

# Embeddings of rearrangement invariant spaces that are not strictly singular

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**Abstract.** We give partial answers to the following conjecture: the natural embedding of a rearrangement invariant space  $E$  into  $L_1([0, 1])$  is strictly singular if and only if  $G$  does not embed into  $E$  continuously, where  $G$  is the closure of the simple functions in the Orlicz space  $L_\Phi$  with  $\Phi(x) = \exp(x^2) - 1$ .

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In this paper we ask the following question. Given a rearrangement invariant space  $E$  on  $[0, 1]$ , when is the natural embedding  $E \subset L_1([0, 1])$  strictly singular. (We refer the reader to [4] for the definition and properties of rearrangement invariant spaces.) We define a linear map between two normed spaces to be *strictly singular* if there does not exist an infinite dimensional subspace of the domain upon which the operator is an isomorphism.

This question is a natural extension of similar work by del Amo, Hernández, Sánchez and Semenov [1], when they considered the problem of which embeddings between rearrangement invariant spaces are not disjointly strictly singular. A positive linear operator between two Banach lattices is *disjointly strictly singular* if there exists an infinite sequence of non-zero disjoint elements in the domain such that the operator is an isomorphism on the span of this sequence. This work [1] contains a number of very sharp results, giving some very clear criteria.

However the question concerning when such maps are strictly singular seems to be more difficult. For this reason, we will restrict ourselves to considering the case when the range is  $L_1([0, 1])$ . Even then, we do not have complete answers, and in this paper, we leave as many questions unanswered as we answer.

The “other end” was investigated by Novikov [7], who showed that the natural embedding  $L_\infty([0, 1]) \subset E$  is strictly singular unless  $E$  is

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equivalent to  $L_\infty([0, 1])$ . (The case when  $E = L_p([0, 1])$  follows from a classical result of Grothendieck, Theorem 5.2 of [9].)

In answering our question, there is one rearrangement invariant space that plays a prominent role. This rearrangement invariant space, denoted by  $G$ , is the closure of the simple functions in the Orlicz space corresponding to the Orlicz function  $e^{x^2} - 1$ . (We define Orlicz spaces below.) The reason why this space plays such a role is as follows. Consider the Rademacher functions on  $[0, 1]$  given by  $r_n(t) = \text{sign}(\sin(2^n \pi t))$ . The following result of Rodin and Semenov [8] is well known.

**THEOREM 1.** *Let  $E$  be a rearrangement invariant space on  $[0, 1]$ . Then the following are equivalent.*

1. *The sequence  $(r_n)$  in  $E$  is equivalent to the unit vector basis of  $\ell_2$ ;*
2.  *$G$  embeds continuously into  $E$ ;*
3. *There is a constant  $c > 0$  such that  $\|\sum_{i=1}^n r_i\|_E \leq c\sqrt{n}$ .*

It is clear that if  $E$  is a rearrangement invariant space on  $[0, 1]$  that contains  $G$  continuously, then the natural embedding  $E \subset L_1$  is not strictly singular. Here, of course, the subspace on which the norms are equivalent is the span of the Rademacher functions.

For this reason, it is natural to pose the following conjecture.

**CONJECTURE 2.** *Let  $E$  be a rearrangement invariant space on  $[0, 1]$ . Then the natural embedding  $E \subset L_1$  is not strictly singular if and only if  $G$  embeds into  $E$  continuously.*

It will become apparent below that the following conjecture implies the previous one.

**CONJECTURE 3.** *Given  $x_1, \dots, x_n \in L_1([0, 1])$ , and a rearrangement invariant space  $E$  on  $[0, 1]$ , there exists signs  $\epsilon_1, \dots, \epsilon_n = \pm 1$  such that*

$$\left\| \sum_{i=1}^n \epsilon_i x_i \right\|_{L_1} \geq c^{-1} \left\| \sum_{i=1}^n r_i \|x_i\|_1 \right\|_E .$$

Unfortunately we are not able to prove either of these conjectures without some additional hypotheses.

First, let us introduce some examples of rearrangement invariant spaces. A function  $\Phi : \mathbb{R} \rightarrow [0, \infty)$  is called an *Orlicz function* if it is convex, even, and takes zero to zero. The *Orlicz space*  $L_\Phi$  is the

collection of all equivalence classes of measurable functions (where the equivalence relation is equal almost everywhere) on  $[0, 1]$  such that the norm:

$$\|x\|_{\Phi} = \inf \left\{ \lambda : \int_0^1 \Phi(x(t)/\lambda) dt \leq 1 \right\}$$

is finite.

Another class of examples is the Lorentz spaces. If  $\varphi : [0, 1] \rightarrow [0, 1]$  is increasing and concave with  $\varphi(0) = 0$  and  $\varphi(1) = 1$ , then the *Lorentz space*  $\Lambda(\varphi)$  consists of all equivalence classes of measurable functions on  $[0, 1]$  for which the norm

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(t) d\varphi(t)$$

is finite. Here, as in the rest of the paper,  $x^*$  denotes the non-increasing rearrangement of  $|x|$ .

One more class of examples is the Marcinkiewicz spaces. If  $\varphi : [0, 1] \rightarrow [0, 1]$  is as above, then the *Marcinkiewicz space*  $M(\varphi) = \Lambda(\varphi)^*$  consists of all equivalence classes of measurable functions on  $[0, 1]$  for which the norm

$$\|x\|_{M(\varphi)} = \sup_{0 < t < 1} \frac{\int_0^t x^*(s) ds}{\varphi(t)}$$

is finite.

By definition,  $G$  is the closure of the simple functions in the Orlicz space corresponding to the Orlicz function  $e^{x^2} - 1$ . However, it can also be shown that it has an equivalent Marcinkiewicz norm, that is, there is a constant  $c > 0$  such that  $c^{-1} \|x\|_G \leq \|x\|_{M(\varphi)} \leq c \|x\|_G$ , where

$$\varphi(t) = \frac{t}{\sqrt{\log(e/t)}}.$$

Another space that will concern us is the space  $G_1 = \Lambda(\varphi)$  where  $\varphi(t) = \frac{2}{\sqrt{\log(e^2/t)}}$ . It is a simple matter to show that  $G_1$  embeds continuously into  $G$ . Furthermore, for characteristic functions, the norms on  $G$  and  $G_1$  coincide.

Let us now consider the concept of  $D$ -convex rearrangement invariant spaces. This notion was introduced by Kalton [2], and studied extensively by Montgomery-Smith and Semenov [6], where several equivalent properties were given. Perhaps the easiest definition to work with is the following. We will say that a rearrangement invariant space  $E$  is  $D$ -convex if there is a family of Orlicz functions  $\Phi_\alpha : \mathbb{R} \rightarrow [0, \infty)$ , and a constant  $c > 0$ , such that

$$c^{-1} \|x\|_E \leq \sup_{\alpha} \|x\|_{\Phi_\alpha} \leq \|x\|_E.$$

Note that the Marcinkiewicz spaces are  $D$ -convex, because for each  $0 < t < 1$  the map  $x \mapsto \int_0^t x^*(s) ds$  is equivalent to the Orlicz norm given by the Orlicz function  $\Phi(s) = (s - t^{-1})^+$ .

The best result we have obtained so far regarding Conjectures 2 and 3 is the following.

**THEOREM 4.** *Conjectures 2 and 3 are true if  $E$  is  $D$ -convex.*

From here we are able to get a weaker version of Conjecture 2.

**THEOREM 5.** *Let  $E$  be a rearrangement invariant space on  $[0, 1]$ . If the natural embedding  $E \subset L_1$  is not strictly singular, then  $G_1$  embeds into  $E$  continuously.*

We proceed with the proofs.

*Proof of first part of Theorem 4.* It is sufficient to consider the case when  $E$  is an Orlicz space  $L_\Phi$ , where  $\Phi$  is an Orlicz function. Suppose that

$$\sup_{\epsilon} \left\| \sum_{i=1}^n \epsilon_i x_i \right\|_{\Phi} \leq 1,$$

where we write  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ . Thus

$$\sup_{\epsilon} \int \Phi \left( \sum_{i=1}^n \epsilon_i x_i(s) \right) ds \leq 1.$$

Let

$$F_{\epsilon} = \{s \in [0, 1] : \text{sign}(x_i(s)) = \epsilon_i\}.$$

If  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  and  $\eta = (\eta_1, \dots, \eta_n)$ , let us set  $\eta\epsilon = (\eta_1\epsilon_1, \dots, \eta_n\epsilon_n)$ . Then

$$\begin{aligned} \text{Ave}_{\eta} \Phi \left( \sum_{i=1}^n \eta_i \|x_i\|_1 \right) &= \text{Ave}_{\eta} \Phi \left( \sum_{\epsilon} \int I_{F_{\eta\epsilon}}(s) \sum_i \eta_i |x_i(s)| ds \right) \\ &= \text{Ave}_{\eta} \Phi \left( \sum_{\epsilon} \int I_{F_{\eta\epsilon}}(s) \sum_i \epsilon_i x_i(s) ds \right) \\ &\leq \text{Ave}_{\eta} \int \Phi \left( \sum_{\epsilon} I_{F_{\eta\epsilon}}(s) \sum_i \epsilon_i x_i(s) \right) ds \\ &= \text{Ave}_{\eta} \int \sum_{\epsilon} I_{F_{\eta\epsilon}}(s) \Phi \left( \sum_i \epsilon_i x_i(s) \right) ds \\ &= \text{Ave}_{\epsilon} \sum_{\eta} \int I_{F_{\eta\epsilon}}(s) \Phi \left( \sum_i \epsilon_i x_i(s) \right) ds \end{aligned}$$

$$\begin{aligned} &\leq \sup_{\epsilon} \int \Phi \left( \sum_i \epsilon_i x_i(s) \right) ds \\ &\leq 1. \end{aligned}$$

Therefore

$$\left\| \sum_i r_i \|x_i\|_1 \right\|_{\Phi} \leq 1.$$

□

*Proof of second part of Theorem 4.* Suppose that there is an infinite dimensional subspace  $F \subset E$  such that the norms of  $L_1$  and  $E$  are equivalent on  $F$ . By Dvoretzky's Theorem (see [5] Chapter 4), for each integer  $n$ , there is an  $n$ -dimensional subspace  $H$  of  $F$  such that  $H$  is 2-isomorphic to Hilbert space. Pick an orthonormal basis  $x_1, \dots, x_n$  for  $H$ . Then there exist signs  $\epsilon_1, \dots, \epsilon_n = \pm 1$  such that

$$\left\| \sum_i r_i \right\|_E \leq c \left\| \sum_i \epsilon_i x_i \right\|_H = c\sqrt{n}.$$

Thus the result follows by Theorem 1.

□

In order to prove Theorem 5, we need the following Lemma. This result may also be found in [3] (Theorem 2.5.7.).

LEMMA 6. *Given a rearrangement invariant space  $E$ , there exists an increasing function  $\varphi : [0, 1] \rightarrow [0, 1]$  such that  $\|x\|_E \geq \|x\|_{M(\varphi)}$ , but if  $x$  takes only values 0 or 1, then  $\|x\|_E = \|x\|_{M(\varphi)}$ .*

*Proof.* Let

$$\varphi(t) = \frac{t}{\|I_{[0,t]}\|_E}.$$

The latter property is obvious. To show the former, recall (see for example [4]) the space  $E'$  to be those functions  $y$  on  $[0, 1]$  for which the norm

$$\|y\|_{E'} = \sup \left\{ \int x(t)y(t) dt : \|x\|_E \leq 1 \right\}$$

is finite. Notice that

$$\|I_{[0,t]}\|_E \cdot \|I_{[0,t]}\|_{E'} = t,$$

that is,  $\|I_{[0,1]}\|_{E'} = \varphi(t)$ . Then the result follows since

$$\int_0^t x^*(s) ds \leq \|x\|_E \|I_{[0,1]}\|_{E'}.$$

□

*Proof of Theorem 5.* Suppose that the embedding  $E \subset L_1([0, 1])$  is not strictly singular. Produce  $\varphi$  as in Lemma 6. Then it is clear that the embedding  $M(\varphi) \subset L_1([0, 1])$  is also not strictly singular. Hence by Theorem 4,  $G$  embeds continuously into  $M(\varphi)$ .

In particular, this means that there is a constant  $c > 0$  such that for all  $t \in [0, 1]$  we have

$$\|I_{[0,t]}\|_E = \|I_{[0,t]}\|_{M(\varphi)} \leq c \|I_{[0,t]}\|_G \leq c\psi(t),$$

where  $\psi(t) = \frac{2}{\sqrt{\log(e^4/t)}}$ . Now, writing  $x^* = \int_0^1 I_{[0,t]} d(-x^*(t))$ , we obtain that

$$\begin{aligned} \|x\|_E &\leq \int_0^1 \|I_{[0,t]}\|_E d(-x^*(t)) \\ &\leq c \int_0^1 \psi(t) d(-x^*(t)) = c \int_0^1 x^*(t) d\psi(t) = c \|x\|_{G_1}. \end{aligned}$$

□

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