4), to get

$$\begin{aligned} 2\mathbb{P}(|S_n| > t) &\geq \mathbb{P}(|s_n|^2 > \tfrac{1}{4}E|s_n|^2) \\ &\geq \frac{9\|s_n\|_2^4}{16\|s_n\|_4^4} \\ &\geq \frac{9\|s_n\|_2^4}{16C^4(\|s_n\|_2 + t)^4} \end{aligned}$$

which is bounded below by a universal constant. $\square$

REMARK. The above theorem allows us to approximate tails of the sums of independent random variables in terms of tails of the individual summands. This follows from the fact that in view of Latała's result p_t can be approximated using only information about marginal distributions, and from the well known inequality

$$\frac{\sum \mathbb{P}(|X_i| > u)}{1 + \sum \mathbb{P}(|X_i| > u)} \le \mathbb{P}(X_n^* > u) \le 2 \frac{\sum \mathbb{P}(|X_i| > u)}{1 + \sum \mathbb{P}(|X_i| > u)}, \tag{2.5}$$

which gives tails of X_n^* in terms of tails of individual summands.

3. Another proof of Latała's result for nonnegative rv's

Here we intend to give another proof of Latała's formula concerning $\|S\|_p$ for nonnegative random variables.

THEOREM 3.1. *Let (X_n) be a sequence of positive independent random variables. Then for all $p \ge 1$ we have that*

$$\kappa |||(X_n)|||_p \le \|S\|_p \le (e^p - 1)^{1/p} |||(X_n)|||_p, \tag{3.1}$$

where $|||(X_n)|||_p$ is given by (1.1), and κ is the positive number for which $f(\kappa) = e$, where

$$f(x) = \sum_{k=0}^{\infty} \frac{(2k+1)^k}{k!} x^k.$$

PROOF. First note that if $|||(X_n)|||_p \le 1$, then since $1 + \sum_n X_n \le \prod_n (1 + X_n)$, we have that $\|S\|_p^p \le e^p - 1$. This proves the second inequality in (3.1) To prove the first, we use certain results concerning decoupling. These ideas appear often in the literature (usually in the context of mean-zero or symmetric random variables, see e.g. Kwapień and Woyczyński (1992).) However, since we will need control of constants, we cite the following, which is a special case of de la Peña, Montgomery-Smith and Szulga (1994, Theorem 2.1.)

LEMMA 3.2. *Let (X_n) be a sequence of real valued independent random variables. Let $(X_n^{(l)})$ be independent copies of (X_n) for $1 \le l \le k$. Furthermore, let $f_{i_1,\dots,i_k}$ be elements of a Banach space such that $f_{i_1,\dots,i_k} = 0$ unless the $i_1, \dots, i_k$ are distinct. Then for any $1 \le p \le \infty$, we have that*

$$\left\| \sum_{i_1,\dots,i_k} f_{i_1,\dots,i_k} X_{i_1} \cdots X_{i_k} \right\|_p \le (2k+1)^k \left\| \sum_{i_1,\dots,i_k} f_{i_1,\dots,i_k} X_{i_1}^{(1)} \cdots X_{i_k}^{(k)} \right\|_p .$$

Now let us finish the proof of Theorem 3.1. Note that

$$\prod_n \mathbb{E}|1 + X_n|^p = \left\| \prod_n (1 + X_n) \right\|_p^p ,$$

and so by Minkowski's inequality we have that

$$\left\| \prod_n (1 + X_n) \right\|_p \le 1 + \sum_{k=1}^{\infty} \left\| \sum_{i_1 < \cdots < i_k} X_{i_1} \cdots X_{i_k} \right\|_p .$$

But if $k \geq 1$

$$\left\| \sum_{i_1 < \cdots < i_k} X_{i_1} \cdots X_{i_k} \right\|_p = \frac{1}{k!} \left\| \sum_{\substack{i_1, \ldots, i_k \\ \text{distinct}}} X_{i_1} \cdots X_{i_k} \right\|_p$$

$$\leq \frac{(2k+1)^k}{k!} \left\| \sum_{\substack{i_1, \ldots, i_k \\ \text{distinct}}} X^{(1)}_{i_1} \cdots X^{(k)}_{i_k} \right\|_p$$

(where $(X_n^{(l)})$ are independent copies of (X_n) for $1 \leq l < \infty$)

$$\leq \frac{(2k+1)^k}{k!} \left\| \sum_{i_1, \ldots, i_k} X^{(1)}_{i_1} \cdots X^{(k)}_{i_k} \right\|_p$$

$$= \frac{(2k+1)^k}{k!} \|S\|_p^k.$$

Hence

$$\left\| \prod_n (1 + X_n) \right\|_p \leq f(\|S\|_p),$$

So, if $\|S\|_p \leq \kappa$, then

$$\left\| \prod_n (1 + X_n) \right\|_p \leq e,$$

that is,

$$|||(X_n)|||_p \leq 1.$$

$\square$

Remark. Our constant in the second inequality of (3.1) is essentially the same as Latała's constant. But in the first inequality our constant, which may numerically be shown to be about 0.1549, is slightly better than Latała's constant, which is about 0.1162.

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