The Distribution of Rademacher Sums

S.J. Montgomery-Smith,
Department of Mathematics,
University of Missouri at Columbia,
Columbia, MO 65211, U.S.A.

Abstract

We find upper and lower bounds for $\Pr\left(\sum \pm x_n \geq t\right)$, where $x_1, x_2, \ldots$ are real numbers. We express the answer in terms of the K-interpolation norm from the theory of interpolation of Banach spaces.

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Introduction

Throughout this paper, we let ε_1, $\varepsilon_2, \ldots$ be independent Bernoulli random variables (that is, $\Pr(\varepsilon_n = 1) = \Pr(\varepsilon_n = -1) = 1/2$). We are going to look for upper and lower bounds for $\Pr\left(\sum \varepsilon_n x_n > t\right)$, where x_1, $x_2, \ldots$ is a sequence of real numbers such that $x = (x_n)_{n=1}^\infty \in l_2$.

Our first upper bound is well known (see, for example, Chapter II, §59 of [5]):

$$\Pr\left(\sum \varepsilon_n x_n > t \left\|x\right\|_2\right) \le e^{-\frac{1}{2}t^2}. \tag{1}$$

However, if $\left\|x\right\|_1 < \infty$, this cannot also provide a good lower bound, because then we have another upper bound:

$$\Pr\left(\sum \varepsilon_n x_n > \left\|x\right\|_1\right) = 0. \tag{2}$$

To look for lower bounds, we might first consider using some version of the central limit theorem. For example, using Theorem 7.1.4 of [2], it can be shown that for some constant c we have

$$\left|\Pr\left(\sum \varepsilon_n x_n > t \left\|x\right\|_2\right) - \tfrac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{1}{2}s^2}\, ds\right| \le c \left(\tfrac{\left\|x\right\|_3}{\left\|x\right\|_2}\right)^3.$$

Thus, for some constant c we have that if $t \le c^{-1}\big(\log \left\|x\right\|_3 / \left\|x\right\|_2\big)^{\frac{1}{2}}$, then

$$\Pr\left(\sum \varepsilon_n x_n > t \left\|x\right\|_2\right) \ge c^{-1} \int_t^\infty e^{-\frac{1}{2}s^2}\, ds \ge \frac{c^{-2} e^{-\frac{1}{2}t^2}}{t}.$$

However, we should hope for far more. From (1) and (2), we could conjecture something like

$$\Pr\left(\sum \varepsilon_n x_n > c^{-1} \inf\{\left\|x\right\|_1, t\left\|x\right\|_2\}\right) \ge c^{-1} e^{-ct^2}.$$

Actually such a conjecture is unreasonable—one should not take infimums of norms, but instead one should consider the following quantity:

$$K(x,t;l_1,l_2) = K_{1,2}(x,t) = \inf\{\, \left\|x'\right\|_1 + t\left\|x''\right\|_2 : x', x'' \in l_2,\ x' + x'' = x\}.$$

This norm is well known to the theory of interpolation of Banach spaces (see, for example [1] or [3]). For small t, this norm looks a lot like $t\left\|x\right\|_2$, but as t gets much larger, it starts to look more like $\left\|x\right\|_1$. In fact, there is a rather nice approximate formula due to T. Holmstedt (Theorem 4.1 of [3]): if we write $(x_n^*)_{n=1}^\infty$ for the sequence $(|x_n|)_{n=1}^\infty$ rearranged into decreasing order, then

$$c^{-1} K_{1,2}(x,t) \le \sum_{n=1}^{\lfloor t^2 \rfloor} x_n^* + t \left(\sum_{n=\lfloor t^2\rfloor+1}^{\infty} (x_n^*)^2\right)^{\frac{1}{2}} \le K_{1,2}(x,t),$$

where c is a universal constant.

In this paper, we will prove the following result.

Theorem. *There is a constant c such that for all $x \in l_2$ and $t > 0$ we have*

$$\Pr\left(\sum \varepsilon_n x_n > K_{1,2}(x,t)\right) \le e^{-\frac{1}{2}t^2}$$

and

$$\Pr\left(\sum \varepsilon_n x_n > c^{-1} K_{1,2}(x,t)\right) \ge c^{-1} e^{-ct^2}.$$

An interesting example is $x = (n^{-1})_{n=1}^\infty$. Then $c^{-1}\log t \le K_{1,2}(x,t) \le c\,\log t$, and hence

$$c^{-1}\exp(-\exp(ct)) \le \Pr\left(\sum \varepsilon_n n^{-1} > t\right) \le c\,\exp(-\exp(c^{-1}t)).$$

This is quite different behaviour then that which we might have expected from the central limit theorem.

We might also consider $x = (n^{-\frac{1}{p}})_{n=1}^\infty$, where $1 < p < 2$. This example leads us to deduce Proposition 2.1 of [7]. More involved methods allow us to rederive the results of [8] (which include the above mentioned result from [7]). We do not go into details.

We also deduce the following corollary.

Corollary. *There is a constant c such that for all $x \in l_2$ and $0 < t \le \|x\|_2 / \|x\|_\infty$ we have*

$$\Pr\left(\sum \varepsilon_n x_n > c^{-1} t \|x\|_2\right) \ge c^{-1} e^{-ct^2}.$$

Proof: It is sufficient to show that there is a constant c such that if $0 < t \le \|x\|_2 / \|x\|_\infty$, then

$$K_{1,2}(x,t) \le t\,\|x\|_2 \le c\,K_{1,2}(x,t).$$

The left hand inequality follows straight away from the definition of $K_{1,2}(x,t)$. The right hand side follows easily from Holmstedt's formula; obviously if $t < 1$, and otherwise because

$$\sum_{n=1}^{\lfloor t^2 \rfloor} x_n^* \ge \lfloor t^2 \rfloor \frac{\|x\|_2}{t} \ge \tfrac{t}{2}\,\|x\|_2\,.$$

□

Proof of Theorem

In order to prove the theorem, we will need some new norms on l_2, and a few lemmas.

Definition. *For $x \in l_2$ and $t > 0$, define the norm*

$$J(x,t;l_\infty,l_2) = J_{\infty,2}(x,t) = \max\{\|x\|_\infty\,,\, t\,\|x\|_2\}.$$

Lemma 1. *For $t > 0$, the spaces $\big(l_2, K_{1,2}(\,\cdot\,,t)\big)$ and $\big(l_2, J_{\infty,2}(\,\cdot\,,t^{-1})\big)$ are dual to one another, that is, for $x \in l_2$ we have*

$$K_{1,2}(x,t) = \sup\left\{ \sum x_n y_n : y \in l_2,\ J_{\infty,2}(y,t^{-1}) \le 1 \right\}.$$

Proof: This is elementary (see, for example Chapter 3, Exercise 1–6 of [1]). □

Definition. *For $x \in l_2$ and $t \in \mathbf{N}$, define the norm*

$$\|x\|_{P(t)} = \sup\left\{ \sum_{m=1}^{t} \left(\sum_{n\in B_m} |x_n|^2 \right)^{\frac{1}{2}} \right\},$$

where the supremum is taken over all disjoint subsets, $B_1,\ B_2, \ldots,\ B_t \subseteq \mathbf{N}$.

Lemma 2. *If $x \in l_2$ and $t^2 \in \mathbf{N}$, then*

$$\|x\|_{P(t^2)} \le K_{1,2}(x,t) \le \sqrt{2}\ \|x\|_{P(t^2)}\,.$$

Proof: To show the first inequality, note that we have

$$\|x\|_{P(t^2)} \le \|x\|_1 \quad \text{and} \quad \|x\|_{P(t^2)} \le t\,\|x\|_2\,.$$

Hence

$$\begin{aligned} K_{1,2}(x,t) &= \inf\{\, \|x'\|_1 + t\,\|x''\|_2 : x' + x'' = x\} \\ &\ge \inf\{\, \|x'\|_{P(t^2)} + \|x''\|_{P(t^2)} : x' + x'' = x\} \\ &\ge \|x\|_{P(t^2)}\,, \end{aligned}$$

where the last step follows by the triangle inequality.

For the second inequality, we start by using Lemma 1. For any $\delta > 0$, let $y \in l_2$ be such that

$$(1-\delta)K_{1,2}(x,t) \le \sum x_n y_n \quad \text{and} \quad J_{\infty,2}(y,t^{-1}) = 1.$$

Now pick numbers $n_0,\, n_1,\, n_2, \ldots,\, n_{t^2} \in \{0,\, 1,\, 2, \ldots,\, \infty\}$ by induction as follows: given $0 = n_0 < n_1 < \ldots < n_m$, let

$$n_{m+1} = 1 + \sup \left\{ \nu : \sum_{n=n_m+1}^{\nu} |y_n|^2 \le 1 \right\}.$$

Since $\|y\|_\infty \le 1$, we have that $\sum_{n=n_m+1}^{n_{m+1}} |y_n|^2 \le 2$. Also, as $\|y\|_2 \le t$, it follows that $n_{t^2} = \infty$. Therefore

$$\begin{aligned}(1-\delta)K_{1,2}(x,t) &\le \sum x_n y_n \\ &\le \sum_{m=1}^{t^2} \left(\sum_{n=n_{m-1}+1}^{n_m} |y_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=n_{m-1}+1}^{n_m} |x_n|^2 \right)^{\frac{1}{2}} \\ &\le \sqrt{2}\, \|x\|_{P(t^2)} .\end{aligned}$$

Since this is true for all $\delta > 0$, the result follows. □

The following lemma is due to Paley and Zygmund.

Lemma 3. *If $x \in l_2$, then given $0 < \lambda < 1$ we have*

$$\Pr\left(\sum \varepsilon_n x_n > \lambda \|x\|_2 \right) \ge \tfrac{1}{3}(1-\lambda^2)^2.$$

Proof: See Chapter 3, Theorem 3 of [4]. □

Now we proceed with the proof of the theorem. First we will show that

$$\Pr\left(\sum \varepsilon_n x_n > K_{1,2}(x,t) \right) \le e^{-\frac{1}{2}t^2}.$$

Given $\delta > 0$, let $x',\, x'' \in l_2$ be such that $x' + x'' = x$, and

$$(1+\delta)K_{1,2}(x,t) > \|x'\|_1 + t\,\|x''\|_2 .$$

Then

$$\begin{aligned}\Pr\left(\sum \varepsilon_n x_n > (1+\delta)K_{1,2}(x,t) \right) &\le \Pr\left(\sum \varepsilon_n x'_n > \|x'\|_1 \right) \\ &\quad + \Pr\left(\sum \varepsilon_n x''_n > t\,\|x''\|_2 \right) \\ &\le 0 + e^{-\frac{1}{2}t^2},\end{aligned}$$

where the last inequality follows from equations (1) and (2) above. Letting $\delta \to 0$, the result follows.

Now we show that for some constant c we have

$$\Pr\left(\sum \varepsilon_n x_n > c^{-1} K_{1,2}(x,t)\right) \geq c^{-1} e^{-ct^2}.$$

First, let us assume that $t^2 \in \mathbf{N}$. Given $\delta > 0$, let $B_1, B_2, \ldots, B_{t^2} \subseteq \mathbf{N}$ be disjoint subsets such that $\bigcup_{m=1}^{t^2} B_m = \mathbf{N}$ and

$$\|x\|_{P(t^2)} \leq (1+\delta) \sum_{m=1}^{t^2} \left(\sum_{n \in B_m} |x_n|^2 \right)^{\frac{1}{2}}.$$

Then

$$\Pr\left(\sum \varepsilon_n x_n > \tfrac{1}{2} K_{1,2}(x,t)\right) \geq \Pr\left(\sum \varepsilon_n x_n > \tfrac{1}{\sqrt{2}} \|x\|_{P(t^2)}\right)$$

$$\geq \Pr\left(\sum_{m=1}^{t^2} \sum_{n \in B_m} \epsilon_n x_n \geq \tfrac{1}{\sqrt{2}} (1+\delta) \sum_{m=1}^{t^2} \left(\sum_{n \in B_m} |x_n|^2 \right)^{\frac{1}{2}} \right)$$

$$\geq \prod_{m=1}^{t^2} \Pr\left(\sum_{n \in B_m} \varepsilon_n x_n \geq \tfrac{1}{\sqrt{2}} (1+\delta) \left(\sum_{n \in B_m} |x_n|^2 \right)^{\frac{1}{2}} \right)$$

$$\geq \left(\tfrac{1}{3} \left(1 - \tfrac{1}{2}(1+\delta)^2\right)^2 \right)^{t^2},$$

where the last step is from Lemma 3. If we let $\delta \to 0$, then we see that

$$\Pr\left(\sum \varepsilon_n x_n > \tfrac{1}{2} K_{1,2}(x,t)\right) \geq \exp\bigl(-(\log 12)\, t^2\bigr).$$

This proves the result for $t^2 \in \mathbf{N}$. For $t \geq 1$, note that

$$K_{1,2}(x,t) \leq K_{1,2}(x, \lceil t \rceil) \qquad \text{and} \qquad \lceil t \rceil^2 \leq 4t^2,$$

and hence the result follows (with $c = 4 \log 12$). For $t < 1$, the result may be deduced straightaway from Holmstedt's formula and Lemma 3. $\square$

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