STABILITY RADIUS AND INTERNAL VERSUS EXTERNAL STABILITY IN BANACH SPACES: AN EVOLUTION SEMIGROUP APPROACH [*]

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Abstract. In this paper the theory of evolution semigroups is developed and used to provide a framework to study the stability of general linear control systems. These include autonomous and nonautonomous systems modeled with unbounded state-space operators acting on Banach spaces. This approach allows one to apply the classical theory of strongly continuous semigroups to time-varying systems. In particular, the complex stability radius may be expressed explicitly in terms of the generator of a (evolution) semigroup. Examples are given to show that classical formulas for the stability radius of an autonomous Hilbert-space system fail in more general settings. Upper and lower bounds on the stability radius are proven for Banach-space systems. In addition, it is shown that the theory of evolution semigroups allows for a straightforward operator-theoretic analysis of internal stability as determined by classical frequency-domain and input-output operators, even for nonautonomous Banach-space systems. In particular, for the nonautonomous setting, internal stability is shown to be equivalent to input-output stability, for stabilizable and detectable systems. For the autonomous setting, an explicit formula for the norm of input-output operator is given.

Key words. evolution semigroups, stability radius, exponential stability, external stability, spectral mapping theorem, transfer function

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0. Introduction. Presented here is a study of stability of infinite-dimensional linear control systems which is based on the relatively recent development of the theory of evolution semigroups. These semigroups have been used in the study of exponential dichotomy of time-varying differential equations and more general hyperbolic dynamical systems; see [6, 22, 23, 26, 29, 34, 42] and the bibliographies therein. The intent of this paper is to show how the theory of evolution semigroups can be used to provide a clarifying perspective, and prove new results, on the uniform exponential stability for general linear control systems, $\dot{x}(t) = A(t)x(t) + B(t)u(t)$, $y(t) = C(t)x(t),\ \ t \geq 0$. The operators $A(t)$ are generally unbounded operators on a Banach space, X, while the operators $B(t)$ and $C(t)$ may act on Banach spaces, U and Y, respectively. In addressing the general settings, difficulties arise both from the time-varying aspect and from a loss of Hilbert-space properties. This presentation, however, provides some relatively simple operator-theoretic arguments for properties that extend classical theorems of autonomous systems in finite dimensions. The topics

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covered here include characterizing internal stability of the nominal system in terms of appropriate input-state-output operators and, subsequently, using these properties to obtain new explicit formulas for bounds on the stability radius. Nonautonomous systems are generally considered, but some results apply only to autonomous ones, such as the upper bound for the stability radius (Section 3.3), the formula for the norm of the input-output operator in Banach spaces (Section 3.4) and a characterization of stability that is related to this formula (Section 4.2).

Although practical considerations usually dictate that U and Y are Hilbert spaces (indeed, finite dimensional), the Banach-space setting addressed here may be motivated by the problem of determining optimal sensor (or actuator) location. For this, it may be natural to consider $U = X$ and $B = I_X$ (or $Y = X$ and $C = I_X$) [4]; if the natural state space X is a Banach space then, as will be shown in this paper, Hilbert-space characterizations of internal stability or its robustness do not apply. We also show that even in the case of Hilbert spaces U and Y, known formulas for the stability radius involving the spaces $L^2(\mathbb{R}_+, U)$ and $L^2(\mathbb{R}_+, Y)$ do not apply if the L^2 norm is replaced by, say, the L^1 norm—see the examples in Subsection 3.5 below. In addition to the general setting of nonautonomous systems on Banach spaces, autonomous and Hilbert-space systems are considered.

For the autonomous case, the primary observation we make about general Banach-space settings versus the classical L^2 and Hilbert-space setting can be explained using the notion of L^p–Fourier multipliers. For this, let $H(s) = C(A - is)^{-1}B$, $s \in \mathbb{R}$, denote the transfer function and $\mathbf{F}$ denote the Fourier transform. The transfer function H is said to be an *L^p–Fourier multiplier* if the operator $u \mapsto \mathbf{F}^{-1}H(\cdot)\mathbf{F}u$ can be extended from the Schwartz class of rapidly decaying U–valued functions to a bounded operator from $L^p(\mathbb{R}; U)$ to $L^p(\mathbb{R}; Y)$; see, e.g., [1] for the definitions. As shown in Theorem 3.11 below, the norm of this operator is equal to the norm of the input-output operator. If U and Y are Hilbert spaces and $p = 2$, then H is an L^2–Fourier multiplier if and only if $\|H(\cdot)\|$ is bounded on $\mathbb{R}$; see formula (3.20). For Banach spaces and/or $p \neq 2$ this latter condition is necessary, but *not sufficient* for H to be an L^p–Fourier multiplier. As a result, our formula (3.18) for the norm of the input-output operator is more involved.

To motivate the methods, recall Lyapunov's stability theorem which says that if A is a bounded linear operator on X and if the spectrum of A is contained in the open left half of the complex plane, then the solution of the autonomous differential equation $\dot{x}(t) = Ax(t)$ on X is uniformly exponentially stable; equivalently, the spectrum $\sigma(e^{tA})$ is contained in the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$, for $t > 0$. This is a consequence of the fact that when A is a bounded operator, then the spectral mapping theorem holds: $\sigma(e^{tA}) \setminus \{0\} = e^{t\sigma(A)}, \quad t \neq 0$. Difficulties with Lyapunov's Theorem arise when the operators, A, are allowed to be unbounded. In particular, it is well known that there exist strongly continuous semigroups $\{e^{tA}\}_{t\geq 0}$ that are *not* uniformly exponentially stable even though $\mathrm{Re}\lambda \leq \omega < 0$ for all $\lambda \in \sigma(A)$; see, e.g., [27, 29, 41]. For nonautonomous equations the situation is worse. Indeed, even for finite-dimensional X it is possible for the spectra of $A(t)$ to be the same for all $t > 0$ *and* contained in the open left half-plane yet the corresponding solutions to $\dot{x}(t) = A(t)x(t)$ are not uniformly exponentially stable (see [14, Exm.7.1] for a classical example). In the development that follows we plan to show how these difficulties can be overcome by the construction of an "evolution semigroup." This is a family of operators defined on a superspace of functions from $\mathbb{R}$ into X, such as $L^p(\mathbb{R}, X)$, $1 \leq p < \infty$, or $C_0(\mathbb{R}, X)$.

Section 1 sets up the notation and provides background information. Section 2 presents the basic properties of the evolution semigroups. Included here is the property that the spectral mapping theorem always holds for these semigroups when they are defined on X-valued functions on the *half-line*, such as $L^p(\mathbb{R}_+, X)$. A consequence of this is a characterization of exponential stability for nonautonomous systems in terms of the invertibility of the generator Γ of the evolution semigroup. This operator, and its role in determining exponential stability, is the basis for many of the subsequent developments. In particular, the semigroup $\{e^{tA}\}_{t\geq 0}$ *is* uniformly exponentially stable provided Re $\lambda < 0$ for all $\lambda \in \sigma(\Gamma)$.

Section 3 addresses the topic of the (complex) stability radius; that is, the size of the smallest disturbance, $\Delta(\cdot)$, under which the perturbation, $\dot{x}(t) = (A(t)+\Delta(t))x(t)$, of an exponentially stable system, $\dot{x}(t) = A(t)x(t)$, looses exponential stability. Results address structured and unstructured perturbations of autonomous and nonautonomous systems in both Banach and Hilbert space settings. Examples are given which highlight some important differences between these settings. Also included in this section is a discussion about the transfer function for infinite-dimensional *time-varying* systems. This concept arises naturally in the context of evolution semigroups.

In Section 4 the explicit relationship between internal and external stability is studied for general linear systems. This material expands on the ideas begun in [37]. A classical result for autonomous systems in Hilbert space is the fact that exponential stability of the nominal system (internal stability) is, under the hypotheses of stabilizability and detectability, equivalent to the boundedness of the transfer function in the right half-plane (external stability). Such a result does not apply to nonautonomous systems and a counterexample shows that this property fails to hold for Banach space systems. Properties from Section 2 provide a natural Banach-space extension of this result: the role of transfer function is replaced by the input-output operator. Moreover, for autonomous systems we provide an explicit formula relating the norm of this input-output operator to that of the transfer function. Finally, we prove two theorems—one for nonautonomous and one for autonomous systems—which characterize internal stability in terms of the various input-state-output operators.

This Introduction concludes with a brief synopsis of the main results. The characterization of uniform exponential stability in terms of an evolution semigroup and its generator is given in Theorem 2.2, Theorem 2.5, and Corollary 2.6. Although these results are essentially known, the proofs are approached in a new way. In particular, Theorem 2.5 identifies the operator $\mathbb{G} = -\Gamma^{-1}$ used to determine stability throughout the paper. Theorem 3.2 records the main observation that the input-output operator, $\mathbb{L} = \mathcal{C}\mathbb{G}\mathcal{B}$, for a general nonautonomous system is related to the inverse of the generator of the evolution semigroup. A very short proof of the known fact that the stability radius for such a system is bounded from below by $\|\mathbb{L}\|^{-1}$ is also provided here. The upper bound for the stability radius, being given in terms of the transfer function, applies only to autonomous systems and is proven in Subsection 3.3. The upper bound, as identified here for Banach spaces, seems to be new although our proof is based on the idea of the Hilbert-space result of [17, Thm. 3.5]. In Subsection 3.3 we also introduce the pointwise stability radius and dichotomy radius. Estimates for the former are provided by Theorems 3.3 and 3.4 while the latter is addressed in Lemma 3.5. Examples 3.13 and 3.15 show that, for autonomous Banach space systems, both inequalities for the upper and lower bounds on the stability radius (see Theorem 3.1) can be strict. In view of the possibility of the strict inequality $\|\mathbb{L}\| > \sup_{s\in\mathbb{R}} \|C(A - is)^{-1}B\|$, Theorem 3.11 provides a new Banach space formula for $\|\mathbb{L}\|$ in terms of A, B, and C. In Section 4 this expression for $\|L\|$ is used to relate state-space versus frequency-domain stability—concepts which are *not* equivalent for Banach-space systems. A special case of this expression gives a new formula for the growth bound of a semigroup on a Banach space; see Theorem 4.4 and the subsequent paragraph. Finally, Theorem 4.3 extends a classical characterization of stability for stabilizable and detectable control systems as it applies to nonautonomous Banach-space settings.

1. Notation and Preliminaries. Throughout the paper, $\mathcal{L}(X,Y)$ will denote the set of bounded linear operators between complex Banach spaces X and Y. If A is a linear operator on X, $\sigma(A)$ will denote the spectrum of A, $\rho(A)$ the resolvent set of A relative to $\mathcal{L}(X)=\mathcal{L}(X,X)$, and $\|A\|_\bullet = \|A\|_{\bullet,X} := \inf\{\|Ax\| : x\in \mathrm{Dom}(A), \|x\| = 1\}$. In particular, if A is invertible in $\mathcal{L}(X)$, $\|A\|_\bullet = 1/\|A^{-1}\|_{\mathcal{L}(X)}$. Also, let $\mathbb{C}_+ = \{\lambda\in\mathbb{C} : \mathrm{Re}\ \lambda > 0\}$.

If A generates a strongly continuous (or C_0) semigroup $\{e^{tA}\}_{t\ge 0}$ on a Banach space X the following notation will be used: $s(A) = \sup\{\mathrm{Re}\lambda : \lambda\in\sigma(A)\}$ denotes the spectral bound; $s_0(A) = \inf\{\omega\in\mathbb{R} :\ \{\lambda : \mathrm{Re}\ \lambda > \omega\}\subset\rho(A) \quad \text{and} \quad \sup_{\mathrm{Re}\ \lambda>\omega}\|(A-\lambda)^{-1}\| < \infty\}$ is the abscissa of uniform boundedness of the resolvent; and $\omega_0(e^{tA}) = \inf\{\omega\in\mathbb{R} :\ \|e^{tA}\|\le Me^{t\omega} \quad \text{for some} \quad M\ge 0 \quad \text{and all} \quad t\ge 0\}$ denotes the growth bound of the semigroup. In general, $s(A)\le s_0(A)\le\omega_0(e^{tA})$ (see, e.g., [29]) with *strict* inequalities possible; see [27, 29, 41] for examples. However, when X is a Hilbert space, the following spectral mapping theorem of L. Gearhart holds (see, e.g., [27, p. 95] or [29, 33]):

THEOREM 1.1. *If A generates a strongly continuous semigroup $\{e^{tA}\}_{t\ge0}$ on a Hilbert space, then $s_0(A) = \omega_0(e^{tA})$. Moreover, $1\in\rho(e^{2\pi A})$ if and only if $i\mathbb{Z}\subset\rho(A)$ and $\sup_{k\in\mathbb{Z}}\|(A-ik)^{-1}\| < \infty$.*

In particular, this result shows that on a Hilbert space X the semigroup $\{e^{tA}\}_{t\ge0}$ is uniformly exponentially stable if and only if $\sup_{\lambda\in\mathbb{C}_+}\|(A-\lambda)^{-1}\| < \infty$ [18].

Now consider operators $A(t)$, $t\ge 0$, with domain $\mathrm{Dom}(A(t))$ in a Banach space X. If the abstract Cauchy problem

$$\dot{x}(t) = A(t)x(t), \quad x(\tau)\in\mathrm{Dom}(A(\tau)), \qquad t\ge\tau\ge 0, \tag{1.1}$$

is well-posed in the sense that there exists an evolution (solving) family of operators $\mathcal{U} = \{U(t,\tau)\}_{t\ge\tau}$ on X which gives a differentiable solution, then $x(\cdot) : t\mapsto U(t,\tau)x(\tau)$, $t\ge\tau$ in $\mathbb{R}$, is differentiable, $x(t)$ is in $\mathrm{Dom}(A(t))$ for $t\ge 0$, and (1.1) holds. The precise meaning of the term evolution family used here is as follows.

DEFINITION 1.2. *A family of bounded operators $\{U(t,\tau)\}_{t\ge\tau}$ on X is called an* evolution family *if*

(i) $U(t,\tau) = U(t,s)U(s,\tau)$ and $U(t,t) = I$ for all $t\ge s\ge\tau$;

(ii) for each $x\in X$ the function $(t,\tau)\mapsto U(t,\tau)x$ is continuous for $t\ge\tau$.

An evolution family $\{U(t,\tau)\}_{t\ge\tau}$ is called exponentially bounded *if, in addition,*

(iii) there exist constants $M\ge 1$, $\omega\in\mathbb{R}$ such that

$$\|U(t,\tau)\|\le Me^{\omega(t-\tau)}, \quad t\ge\tau.$$

REMARKS 1.3.

(a) An evolution family $\{U(t,\tau)\}_{t\ge\tau}$ is called uniformly exponentially stable *if in part (iii), ω can be taken to be strictly less than zero.*

(b) Evolution families appear as solutions for abstract Cauchy problems (1.1). *Since the definition requires that $(t,\tau)\mapsto U(t,\tau)$ is merely strongly continuous the operators $A(t)$ in* (1.1) *can be unbounded.*

(c) *In the autonomous case where* $A(t) \equiv A$ *is the infinitesimal generator of a strongly continuous semigroup* $\{e^{tA}\}_{t\geq 0}$ *on* X *then* $U(t,\tau) = e^{(t-\tau)A}$*, for* $t \geq \tau$*, is a strongly continuous exponentially bounded evolution family.*

(d) *The existence of a* differentiable *solution to* (1.1) *plays little role in this paper, so the starting point will usually not be the equation* (1.1)*, but rather the existence of an exponentially bounded evolution family.*

In the next section we will define the evolution semigroup relevant to our interests for the nonautonomous Cauchy problem (1.1) on the half-line, $\mathbb{R}_+ = [0,\infty)$. For now, we begin by considering the autonomous equation $\dot{x}(t) = Ax(t)$, $t \in \mathbb{R}$, where A is the generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ on X. If $\mathcal{F}_\mathbb{R}$ is a space of X-valued functions, $f : \mathbb{R} \to X$, define

$$(E^t_\mathbb{R} f)(\tau) = e^{tA} f(\tau - t), \quad \text{for } f \in \mathcal{F}_\mathbb{R}. \tag{1.2}$$

If $\mathcal{F}_\mathbb{R} = L^p(\mathbb{R}, X)$, $1 \leq p < \infty$, or $\mathcal{F}_\mathbb{R} = C_0(\mathbb{R}, X)$, the space of continuous functions vanishing at infinities (or another Banach function space as in [34]) this defines a strongly continuous semigroup of operators $\{E^t_\mathbb{R}\}_{t\geq 0}$ whose generator will be denoted by $\Gamma_\mathbb{R}$. In the case $\mathcal{F}_\mathbb{R} = L^p(\mathbb{R}, X)$, $\Gamma_\mathbb{R}$ is the closure (in $L^p(\mathbb{R}, X)$) of the operator $-d/dt + \mathcal{A}$ where $(\mathcal{A}f)(t) = Af(t)$ and

$$\begin{aligned}\mathrm{Dom}(-d/dt + \mathcal{A}) &= \mathrm{Dom}(-d/dt) \cap \mathrm{Dom}(\mathcal{A}) \\ &= \{v \in L^p(\mathbb{R}, X) : v \in AC(\mathbb{R}, X),\, v' \in L^p(\mathbb{R}, X), \\ &\qquad v(s) \in \mathrm{Dom}(\mathcal{A}) \text{ for almost every } s, \text{ and } -v' + Av \in L^p(\mathbb{R}, X)\}.\end{aligned}$$

The important properties of this “evolution semigroup” are summarized in the following remarks; see [22] and also further developments in [29, 34, 42]. The unit circle in $\mathbb{C}$ is denoted here by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

REMARKS 1.4. *The spectrum* $\sigma(E^t_\mathbb{R})$*, for* $t > 0$*, is invariant with respect to rotations centered at the origin, and* $\sigma(\Gamma_\mathbb{R})$ *is invariant with respect to translations along* $i\mathbb{R}$*. Moreover, the following are equivalent.*

(i) $\sigma(e^{tA}) \cap \mathbb{T} = \emptyset$ *on* X*;*

(ii) $\sigma(E^t_\mathbb{R}) \cap \mathbb{T} = \emptyset$ *on* $\mathcal{F}_\mathbb{R}$*;*

(iii) $0 \in \rho(\Gamma_\mathbb{R})$ *on* $\mathcal{F}_\mathbb{R}$*.*

As a consequence,

$$\sigma(E^t_\mathbb{R}) \setminus \{0\} = e^{t\sigma(\Gamma_\mathbb{R})}, \quad t > 0. \tag{1.3}$$

Note that $\{E^t_\mathbb{R}\}_{t\geq 0}$ has the spectral mapping property (1.3) on $\mathcal{F}_\mathbb{R}$ even if the underlying semigroup $\{e^{tA}\}_{t\geq 0}$ does not have the spectral mapping property on X. In the latter case, it may be that the exponential stability of the solutions to $\dot{x} = Ax$ on $\mathbb{R}$ are not determined by the spectrum of A. However, such stability *is determined* by the spectrum of $\Gamma_\mathbb{R}$. This is made explicit by the following corollary of Remarks 1.4: The spectral bound $s(\Gamma_\mathbb{R})$ and the growth bound $\omega_0(E^t_\mathbb{R})$ for the evolution semigroup coincide and are equal to the growth bound of $\{e^{tA}\}_{t\geq 0}$:

$$s(\Gamma_\mathbb{R}) = \omega_0(E^t_\mathbb{R}) = \omega_0(e^{tA}).$$

One of the difficulties related to nonautonomous problems is that their associated evolution families are two-parameter families of operators. From this point of view, it would be of interest to define a *one-parameter* semigroup that is associated to the solutions of the nonautonomous Cauchy problem (1.1). For such a semigroup to be useful, its properties should be closely connected to the asymptotic behavior of the original nonautonomous problem. Ideally, this semigroup would have a generator that plays the same significant role in determining the stability of the solutions as the operator A played in Lyapunov's classical stability theorem for finite-dimensional autonomous systems, $\dot{x} = Ax$. This can, in fact, be done and the operator of interest is the generator of the following *evolution semigroup* that is induced by the two-parameter evolution family: if $\mathcal{U} = \{U(t,\tau)\}_{t\geq\tau}$ is an evolution family, define operators $E^t_{\mathbb{R}}$, $t \geq 0$, on $\mathcal{F}_{\mathbb{R}} = L^p(\mathbb{R}, X)$ or $\mathcal{F}_{\mathbb{R}} = C_0(\mathbb{R}, X)$ by

$$(E^t_{\mathbb{R}} f)(\tau) = U(\tau, \tau - t) f(\tau - t), \quad \tau \in \mathbb{R}, \quad t \geq 0. \tag{1.4}$$

When $\mathcal{U}$ exponentially bounded, this defines a strongly continuous evolution semigroup on $\mathcal{F}_{\mathbb{R}}$ whose generator will be denoted by $\Gamma_{\mathbb{R}}$. As shown in [22] and [34] the spectral mapping theorem (1.3) holds for this semigroup. Moreover, the existence of an exponential dichotomy for solutions to $\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R}$, is characterized by the condition that $\Gamma_{\mathbb{R}}$ is invertible on $\mathcal{F}_{\mathbb{R}}$. Note that in the autonomous case where $U(t,\tau) = e^{(t-\tau)A}$, this is the evolution semigroup defined in (1.2). In the nonautonomous case, the construction of an evolution semigroup is a way to "autonomize" a time-varying Cauchy problem by replacing the time-dependent differential equation $\dot{x} = A(t)x$ on X by an autonomous differential equation $\dot{f} = \Gamma f$ on a superspace of X-valued functions.

2. Evolution Semigroups and Cauchy Problems. In order to tackle the problem of characterizing the exponential stability of solutions to the nonautonomous Cauchy problem (1.1) on the half-line, $\mathbb{R}_+$, the following variant of the above evolution semigroup is needed. As before, let $\{U(t,\tau)\}_{t\geq\tau}$ be an exponentially bounded evolution family, and define operators E^t, $t \geq 0$, on functions $f : \mathbb{R}_+ \to X$ by

$$(E^t f)(\tau) = \begin{cases} U(\tau, \tau - t) f(\tau - t), & 0 \leq t \leq \tau \\ 0, & 0 \leq \tau < t. \end{cases} \tag{2.1}$$

This defines a strongly continuous semigroup of operators on the space of functions $\mathcal{F} = L^p(\mathbb{R}_+, X)$, and the generator of this *evolution semigroup* will be denoted by Γ. This also defines a strongly continuous semigroup on $C_{00}(\mathbb{R}_+, X) = \{f \in C_0(\mathbb{R}_+, X) : f(0) = 0\}$. For more information on evolution semigroups on the half-line see also [26, 28, 29, 42].

2.1. Stability. The primary goal of this subsection is to identify the useful properties of the semigroup of operators defined in (2.1) which will be used in the subsequent sections. In particular, the following spectral mapping theorem will allow this semigroup to be used in characterizing the exponential stability of solutions to (1.1) on $\mathbb{R}_+$. See also [34, 42] for different proofs. The spectral symmetry portion of this theorem is due to R. Rau [38].

THEOREM 2.1. *Let $\mathcal{F}$ denote $C_{00}(\mathbb{R}_+, X)$ or $L^p(\mathbb{R}_+, X)$. The spectrum $\sigma(\Gamma)$ is a half plane, the spectrum $\sigma(E^t)$ is a disk centered at the origin, and*

$$e^{t\sigma(\Gamma)} = \sigma(E^t) \setminus \{0\}, \quad t > 0. \tag{2.2}$$

Proof. The arguments for the two cases $\mathcal{F} = C_{00}(\mathbb{R}_+, X)$ and $\mathcal{F} = L^p(\mathbb{R}_+, X)$ are similar, so only the first one is considered here.

We first note that $\sigma(\Gamma)$ is invariant under translations along $i\mathbb{R}$ and $\sigma(E^t)$ is invariant under rotations about zero. This spectral symmetry is a consequence of the fact that for $\xi \in \mathbb{R}$,

$$E^t e^{i\xi\cdot} f = e^{i\xi\cdot} e^{-i\xi t} E^t f, \quad \text{and} \quad \Gamma e^{i\xi\cdot} = e^{i\xi\cdot}(\Gamma - i\xi). \tag{2.3}$$

The inclusion $e^{t\sigma(\Gamma)} \subseteq \sigma(E^t) \setminus \{0\}$ follows from the standard spectral inclusion for strongly continuous semigroups [27]. In view of the spectral symmetry, it suffices to show that $\sigma(E^t) \cap \mathbb{T} = \emptyset$ whenever $0 \in \rho(\Gamma)$. To this end, we replace the Banach space X in Remarks 1.4 by $C_{00}(\mathbb{R}_+, X)$ and consider two semigroups $\{\tilde{E}^t\}_{t\geq 0}$ and $\{\mathcal{E}^t\}_{t\geq 0}$ with generators $\tilde{\Gamma}$ and $\mathcal{G}$, respectively, acting on the space $C_0(\mathbb{R}, C_{00}(\mathbb{R}_+, X))$. These semigroups are defined by

$$(\tilde{E}^t h)(\tau,\theta) = \begin{cases} U(\theta,\theta - t)h(\tau - t, \theta - t) & \text{for} \quad \theta \geq t, \\ 0 & \text{for} \quad 0 \leq \theta < t, \end{cases}$$

$$(\mathcal{E}^t h)(\tau,\theta) = \begin{cases} U(\theta,\theta - t)h(\tau, \theta - t) & \text{for} \quad \theta \geq t, \\ 0, & \text{for} \quad 0 \leq \theta \leq t, \end{cases}$$

where $\tau \in \mathbb{R}$ and $h(\tau,\cdot) \in C_{00}(\mathbb{R}_+, X)$. Note that if $H \in C_0(\mathbb{R}, C_{00}(\mathbb{R}_+, X))$, then $h(\tau,\cdot) := H(\tau) \in C_{00}(\mathbb{R}_+, X)$ and we recognize $\{\tilde{E}^t\}_{t\geq 0}$ as the evolution semigroup induced by $\{E^t\}_{t\geq 0}$, as in (1.2):

$$(\tilde{E}^t H)(\tau) = E^t H(\tau - t).$$

Also, the semigroup $\{\mathcal{E}^t\}_{t\geq 0}$ is the family of multiplication operators given by

$$(\mathcal{E}^t H)(\tau) = E^t H(\tau).$$

The generator $\mathcal{G}$ of this semigroup is the operator of multiplication by Γ: $(\mathcal{G}H)(\tau) = \Gamma(H(\tau))$, where $H(\tau) \in \text{Dom}(\Gamma)$ for $\tau \in \mathbb{R}$. In particular, if $0 \in \rho(\Gamma)$ on $\mathcal{F}$, then $(\mathcal{G}^{-1}H)(\tau) = \Gamma^{-1}(H(\tau))$, and so $0 \in \rho(\mathcal{G})$.

Let J denote the isometry on $C_0(\mathbb{R}, C_{00}(\mathbb{R}_+, X))$ given by $(Jh)(\tau,\theta) = h(\tau+\theta,\theta)$ for $\tau \in \mathbb{R}$, $\theta \in \mathbb{R}_+$. Then J satisfies the identity:

$$(\mathcal{E}^t Jh)(\tau,\theta) = (J\tilde{E}^t h)(\tau,\theta), \quad \tau \in \mathbb{R}, \quad \theta \in \mathbb{R}_+.$$

It follows that $\mathcal{G}JH = J\tilde{\Gamma}H$ for $H \in \text{Dom}(\tilde{\Gamma})$, and $J^{-1}\mathcal{G}H = \tilde{\Gamma}J^{-1}H$ for $H \in \text{Dom}(\mathcal{G})$. Consequently $\sigma(\mathcal{G}) = \sigma(\tilde{\Gamma})$ on $C_0(\mathbb{R}, C_{00}(\mathbb{R}_+, X))$. In particular, $0 \in \rho(\tilde{\Gamma})$. Therefore, $\sigma(E^t) \cap \mathbb{T} = \emptyset$ follows from Remarks 1.4 applied to the semigroup $\{E^t\}_{t\geq 0}$ on $\mathcal{F}$ in place of $\{e^{tA}\}_{t\geq 0}$ on X.

The facts that $\sigma(\Gamma)$ is a half plane and $\sigma(E^t)$ is a disk follow from the spectral mapping property (2.2) and [38, Proposition 2]. ☐

An important consequence of this theorem is the property that the growth bound $\omega_0(E^t)$ equals the spectral bound $s(\Gamma)$. This leads to the following simple result on stability.

THEOREM 2.2. *Let $\mathcal{F}$ denote $C_{00}(\mathbb{R}_+, X)$ or $L^p(\mathbb{R}_+, X)$. An exponentially bounded evolution family $\{U(t,\tau)\}_{t\geq\tau}$ is exponentially stable if and only if the growth bound $\omega_0(E^t)$ of the induced evolution semigroup on $\mathcal{F}$ is negative.*

Proof. Let $\mathcal{F} = C_{00}(\mathbb{R}_+, X)$. If $\{U(t,\tau)\}_{t\geq\tau}$ is exponentially stable, then there exist $M > 1$, $\beta > 0$ such that $\|U(t,\tau)\|_{\mathcal{L}(X)} \leq Me^{-\beta(t-\tau)}$, $t \geq \tau$. For $\tau \geq 0$ and

$f \in C_{00}(\mathbb{R}_+, X)$,

$$\begin{aligned}\|E^\tau f\|_{C_{00}(\mathbb{R}_+,X)} = \sup_{t>0} \|E^\tau f(t)\|_X &= \sup_{t>\tau} \|U(t, t-\tau) f(t-\tau)\|_X \\ &\le \sup_{t>\tau} \|U(t, t-\tau)\|_{\mathcal{L}(X)} \|f(t-\tau)\|_X \\ &\le M e^{-\beta\tau} \|f\|_{C_{00}(\mathbb{R}_+,X)}.\end{aligned}$$

Conversely, assume there exist $M > 1$, $\alpha > 0$ such that $\|E^t\| \le M e^{-\alpha t}$, $t \ge 0$. Let $x \in X$, $\|x\| = 1$. For fixed $t > \tau > 0$, choose $f \in C_{00}(\mathbb{R}_+, X)$ such that $\|f\|_{C_{00}(\mathbb{R}_+,X)} = 1$ and $f(\tau) = x$. Then,

$$\begin{aligned}\|U(t,\tau)x\|_X = \|U(t,\tau) f(\tau)\|_X &= \|E^{(t-\tau)} f(t)\|_X \\ &\le \sup_{\theta>0} \|E^{(t-\tau)} f(\theta)\|_X \\ &= \|E^{(t-\tau)} f\|_{C_{00}(\mathbb{R}_+,X)} \\ &\le M e^{-\alpha(t-\tau)}.\end{aligned}$$

A similar argument works for $\mathcal{F} = L^p(\mathbb{R}_+, X)$. □

The remainder of this subsection focuses on the operator used for determining exponential stability. In fact, stability is characterized by the boundedness of this operator which, as seen below, is equivalent to the invertibility of Γ, the generator of the evolution semigroup. We begin with the autonomous case.

R. Datko and J. van Neerven have characterized the exponential stability of solutions for autonomous equations $\dot{x} = Ax$, $t \ge 0$, in terms of a convolution operator, $\mathbb{G}$, induced by $\{e^{tA}\}_{t\ge 0}$. In this autonomous setting,

$$(E^t f)(\tau) = \begin{cases} e^{tA} f(\tau - t), & 0 \le t \le \tau \\ 0, & 0 \le \tau < t, \end{cases} \tag{2.4}$$

and the convolution operator takes the following form: for $f \in L^1_{loc}(\mathbb{R}_+, X)$,

$$(\mathbb{G}f)(t) := \int_0^t e^{\tau A} f(t-\tau)\, d\tau = \int_0^\infty (E^\tau f)(t)\, d\tau, \quad t \ge 0. \tag{2.5}$$

For reader's convenience we cite Theorem 1.3 of [28] (see also [12]) in the following remarks.

Remarks 2.3. *If $\{e^{tA}\}_{t\ge 0}$ is a strongly continuous semigroup on X, and $1 \le p < \infty$, then the following are equivalent:*

(i) $\omega_0(e^{tA}) < 0$;

(ii) $\mathbb{G}f \in L^p(\mathbb{R}_+, X)$ *for all* $f \in L^p(\mathbb{R}_+, X)$;

(iii) $\mathbb{G}f \in C_0(\mathbb{R}_+, X)$ *for all* $f \in C_0(\mathbb{R}_+, X)$.

Remarks 2.4.

(a) *Note that condition (ii) is equivalent to the boundedness of* $\mathbb{G}$ *on* $L^p(\mathbb{R}_+, X)$. *To see this, it suffices to show that the map* $f \mapsto \mathbb{G}f$ *is a closed operator on* $L^p(\mathbb{R}_+, X)$*, and then apply the closed graph theorem. For this, let* $f_n \to f$ *and* $\mathbb{G}f_n \to g$ *in* $L^p(\mathbb{R}_+, X)$. *Then* $(\mathbb{G}f_n)(t) \to (\mathbb{G}f)(t)$ *for each* $t \in \mathbb{R}$. *Also, every norm-convergent sequence in* $L^p(\mathbb{R}_+, X)$ *contains a subsequence that converges pointwise almost everywhere. Thus,* $(\mathbb{G}f_{n_k})(t) \to g(t)$ *for almost all* t. *This implies that* $\mathbb{G}f = g$*, as claimed.*

(b) *Also, condition (iii) is equivalent to the boundedness of* $\mathbb{G}$ *on* $C_0(\mathbb{R}_+, X)$. *This follows from the uniform boundedness principle applied to the operators* $\mathbb{G}_t : f \mapsto \int_0^t e^{\tau A} f(t-\tau)\, d\tau$.

We now extend this result so that it may be used to describe exponential stability for a nonautonomous equation. For this define an operator $\mathbb{G}$ in an analogous way: let $\{U(t,\tau)\}_{t\geq\tau}$ be an evolution family and $\{E^t\}_{t\geq 0}$ the evolution semigroup in (2.1). Then define $\mathbb{G}$ for $f \in L^1_{loc}(\mathbb{R}_+, X)$ as

$$\begin{aligned}(\mathbb{G}f)(t) &:= \int_0^\infty (E^\tau f)(t)\, d\tau = \int_0^t U(t, t-\tau) f(t-\tau)\, d\tau \\ &= \int_0^t U(t,\tau) f(\tau)\, d\tau, \qquad t \geq 0.\end{aligned} \tag{2.6}$$

For $\mathbb{G}$ acting on $\mathcal{F} = C_{00}(\mathbb{R}_+, X)$ or $L^p(\mathbb{R}_+, X)$, standard semigroup properties show that $\mathbb{G}$ equals $-\Gamma^{-1}$ provided the semigroup $\{E^t\}_{t\geq 0}$ or the evolution family is uniformly exponentially stable. Parts (i) $\Leftrightarrow$ (ii) of Remarks 2.3 and the nonautonomous version below are the classical results by R. Datko [12]. Our proof uses the evolution semigroup and creates a formally autonomous problem so that Remarks 2.3 can be applied.

THEOREM 2.5. *The following are equivalent for the evolution family of operators* $\{U(t,\tau)\}_{t\geq\tau}$ *on* X.

(i) $\{U(t,\tau)\}_{t\geq\tau}$ *is exponentially stable;*
(ii) $\mathbb{G}$ *is a bounded operator on* $L^p(\mathbb{R}_+, X)$*;*
(iii) $\mathbb{G}$ *is a bounded operator on* $C_0(\mathbb{R}_+, X)$.

Before proceeding with the proof, note that statement (ii) is equivalent to the statement: $\mathbb{G}f \in L^p(\mathbb{R}_+, X)$ for each $f \in L^p(\mathbb{R}_+, X)$. This is seen as in Remark 2.4, above. See also [5] for similar facts.

Proof. By Theorem 2.2, (i) implies that $\{E^t\}_{t\geq 0}$ is exponentially stable, and formula (2.6) implies (ii) and (iii). The implication (ii)$\Rightarrow$(i) will be proved here; the argument for (iii)$\Rightarrow$(i) is similar. The main idea is again to use the "change-of-variables" technique, as in the proof of Theorem 2.1.

Consider the operator $\tilde{\mathbb{G}}$ on $L^p(\mathbb{R}, L^p(\mathbb{R}_+, X)) = L^p(\mathbb{R}\times\mathbb{R}_+, X)$ defined as multiplication by $\mathbb{G}$. More precisely, for $h \in L^p(\mathbb{R}\times\mathbb{R}_+, X)$ with $\mathbf{h}(\theta) := h(\theta, \cdot) \in L^p(\mathbb{R}_+, X)$, define

$$(\tilde{\mathbb{G}}h)(\theta, t) = \mathbb{G}(\mathbf{h}(\theta))(t) = \int_0^t U(t, t-\tau) h(\theta, t-\tau)\, d\tau, \quad t \in \mathbb{R}_+, \quad \theta \in \mathbb{R}.$$

In view of statement (ii), this operator is bounded. For the isometry J defined on the space $L^p(\mathbb{R}, L^p(\mathbb{R}_+, X))$ by $(Jh)(\theta, t) = h(\theta + t, t)$, we have

$$(J^{-1}\tilde{\mathbb{G}}Jh)(\theta, t) = \int_0^t U(t, t-\tau) h(\theta - \tau, t-\tau)\, d\tau. \tag{2.7}$$

Next, let $\{E^t\}_{t\geq 0}$ be the evolution semigroup (2.1) induced by $\{U(t,\tau)\}_{t\geq\tau}$, and define $\mathbb{G}_*$ to be the operator of convolution with this semigroup as in (2.5); that is,

$$(\mathbb{G}_*\mathbf{h})(\theta) = \int_0^\infty E^\tau \mathbf{h}(\theta - \tau)\, d\tau, \quad \mathbf{h} \in L^p(\mathbb{R}, L^p(\mathbb{R}_+, X)). \tag{2.8}$$

If $h(\theta,\cdot)=\mathbf{h}(\theta)\in L^p(\mathbb{R}_+,X)$, then by definition (2.1), evaluating (2.8) at t gives

(2.9)
$$[(\mathbb{G}_*\mathbf{h})(\theta)]\,(t)=(\mathbb{G}_*h)(\theta,t)=\int_0^t U(t,t-\tau)h(\theta-\tau,t-\tau)\,d\tau,\quad t\in\mathbb{R}_+,\ \theta\in\mathbb{R}.$$

From (2.7) it follows that $\mathbb{G}_*=J^{-1}\tilde{\mathbb{G}}J$ is a bounded operator on $L^p(\mathbb{R},L^p(\mathbb{R}_+,X))$.

Now, each function $\mathbf{h}_+\in L^p(\mathbb{R}_+,L^p(\mathbb{R}_+,X))$ is an $L^p(\mathbb{R}_+,X)$-valued function on the half line $\mathbb{R}_+$. We extend each such $\mathbf{h}_+$ to a function $\mathbf{h}\in L^p(\mathbb{R},L^p(\mathbb{R}_+,X))$ by setting $\mathbf{h}(\theta)=\mathbf{h}_+(\theta)$ for $\theta\geq 0$ and $\mathbf{h}(\theta)=0$ for $\theta<0$. Note that $\mathbb{G}_*\mathbf{h}\in L^p(\mathbb{R},L^p(\mathbb{R}_+,X))$ because $\mathbb{G}_*$ is bounded on $L^p(\mathbb{R},L^p(\mathbb{R}_+,X))$. Consider the function $\mathbf{f}_+:\mathbb{R}_+\to L^p(\mathbb{R}_+,X)$ defined by

$$\mathbf{f}_+(t)=\int_0^t E^\tau\mathbf{h}_+(t-\tau)\,d\tau=\int_0^\infty E^\tau\mathbf{h}(t-\tau)\,d\tau,\quad t\in\mathbb{R}_+.$$

To complete the proof of the theorem, it suffices to prove the following claim:

$$\mathbf{f}_+\in L^p(\mathbb{R}_+,L^p(\mathbb{R}_+,X)).$$

Indeed, the operator $\mathbf{h}_+\mapsto\mathbf{f}_+$ is the convolution operator as in (2.5) defined by the semigroup operators E^t instead of e^{tA}. An application of Remarks 2.3 to E^t on $L^p(\mathbb{R}_+,X)$ (in place of e^{tA} on X) shows that the semigroup $\{E^t\}_{t\geq 0}$ is exponentially stable on $L^p(\mathbb{R}_+,X)$ provided:

$$\mathbf{f}_+\in L^p(\mathbb{R}_+,L^p(\mathbb{R}_+,X))\quad\text{for each }\mathbf{h}_+\in L^p(\mathbb{R}_+,L^p(\mathbb{R}_+,X)).$$

But if $\{E^t\}_{t\geq 0}$ is exponentially stable, the evolution family $\{U(t,\tau)\}_{t\geq\tau}$ is exponentially stable by Theorem 2.2.

To prove the claim, apply formula (2.9) for $h(\theta,t)=h_+(\theta,t),\theta\geq 0$ and $h(\theta,t)=0$, $\theta<0$, $t\in\mathbb{R}_+$, where $\mathbf{h}_+(\theta)=h_+(\theta,\cdot)$. This gives

$$(\mathbb{G}_*h)(\theta,t)=\begin{cases}\int_0^{\min\{\theta,t\}}U(t,t-\tau)h_+(\theta-\tau,t-\tau)\,d\tau & \text{for}\quad\theta\geq 0,\ t\in\mathbb{R}_+\\ (\mathbb{G}_*h)(\theta,t)=0, & \text{for}\quad\theta<0,\ t\in\mathbb{R}_+\end{cases}$$

Thus, the function

$$\theta\mapsto(\mathbb{G}_*h)(\theta,\cdot)=(\mathbb{G}_*\mathbf{h})(\theta)\in L^p(\mathbb{R}_+,X)$$

is in the space $L^p(\mathbb{R}_+,L^p(\mathbb{R}_+,X))$. On the other hand, denoting $f_+(\theta,\cdot):=\mathbf{f}_+(\theta)\in L^p(\mathbb{R}_+,X)$, we have that

$$f_+(\theta,t)=\int_0^{\min\{\theta,t\}}U(t,t-\tau)h_+(\theta-\tau,t-\tau)\,d\tau,\quad\theta,t\in\mathbb{R}_+.$$

Thus, $\theta\mapsto f_+(\theta,\cdot)=(\mathbb{G}_*h)(\theta,\cdot)$ is a function in $L^p(\mathbb{R}_+,L^p(\mathbb{R}_+,X))$, and the claim is proved. □

This theorem makes explicit, in the case of the half line $\mathbb{R}_+$, the relationship between the stability of an evolution family $\{U(t,\tau)\}_{t\geq\tau}$ and the generator, Γ, of the corresponding evolution semigroup (2.1). Indeed, as shown above, stability is equivalent to the boundedness of $\mathbb{G}$, in which case $\mathbb{G}=-\Gamma^{-1}$. Combining Theorems 2.1, 2.2 and 2.5 yields the following corollary.

Corollary 2.6. *Let $\{U(t,\tau)\}_{t\geq\tau}$ be an exponentially bounded evolution family and let Γ denote the generator of the induced evolution semigroup on $L^p(\mathbb{R}_+,X)$, $1\leq p<\infty$, or $C_{00}(\mathbb{R}_+,X)$. The following are equivalent:*

(i) $\{U(t,\tau)\}_{t\geq\tau}$ *is exponentially stable;*
(ii) Γ *is invertible with* $\Gamma^{-1} = -\mathbb{G}$*;*
(iii) $s(\Gamma) < 0$.

For more information on stability and dichotomy of evolution families on the semiaxis see [26].

2.2. Perturbations and robust stability. This subsection briefly considers perturbations of (1.1) of the form

$$\dot{x}(t) = (A(t) + D(t))x(t), \quad t \geq 0. \tag{2.10}$$

It will not, however, be assumed that (2.10) has a differentiable solution. For example, let $\{e^{tA_0}\}_{t\geq 0}$ be a strongly continuous semigroup generated by A_0, let $A_1(t) \in \mathcal{L}(X)$ for $t \geq 0$, and define $A(t) = A_0 + A_1(t)$. Then even if $t \mapsto A_1(t)$ is continuous, the Cauchy problem (1.1) may not have a differentiable solution for all initial conditions $x(0) = x \in \mathrm{Dom}(A) = \mathrm{Dom}(A_0)$ (see, e.g.,[31]). Therefore we will want our development to allow for equations with solutions that exist only in the following mild sense.

Let $\{U(t,\tau)\}_{t\geq\tau}$ be an evolution family of operators corresponding to a solution of (1.1), and consider the nonautonomous inhomogeneous equation

$$\dot{x}(t) = A(t)x(t) + f(t), \qquad t \geq 0, \tag{2.11}$$

where f is a locally integrable X-valued function on $\mathbb{R}_+$. A function $x(\cdot)$ is a mild solution of (2.11) with initial value $x(\theta) = x_\theta \in \mathrm{Dom}(A(\theta))$ if

$$x(t) = U(t,\theta)x_\theta + \int_\theta^t U(t,\tau)f(\tau)\,d\tau, \quad t \geq \theta.$$

Given operators $D(t)$, the existence of mild solutions to an additively perturbed equation (2.10) corresponds to the existence of an evolution family $\{U_1(t,\tau)\}_{t\geq\tau}$ satisfying

$$U_1(t,\theta)x = U(t,\theta)x + \int_\theta^t U(t,\tau)D(\tau)U_1(\tau,\theta)x\,d\tau. \tag{2.12}$$

for all $x \in X$. It will be assumed that the perturbation operators, $D(t)$, are strongly measurable and essentially bounded functions of t. In view of this, we use the notation $\mathcal{L}_s(X)$ to denote the set $\mathcal{L}(X)$ endowed with the strong operator topology and use $L^\infty(\mathbb{R}_+, \mathcal{L}_s(X))$ to denote the set of bounded, strongly measurable $\mathcal{L}(X)$-valued functions on $\mathbb{R}_+$. A function $D(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(X))$ induces a multiplication operator $\mathcal{D}$ defined by $\mathcal{D}x(t) = D(t)x(t)$, for $x(\cdot) \in L^p(\mathbb{R}_+, X)$. In fact, $\mathcal{D}$ is a bounded operator on $L^p(\mathbb{R}_+, X)$ with $\|\mathcal{D}\| \leq \|D(\cdot)\|_\infty := \operatorname{ess\,sup}_{t\in\mathbb{R}_+} \|D(t)\|$.

Evolution semigroups induced by an evolution family as in (2.1) have been studied by several authors who have characterized such semigroups in terms of their generators on general Banach function spaces of X-valued functions (see [36, 42] and the bibliography therein). The sets $\mathcal{F} = L^p(\mathbb{R}_+, X)$ or $\mathcal{F} = C_{00}(\mathbb{R}_+, X)$ considered here are examples of more general "Banach function spaces." In the development that follows we use a theorem of R. Schnaubelt [42] (see also Räbiger et al. [35, 36]) which shows exactly when a strongly continuous semigroup on $\mathcal{F}$ arises from a strongly continuous evolution family on X. We state a version of this result which will be used below; a more general version is proven in [36]. The set $C_c^1(\mathbb{R}_+)$ consists of differentiable functions on $\mathbb{R}_+$ that have compact support.

THEOREM 2.7. *Let* $\{T^t\}_{t\geq 0}$ *be a strongly continuous semigroup generated by* Γ *on* $\mathcal{F}$*. The following are equivalent:*

(i) $\{T^t\}_{t\geq 0}$ *is an evolution semigroup; i.e., there exists an exponentially bounded evolution family so that* T^t *is defined as in* (2.1)*;*

(ii) there exists a core, $\mathcal{C}$*, of* Γ *such that for all* $\varphi \in C_c^1(\mathbb{R}_+)$*, and* $f \in \mathcal{C}$*, it follows that* $\varphi f \in \mathrm{Dom}(\Gamma)$ *and* $\Gamma(\varphi f) = -\varphi' f + \varphi \Gamma f$*. Moreover, there exists* $\lambda \in \rho(\Gamma)$ *such that* $R(\lambda, \Gamma) : \mathcal{F} \to C_{00}(\mathbb{R}_+, X)$ *is continuous with dense range.*

Now let $\{U(t,\tau)\}_{t\geq\tau}$ be an evolution family on X and let Γ be the generator of the corresponding evolution semigroup, $\{E^t\}_{t\geq 0}$, as in (2.1). If $D(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(X))$, then the multiplication operator $\mathcal{D}$ is a bounded operator on $\mathcal{F} = L^p(\mathbb{R}_+, X)$. Since a bounded perturbation of a generator of a strongly continuous semigroup is itself such a generator, the operator $\Gamma_1 = \Gamma + \mathcal{D}$ generates a strongly continuous semigroup, $\{E_1^t\}_{t\geq 0}$ on $\mathcal{F}$ (see, e.g., [30]). In fact, Γ_1 generates an *evolution* semigroup, see [35, 42]:

PROPOSITION 2.8. *Let* $D(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(X))$*., and let* $\{U(t,\tau)\}_{t\geq\tau}$ *be an exponentially bounded evolution family. Then there exists a unique evolution family* $\mathcal{U}_1 = \{U_1(t,\tau)\}_{t\geq\tau}$ *which solves the integral equation* (2.12)*. Moreover,* $\mathcal{U}_1$ *is exponentially stable if and only if* $\Gamma + \mathcal{D}$ *is invertible.*

Proof. As already observed, $\Gamma_1 = \Gamma + \mathcal{D}$ generates a strongly continuous semigroup, $\{E_1^t\}_{t\geq 0}$ on $\mathcal{F}$. To see that this is, in fact, an evolution semigroup, note that for $\lambda \in \rho(\Gamma) \cap \rho(\Gamma_1)$,

$$\mathrm{Range}(R(\lambda,\Gamma)) = \mathrm{Dom}(\Gamma) = \mathrm{Dom}(\Gamma + \mathcal{D}) = \mathrm{Range}(R(\lambda, \Gamma_1))$$

is dense in $C_{00}(\mathbb{R}_+, X)$. Also, if $\mathcal{C}$ is a core for Γ, then it is a core for Γ_1, and so for $\varphi \in C_c^1(\mathbb{R})$, $f \in \mathcal{C}$,

$$\Gamma_1(\varphi f) = \Gamma(\varphi f) + \mathcal{D}(\varphi f) = -\varphi' f + \varphi \Gamma f + \varphi \mathcal{D} f = -\varphi' f + \varphi(\Gamma + \mathcal{D}) f.$$

Consequently, Corollary 2.7 shows that $\{E_1^t\}_{t\geq 0}$ corresponds to an evolutionary family, $\{U_1(t,\tau)\}_{t\geq\tau}$. Moreover, $x(t) = U_1(t,\tau)x(\tau)$ is seen to define a mild solution to (2.10). Indeed,

$$E_1^t f = E^t f + \int_0^t E^{(t-\tau)} \mathcal{D} E_1^\tau f \, d\tau, \tag{2.13}$$

holds for all $f \in F$. In particular, for $x \in X$, and any $\varphi \in C_c^1(\mathbb{R})$, setting $f = \varphi \otimes x$ in (2.13), where $\varphi \otimes x(t) = \varphi(t)x$, and using a change of variables leads to

$$\varphi(\theta) U_1(t,\theta) x = \varphi(\theta) U(t,\theta) x + \varphi(\theta) \int_\theta^t U(t,\tau) D(\tau) U_1(\tau,\theta) x \, d\tau.$$

Therefore, (2.12) holds for all $x \in X$.

Finally, Theorem 2.6 shows that $\mathcal{U}_1$ is exponentially stable if and only if Γ_1 is invertible. ☐

The existence of mild solutions under bounded perturbations of this type is well known (see, e.g., [9]), but an immediate consequence of the approach given here is the property of robustness for the stability of $\{U(t,\tau)\}_{t\geq\tau}$. Indeed, by continuity properties of the spectrum of an operator Γ, there exists $\epsilon > 0$ such that Γ_1 is invertible whenever $\|\Gamma_1 - \Gamma\| < \epsilon$; that is, $\{U_1(t,\tau)\}_{t\geq\tau}$ is exponentially stable whenever $\|D(\cdot)\|_\infty < \epsilon$. Also, the type of proof presented here can be extended to address the case of unbounded perturbations. For an example of this, we refer to [36]. Finally, and most important to the present paper, is the fact that this approach provides insight into the concept of the stability radius. This topic is studied next.

3. Stability Radius. The goal of this section is to use the previous development to study the (complex) stability radius of an exponentially stable system. Loosely speaking, this is a measurement on the size of the smallest operator under which the additively perturbed system looses exponential stability. This is an important concept for linear systems theory and was introduced by D. Hinrichsen and A. J. Pritchard as the basis for a state-space approach to studying robustness of linear time-invariant [16] and time-varying systems [15, 17, 32]. A systematic study of various stability radii in the spirit of the current paper has recently be given by A. Fischer and J. van Neerven [13].

3.1. General estimates. In this subsection we give estimates for the stability radius of general nonautonomous systems on Banach spaces. The perturbations considered here are additive "structured" perturbations of output feedback type. That is, let U and Y be Banach spaces and let $\Delta(t) : Y \to U$ denote an unknown disturbance operator. The operators $B(t) : U \to X$ and $C(t) : X \to Y$ describe the structure of the perturbation in the following (formal) sense: if $u(t) = \Delta(t)y(t)$ is viewed as a feedback for the system

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(s) = x_s \in \mathrm{Dom}(A(s)),\\ &y(t) = C(t)x(t), \quad t \geq s \geq 0, \end{aligned} \tag{3.1}$$

then the nominal system $\dot{x}(t) = A(t)x(t)$ is subject to the structured perturbation:

$$\dot{x}(t) = (A(t) + B(t)\Delta(t)C(t))x(t), \quad t \geq 0. \tag{3.2}$$

In this section B and C do not represent input and output operators, rather they describe the structure of the uncertainty of the system. Also, systems considered throughout this paper are not assumed to have differentiable solutions and so (3.2) is to be interpreted in the mild sense as described in (2.12) where $D(t) = B(t)\Delta(t)C(t)$. Similarly, (3.1) is interpreted in the mild sense; that is, there exists a strongly continuous exponentially bounded evolution family $\{U(t,\tau)\}_{t\geq\tau}$ on a Banach space X which satisfies

$$\begin{aligned} &x(t) = U(t,s)x(s) + \int_s^t U(t,\tau)B(\tau)u(\tau)\,d\tau,\\ &y(t) = C(t)x(t), \qquad t \geq s \geq 0. \end{aligned} \tag{3.3}$$

In the case of time-invariant systems, equation (3.3) takes the form

$$\begin{aligned} &x(t) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau,\\ &y(t) = Cx(t), \qquad t \geq 0, \end{aligned} \tag{3.4}$$

where $\{e^{tA}\}_{t\geq 0}$ is a strongly continuous semigroup on X generated by A, $x(0) = x_0 \in \mathrm{Dom}(A)$.

It should be emphasized that we will not address questions concerning the existence of solutions for a perturbed system (3.2) beyond the point already discussed in Proposition 2.8. In view of that proposition, we make the following assumptions: B, C and Δ are strongly measurable and essentially bounded functions of t; i.e., $B(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(U,X))$, $C(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(X,Y))$ and $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y,U))$. As such, they induce bounded multiplication operators, $\mathcal{B}$, $\mathcal{C}$ and $\tilde{\Delta}$ acting on the spaces $L^p(\mathbb{R}_+, U)$, $L^p(\mathbb{R}_+, X)$ and $L^p(\mathbb{R}_+, Y)$, respectively.

Next, for an exponentially bounded evolution family $\{U(t,\tau)\}_{t\geq\tau}$, define the "input-output" operator $\mathbb{L}$ on functions $u:\mathbb{R}_+\to U$ by the rule

$$(\mathbb{L}u)(t)=C(t)\int_0^t U(t,\tau)B(\tau)u(\tau)\,d\tau.$$

Using the above notation note that $\mathbb{L}=\mathcal{C}\mathbb{G}\mathcal{B}$. Much of the stability analysis that follows is based on this observation in combination with Theorem 2.5 which shows that the operator $\mathbb{G}$ completely characterizes stability of the corresponding evolution family.

We now turn to the definition of the stability radius. For this let $\mathcal{U}=\{U(t,\tau)\}_{t\geq\tau}$ be an exponentially stable evolution family on X. Set $\mathcal{D}=\mathcal{B}\tilde{\Delta}\mathcal{C}$ and let $\mathcal{U}_\Delta=\{U_\Delta(t,\tau)\}_{t\geq\tau}$ denote the evolution family corresponding to solutions of the perturbed equation (2.10). That is, $\mathcal{U}_\Delta$ satisfies

$$U_\Delta(t,s)x=U(t,s)x+\int_s^t U(t,\tau)B(\tau)\Delta(\tau)C(\tau)U_\Delta(\tau,s)x\,d\tau,\qquad x\in X.$$

Define the (complex) *stability radius* for $\mathcal{U}$ with respect to the perturbation structure $(B(\cdot),C(\cdot))$ as the quantity

$$r_{stab}(\mathcal{U},B,C)=\sup\{r\geq 0:\|\Delta(\cdot)\|_\infty\leq r\Rightarrow\mathcal{U}_\Delta\text{ is exponentially stable}\}.$$

This definition applies to both nonautonomous and autonomous systems, though in the latter case the notation $r_{stab}(\{e^{tA}\},B,C))$ will be used to distinguish the case where all the operators except $\Delta(t)$ are independent of t. We will have occasion to consider the *constant stability radius* which is defined for the case in which $\Delta(t)\equiv\Delta$ is constant; this will be denoted by $rc_{stab}(\{e^{tA}\},B,C))$ or $rc_{stab}(\mathcal{U},B,C))$, depending on the context. The above remarks concerning $\Gamma+\mathcal{D}$ (see Proposition 2.8), when combined with Theorem 2.2, make it clear that

$$r_{stab}(\mathcal{U},B,C)=\sup\{r\geq 0:\|\Delta(\cdot)\|_\infty\leq r\Rightarrow\Gamma+\mathcal{B}\tilde{\Delta}\mathcal{C}\text{ is invertible}\}. \tag{3.5}$$

It is well known that for autonomous systems in which U and Y are Hilbert spaces and $p=2$, the stability radius may be expressed in terms of the norm of the input-output operator or the transfer function:

$$\frac{1}{\|\mathbb{L}\|_{\mathcal{L}(L^2)}}=r_{stab}(\{e^{tA}\},B,C)=\frac{1}{\sup_{s\in\mathbb{R}}\|C(A-is)^{-1}B\|}; \tag{3.6}$$

see, e.g., [17, Theorem 3.5]. For nonautonomous equations, a scalar example given in Example 4.4 of [15] shows that, in general, a strict inequality $1/\|\mathbb{L}\|<r_{stab}(\mathcal{U},B,C)$ may hold. Moreover, even for autonomous systems, when Banach spaces are allowed or when $p\neq 2$, Example 3.13 and Example 3.15 below will show that neither of the equalities in (3.6) necessarily hold. Subsection 3.3 below focuses on autonomous equations and a primary objective there is to prove the following result.

THEOREM 3.1. *For the general autonomous systems,*

$$\frac{1}{\|\mathbb{L}\|_{\mathcal{L}(L^p)}}\leq r_{stab}(\{e^{tA}\},B,C)\leq\frac{1}{\sup_{s\in\mathbb{R}}\|C(A-is)^{-1}B\|},\quad 1\leq p<\infty. \tag{3.7}$$

As seen next, the lower bound here holds for general nonautonomous systems and may be proven in a very direct way using the make-up of the operator $\mathbb{L}=\mathcal{C}\mathbb{G}\mathcal{B}$. This lower bound is also proven in [17, Theorem 3.2] using a completely different approach.

THEOREM 3.2. *Assume $\mathcal{U}$ is an exponentially stable evolution family and let Γ denote the generator of the corresponding evolution semigroup. If*

$$B(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(U,X)) \quad and \quad C(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(X,Y)),$$

then $\mathbb{L}$ is a bounded operator from $L^p(\mathbb{R}_+, U)$ to $L^p(\mathbb{R}_+, Y)$, $1 \le p < \infty$, the formula

$$\mathbb{L} = \mathcal{C}\mathbb{G}\mathcal{B} = -\mathcal{C}\Gamma^{-1}\mathcal{B}$$

holds, and

$$\frac{1}{\|\mathbb{L}\|} \le r_{stab}(\mathcal{U}, B, C). \tag{3.8}$$

In the "unstructured" case, where $U = Y = X$ and $B = C = I$, one has

$$\mathbb{L} = -\Gamma^{-1}, \qquad and \qquad \frac{1}{\|\Gamma^{-1}\|} \le r_{stab}(\mathcal{U}, I, I) \le \frac{1}{r(\Gamma^{-1})},$$

where $r(\cdot)$ denotes the spectral radius.

Proof. Since $\mathcal{U}$ is exponentially stable, Γ is invertible and $\Gamma^{-1} = -\mathbb{G}$. The required formula for $\mathbb{L}$ follows from (2.6).

Set $\mathcal{H} := \Gamma^{-1}\mathcal{B}\tilde{\Delta}$. To prove (3.8), let $\Delta(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y,U))$ and suppose that $\|\Delta(\cdot)\|_\infty < 1/\|\mathbb{L}\|$. Then $\|\mathbb{L}\tilde{\Delta}\| < 1$, and hence $I - \mathbb{L}\tilde{\Delta} = I + \mathcal{C}\Gamma^{-1}\mathcal{B}\tilde{\Delta}$ is invertible on $L^p(\mathbb{R}_+, Y)$. That is, $I + \mathcal{C}\mathcal{H}$ is invertible on $L^p(\mathbb{R}_+, Y)$, and hence $I + \mathcal{H}\mathcal{C}$ is invertible on $L^p(\mathbb{R}_+, X)$ (with inverse $(I - \mathcal{H}(I + \mathcal{C}\mathcal{H})^{-1}\mathcal{C})$). Now,

$$\Gamma + \mathcal{B}\tilde{\Delta}\mathcal{C} = \Gamma(I + \Gamma^{-1}\mathcal{B}\tilde{\Delta}\mathcal{C}) = \Gamma(I + \mathcal{H}\mathcal{C})$$

and so $\Gamma + \mathcal{B}\tilde{\Delta}\mathcal{C}$ is invertible. It follows from the expression (3.5) that $1/\|\mathbb{L}\| \le r_{stab}(\mathcal{U}, B, C)$.

For the last assertion, suppose that $r_{stab}(\mathcal{U}, I, I) > 1/r(\Gamma^{-1})$. Then there exists λ such that $|\lambda| = r(\Gamma^{-1})$ and $\lambda + \Gamma^{-1}$ is not invertible. But then setting $\tilde{\Delta} \equiv \frac{1}{\lambda}$ gives $\|\tilde{\Delta}\| = \frac{1}{|\lambda|} < r_{stab}(\mathcal{U}, I, I)$, and so $\Gamma + \tilde{\Delta} = \tilde{\Delta}(\lambda + \Gamma^{-1})\Gamma$ is invertible, a contradiction. ∎

3.2. The transfer function for nonautonomous systems. In this subsection we consider a time-varying version of equation (3.6) and then observe that the concept of a transfer function, or frequency-response function, arises naturally from these ideas. For this we assume in this subsection that X, U and Y are Hilbert spaces and $p = 2$.

Let $\{U(t,\tau)\}_{t \ge \tau}$ be an uniformly exponentially stable evolution family and let $\{E^t\}_{t \ge 0}$ be the induced evolution semigroup with generator Γ on $L^2(\mathbb{R}_+, X)$. Recall that $\mathcal{B}$ and $\mathcal{C}$ denote multiplication operators, with respective multipliers $B(\cdot)$ and $C(\cdot)$, that act on the spaces $L^2(\mathbb{R}_+, U)$ and $L^2(\mathbb{R}_+, X)$, respectively. Let $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ denote operators of multiplication induced by $\mathcal{B}$ and $\mathcal{C}$, respectively; e.g., $(\tilde{\mathcal{B}}\mathbf{u})(t) = \mathcal{B}(\mathbf{u}(t))$, for $\mathbf{u} : \mathbb{R}_+ \to L^2(\mathbb{R}_+, U)$. Now consider the operator $\mathbb{G}_*$ as defined in equation (2.8) and note that operator $\mathbb{L}_* := \tilde{\mathcal{C}}\mathbb{G}_*\tilde{\mathcal{B}}$ may be viewed (formally) as an input-output operator for the "autonomized" system: $\dot{f} = \Gamma f + \mathcal{B}u$, $g = \mathcal{C}f$, where the state space is $L^2(\mathbb{R}_+, X)$. It follows from the known Hilbert-space equalities in (3.6) that

$$\frac{1}{\|\mathbb{L}_*\|} = r_{stab}(\{E^t\}, \mathcal{B}, \mathcal{C}) = \frac{1}{\sup_{s \in \mathbb{R}} \|\mathcal{C}(\Gamma - is)^{-1}\mathcal{B}\|}.$$

Note, however, that the rescaling identities (2.3) for Γ imply that

$$\|\mathbb{L}_*\| = \|\mathcal{C}(\Gamma - is)^{-1}\mathcal{B}\| = \|\mathcal{C}\Gamma^{-1}\mathcal{B}\| = \|\mathbb{L}\|,$$

and so the stability radius for the evolution semigroup is also $1/\|\mathbb{L}\|$. In view of the above-mentioned nonautonomous scalar example for which $1/\|\mathbb{L}\| < r_{stab}(\mathcal{U}, B, C)$ we see that even though the evolution semigroup (or its generator) completely determines the exponential stability of a system, it does not provide a formula for the stability radius.

However, the operator $\mathcal{C}(\Gamma - is)^{-1}\mathcal{B}$ appearing above suggests that the transfer function for time-varying systems arises naturally when viewed in the context of evolution semigroups. Several authors have considered the concept of a tranasfer function for nonautonomous systems but the work of J. Ball, I. Gohberg, and M.A. Kaashoek [3] seems to be the most comprehensive in providing a system-theoretic input-output interpretation for the value of such a transfer function at a point. Their interpretation justifies the term *frequency response* function for time-varying finite-dimensional systems with "time-varying complex exponential inputs." Our remarks concerning the frequency response for time-varying infinite-dimensional systems will be restricted to inputs of the form $u(t) = u_0 e^{\lambda t}$.

For motivation, consider the input-output operator $\mathbb{L}$ associated with an autonomous system (3.4) where the nominal system is exponentially stable. The *transfer function* of $\mathbb{L}$ is the unique bounded analytic $\mathcal{L}(U, Y)$-valued function, H, defined on $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \text{Re } \lambda > 0\}$ such that for any $u \in L^2(\mathbb{R}_+, U)$,

$$(\widehat{\mathbb{L}u})(\lambda) = H(\lambda)\hat{u}(\lambda), \qquad \lambda \in \mathbb{C}_+,$$

where $\widehat{\ }$ denotes the Laplace transform (see, e.g., [44]). In this autonomous setting, A generates a uniformly exponentially stable strongly continuous semigroup, and $\mathbb{L} = \mathcal{C}\mathbb{G}\mathcal{B}$ where $\mathbb{G}$ is the operator of convolution with the semigroup operators e^{tA} (see (2.5)). Standard arguments show that $(\widehat{\mathbb{L}u})(\lambda) = C(\lambda - A)^{-1}B\hat{u}(\lambda)$; that is, $H(\lambda) = C(\lambda - A)^{-1}B$.

Now let $\mathbb{L}$ be the input-output operator for the nonautonomous system (3.3). We wish to identify the transfer function of $\mathbb{L}$ as the Laplace transform of the appropriate operator. We are guided by the fact that, just as $(\lambda - A)^{-1}$ may be expressed as the Laplace transform of the semigroup generated by A, the operator $(\lambda - \Gamma)^{-1}$ is the Laplace transform of the evolution semigroup. For nonautonomous systems, $\mathbb{L}$ is again given by $\mathcal{C}\mathbb{G}\mathcal{B}$, although now $\mathbb{G}$ from (2.6) is not, generally, a convolution operator. So instead recall the operator $\mathbb{G}_*$ from (2.8) which *is* the operator of convolution with the evolution semigroup $\{E^t\}_{t\geq 0}$. As noted above, the operator $\mathbb{L}_* := \tilde{\mathcal{C}}\mathbb{G}_*\tilde{\mathcal{B}}$ may be viewed as an input-output operator for an autonomous system (where the state space is $L^2(\mathbb{R}_+, X)$). Therefore, the autonomous theory applies directly to show that, for $\mathbf{u} \in L^2(\mathbb{R}_+, L^2(\mathbb{R}_+, U))$,

$$(\widehat{\mathbb{L}_*\mathbf{u}})(\lambda) = \mathcal{C}(\lambda - \Gamma)^{-1}\mathcal{B}\hat{\mathbf{u}}(\lambda). \tag{3.9}$$

In other words, the transfer function for $\mathbb{L}_*$ is $\mathcal{C}(\lambda - \Gamma)^{-1}\mathcal{B}$, where

$$\mathcal{C}(\lambda - \Gamma)^{-1}\mathcal{B}u = \mathcal{C}\int_0^\infty e^{-\lambda\tau}E^\tau\mathcal{B}u\, d\tau, \qquad u \in L^2(\mathbb{R}_+, U).$$

Evaluating these expressions at $t \in \mathbb{R}_+$ gives

$$[\mathcal{C}(\lambda - \Gamma)^{-1}\mathcal{B}u](t) = \int_0^t C(t)U(t,\tau)B(\tau)u(\tau)e^{-\lambda(t-\tau)}\, d\tau. \tag{3.10}$$

It is natural to call $\mathcal{C}(\lambda - \Gamma)^{-1}\mathcal{B}$ the transfer function for the nonautonomous system. Moreover, the following remarks show that, by looking at the right-hand side of (3.10), this gives a natural "frequency response" function for nonautonomous systems. To see this, we first consider autonomous systems and note that the definition of the transfer function for an autonomous system can be extended to allow for a class of "Laplace transformable" functions that are in $L^2_{loc}(\mathbb{R}_+, U)$ (see, e.g., [44]). This class includes constant functions of the form $v_0(t) = u_0$, $t \geq 0$, for a given $u_0 \in U$. If a periodic input signal of the form $u(t) = u_0 e^{i\omega t}$, $t \geq 0$, (for some $u_0 \in U$ and $\omega \in \mathbb{R}$) is fed into an autonomous system with initial condition $x(0) = x_0$, then, by definition of the input-output operator, we have

$$(\mathbb{L}u)(t) = C(i\omega - A)^{-1}Bu_0 \cdot e^{i\omega t} - Ce^{tA}x_0, \quad (\mathbb{L}u)(t) = C\int_0^t e^{(t-s)A}Bu(s)\,ds.$$

Thus, the output

$$y(t; u(\cdot), x_0) = (\mathbb{L}u)(t) + Ce^{tA}x_0 = C(i\omega - A)^{-1}Bu_0 \cdot e^{i\omega t}$$

has the same frequency as the input. In view of this, the function $C(i\omega - A)^{-1}B$ is sometimes called the frequency response function. Now recall that the semigroup $\{e^{tA}\}_{t\geq 0}$ is stable and so $\lim_{t\to\infty} \|Ce^{tA}x_0\| = 0$. On the other hand, consider $v(t) = u_0$ and (formally) apply $\mathcal{C}(i\omega - \Gamma)^{-1}\mathcal{B}$ to this v. For $x_0 = (i\omega - A)^{-1}Bu_0$, a calculation based on the Laplace transform formula for the resolvent of the generator (applied to the evolution semigroup $\{E^t\}_{t\geq 0}$) yields the identity

$$\left[\mathcal{C}(i\omega - \Gamma)^{-1}\mathcal{B}u_0\right](t) = C(i\omega - A)^{-1}Bu_0 - Ce^{tA}x_0 \cdot e^{-i\omega t}.$$

Let us consider this expression $[\mathcal{C}(i\omega - \Gamma)^{-1}\mathcal{B}u_0](t)$ in the nonautonomous case. By equation (3.10), this coincides with the frequency response function for time-varying systems which is defined in [3, Corollary 3.2] by the formula

$$\int_0^t C(t)U(t,\tau)B(\tau)u_0 e^{i\omega(\tau - t)}\,d\tau.$$

Also, as noted in this reference, the result of our derivation agrees the Arveson frequency response function as it appears in [43]. We recover it here explicitly as the Laplace transform of an input-output operator (see equation (3.9)).

3.3. Autonomous systems. In this subsection we give the proof of (3.7) when X, U and Y are Banach spaces. In the process, however, we also consider two other "stability radii": a pointwise stability radius and a dichotomy radius.

First, we give a generalization to Banach spaces of Theorem 1.1 (cf. [22]). Here, $\mathcal{F}_{per}$ denotes the Banach space $L^p([0, 2\pi], X)$, $1 \leq p < \infty$. If $\{e^{tA}\}_{t\geq 0}$ is a strongly continuous semigroup on X, $\{E^t_{per}\}_{t\geq 0}$ will denote the evolution semigroup defined on $\mathcal{F}_{per}$ by the rule $E^t_{per}f(s) = e^{tA}f([s-t](\operatorname{mod} 2\pi))$; its generator will be denoted by Γ_{per}. The symbol Λ will be used to denote the set of all finite sequences $\{v_k\}_{k=-N}^N$ in X or $\mathcal{D}(A)$, or $\{u_k\}_{k=-N}^N$ in U.

Theorem 3.3. *Let A generate a C_0 semigroup $\{e^{tA}\}_{t\geq 0}$ on X. Let B and C be as above, and $\Delta \in (Y, U)$. Let $\{e^{t(A+B\Delta C)}\}_{t\geq 0}$ be the strongly continuous semigroup generated by $A + B\Delta C$. Then the following are equivalent:*

(i) $1 \in \rho(e^{2\pi(A+B\Delta C)})$;

(ii) $i\mathbb{Z} \subset \rho(A+B\Delta C)$ *and*

$$\sup_{\{v_k\}\in\Lambda} \frac{\|\sum_k (A-ik+B\Delta C)^{-1} v_k e^{ik(\cdot)}\|_{\mathcal{F}_{per}}}{\|\sum_k v_k e^{ik(\cdot)}\|_{\mathcal{F}_{per}}} < \infty;$$

(iii) $i\mathbb{Z} \subset \rho(A+B\Delta C)$ *and*

$$\inf_{\{v_k\}\in\Lambda} \frac{\|\sum_k (A-ik+B\Delta C) v_k e^{ik(\cdot)}\|_{\mathcal{F}_{per}}}{\|\sum_k v_k e^{ik(\cdot)}\|_{\mathcal{F}_{per}}} > 0.$$

Further, if Γ_{per} denotes the generator of the evolution semigroup on $\mathcal{F}_{per}$, as above, and if $1 \in \rho(e^{2\pi A})$, then Γ_{per} is invertible and

$$\|\mathcal{C}\Gamma_{per}^{-1}\mathcal{B}\| = \sup_{\{u_k\}\in\Lambda} \frac{\|\sum_k C(A-ik)^{-1}Bu_k e^{ik(\cdot)}\|_{L^p([0,2\pi],Y)}}{\|\sum_k u_k e^{ik(\cdot)}\|_{L^p([0,2\pi],U)}} \tag{3.11}$$

where $\mathcal{C}\Gamma_{per}^{-1}\mathcal{B} \in \mathcal{L}(L^p([0,2\pi],U), L^p([0,2\pi],Y))$.

Proof. The equivalence of *(i)–(iii)* follows as in Theorem 2.3 of [22]. For the last statement, let $\{u_k\}$ be a finite set in U and consider functions f and g of the form

$$f(s) = \sum_k (A-ik)^{-1}Bu_k e^{iks}, \quad \text{and} \quad g(s) = \sum_k Bu_k e^{iks}.$$

Then $f = \Gamma_{per}^{-1} g$. For,

$$\begin{aligned}(\Gamma_{per} f)(s) &= \frac{d}{dt}\bigg|_{t=0} e^{tA} f([s-t]\mathrm{mod} 2\pi) \\ &= \sum_k [A(A-ik)^{-1}Bu_k e^{iks} - ik(A-ik)^{-1}Bu_k e^{iks}] = g(s).\end{aligned}$$

For functions of the form $h(s) = \sum_k u_k e^{iks}$, where $\{u_k\}_k$ is a finite set in U, we have $\mathcal{C}\Gamma^{-1}\mathcal{B}h = \sum_k C(A-ik)^{-1}Bu_k e^{ik(\cdot)}$. Taking the supremum over all such functions gives:

$$\begin{aligned}\|\mathcal{C}\Gamma_{per}^{-1}\mathcal{B}\| &= \sup_h \frac{\|\mathcal{C}\Gamma_{per}^{-1}\mathcal{B}h\|}{\|h\|} \\ &= \sup_{\{u_k\}\in\Lambda} \frac{\|\sum_k C(A-ik)^{-1}Bu_k e^{ik(\cdot)}\|_{L^p([0,2\pi],Y)}}{\|\sum_k u_k e^{ik(\cdot)}\|_{L^p([0,2\pi],U)}}.\end{aligned}$$

□

In view of these facts we introduce a "pointwise" variant of the constant stability radius: for $t_0 > 0$ and $\lambda \in \rho(e^{t_0 A})$, define the *pointwise stability radius*

$$rc_{stab}^{\lambda}(e^{t_0 A}, B, C) := \sup\{r > 0 : \|\Delta\|_{\mathcal{L}(Y,U)} \le r \Rightarrow \lambda \in \rho(e^{t_0(A+B\Delta C)})\}.$$

By rescaling, the study of this quantity can be reduced to the case of $\lambda = 1$ and $t_0 = 2\pi$. Indeed,

$$rc_{stab}^{\lambda}(e^{t_0 A}, B, C) = \frac{2\pi}{t_0} rc_{stab}^{\lambda}(e^{2\pi A'}, B, C), \quad \text{where} \quad A' = \frac{t_0}{2\pi}A.$$

Also, after writing $\lambda = |\lambda| e^{i\theta}$ ($\theta \in \mathbb{R}$), note that

$$rc^{\lambda}_{stab}(e^{2\pi A}, B, C) = rc^{1}_{stab}(e^{2\pi A''}, B, C), \qquad \text{for} \qquad A'' = A - \frac{1}{2\pi}(\ln|\lambda| + i\theta).$$

Therefore,

$$rc^{\lambda}_{stab}(e^{t_0 A}, B, C) = \frac{2\pi}{t_0} rc^{1}_{stab}(e^{2\pi A'''}, B, C)$$

for

$$A''' = \frac{1}{2\pi}(t_0 A - \ln|\lambda| - i\theta).$$

In the following theorem we estimate $rc^{1}_{stab}(e^{2\pi A}, B, C)$. The idea for the proof goes back to [17]. See also further developments in [13].

THEOREM 3.4. *Let $\{e^{tA}\}_{t\geq 0}$ be a strongly continuous semigroup generated by A on X, and assume $1 \in \rho(e^{2\pi A})$. Let Γ_{per} denote the generator of the induced evolution semigroup on $\mathcal{F}_{per}$. Let $B \in \mathcal{L}(U, X)$, and $C \in \mathcal{L}(X, Y)$. Then*

$$\frac{1}{\|\mathcal{C}\Gamma^{-1}_{per}\mathcal{B}\|} \leq rc^{1}_{stab}(e^{2\pi A}, B, C) \leq \frac{1}{\sup_{k\in\mathbb{Z}} \|C(A - ik)^{-1}B\|}. \tag{3.12}$$

If U and Y are a Hilbert spaces and $p = 2$, then equalities hold in (3.12).

Proof. The first inequality follows from an argument as in Theorem 3.2. For the second inequality, let $\epsilon > 0$, and choose $\bar{u} \in U$ with $\|\bar{u}\| = 1$ and $k_0 \in \mathbb{Z}$ such that

$$\|C(A - ik_0)^{-1}B\bar{u}\|_Y \geq \sup_{k\in\mathbb{Z}} \|C(A - ik)^{-1}B\| - \epsilon > 0.$$

Using the Hahn-Banach Theorem, choose $y^* \in Y^*$ with $\|y^*\| \leq 1$ such that

$$\left\langle y^*, \frac{C(A - ik_0)^{-1}B\bar{u}}{\|C(A - ik_0)^{-1}B\bar{u}\|_Y} \right\rangle = 1.$$

Define $\Delta \in \mathcal{L}(Y, U)$ by

$$\Delta y = -\frac{\langle y^*, y\rangle}{\|C(A - ik_0)^{-1}B\bar{u}\|_Y}\bar{u}, \quad y \in Y.$$

We note that

$$\Delta C(A - ik_0)^{-1}B\bar{u} = -\frac{\langle y^*, C(A - ik_0)^{-1}B\bar{u}\rangle}{\|C(A - ik_0)^{-1}B\bar{u}\|_Y}\bar{u} = -\bar{u}, \tag{3.13}$$

and

$$\|\Delta\| \leq \frac{1}{\|C(A - ik_0)^{-1}B\bar{u}\|_Y} \leq \frac{1}{\sup_{k\in\mathbb{Z}} \|C(A - ik_0)^{-1}B\bar{u}\|_Y - \epsilon}. \tag{3.14}$$

Now set $\bar{v} := (A - ik_0)^{-1}B\bar{u}$ in X. By (3.13), $\Delta C\bar{v} = -\bar{u}$, and so

$$(A - ik_0 + B\Delta C)\bar{v} = (A - ik_0)\bar{v} + B\Delta C\bar{v} = B\bar{u} - B\bar{u} = 0.$$

Therefore,

$$\inf_{\{v_k\}\in\Lambda} \frac{\|\sum_k (A-ik+B\Delta C)v_k e^{ik(\cdot)}\|_{\mathcal{F}_{per}}}{\|\sum_k u_k e^{ik(\cdot)}\|_{\mathcal{F}_{per}}} \le \frac{\|(A-ik_0+B\Delta C)\bar{v}e^{ik_0(\cdot)}\|_{\mathcal{F}_{per}}}{\|\bar{v}e^{ik_0(\cdot)}\|_{\mathcal{F}_{per}}} = 0.$$

By Theorem 3.3, $1 \notin \rho(e^{2\pi(A+B\Delta C)})$. This shows that $rc^1_{stab}(e^{2\pi A}, B, C) \le \|\Delta\|$.

To finish the proof, suppose that $rc^1_{stab}(e^{2\pi A}, B, C) > (\sup_{k\in\mathbb{Z}} \|C(A-ik)^{-1}B\|)^{-1}$. Then with $r := (\sup_{k\in\mathbb{Z}} \|C(A-ik)^{-1}B\bar{u}\|_Y - \epsilon)^{-1}$, and $\epsilon > 0$ chosen to be sufficiently small, one has

$$\frac{1}{\sup_{k\in\mathbb{Z}} \|C(A-ik)^{-1}B\bar{u}\|_Y} < r < rc^1_{stab}(e^{2\pi A}, B, C).$$

But then by (3.14), $\|\Delta\| \le r < rc^1_{stab}(e^{2\pi A}, B, C)$, which is a contradiction.

For the last statement of the theorem, note that Parseval's formula applied to (3.11) gives

$$\|\mathcal{C}\Gamma^{-1}_{per}\mathcal{B}\| = \sup_{\{u_k\}\in\Lambda} \frac{\left(\sum_k \|C(A-ik)^{-1}Bu_k\|^2_Y\right)^{1/2}}{\left(\sum_k \|u_k\|^2_U\right)^{1/2}} \le \sup_{k\in\mathbb{Z}} \|C(A-ik)^{-1}B\|. \tag{3.15}$$

Therefore,

$$\frac{1}{\|\mathcal{C}\Gamma^{-1}_{per}\mathcal{B}\|} \ge \frac{1}{\sup_{k\in\mathbb{Z}} \|C(A-ik)^{-1}B\|}$$

and hence equalities hold in (3.12). □

Next we consider the following "hyperbolic" variant of the constant stability radius. Recall, that a strongly continuous semigroup $\{e^{tA}\}_{t\ge 0}$ on X is called *hyperbolic* if

$$\sigma(e^{tA}) \cap \mathbb{T} = \emptyset, \qquad \text{where } \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\},$$

for some (and, hence, for all) $t > 0$ (see, e.g., [29]). The hyperbolic semigroups are those for which the differential equation $\dot{x} = Ax$ has exponential dichotomy (see, e.g., [11]) with the dichotomy projection P being the Riesz projection corresponding to the part of spectrum of e^A that lies in the open unit disc.

For a given hyperbolic semigroup $\{e^{tA}\}_{t\ge 0}$ and operators B, C we define the *constant dichotomy radius* as:

$$\begin{aligned} rc_{dich}(\{e^{tA}\}, B, C) := \sup\{r \ge 0 : \|\Delta\|_{\mathcal{L}(Y,U)} \le r \text{ implies} \\ \sigma(e^{t(A+B\Delta C)}) \cap \mathbb{T} = \emptyset \text{ for all } t > 0\}. \end{aligned}$$

The dichotomy radius measures the size of the smallest $\Delta \in \mathcal{L}(Y, U)$ for which the perturbed equation $\dot{x} = [A + B\Delta C]x$ looses the exponential dichotomy.

Now for any $\xi \in [0, 1]$, consider the rescaled semigroup generated by $A_\xi := A - i\xi$ consisting of operators $e^{tA_\xi} = e^{-i\xi t}e^{tA}$, $t \ge 0$. The pointwise stability radius can be related to the dichotomy radius as follows.

LEMMA 3.5. *Let $\{e^{tA}\}_{t\ge 0}$ be a hyperbolic semigroup. Then*

$$rc_{dich}(\{e^{tA}\}, B, C) = \inf_{\xi\in[0,1]} rc^1_{stab}(e^{2\pi A_\xi}, B, C).$$

Proof. Denote the left-hand side by α and the right-hand side by β. First fix $r < \beta$. Let $\xi \in [0,1]$. If $\|\Delta\| \le r$, then $1 \in \rho(e^{2\pi(A_\xi + B\Delta C)})$ and so $e^{i\xi 2\pi} \in \rho(e^{2\pi(A+B\Delta C)})$ for all $\xi \in [0,1]$. That is, $e^{is} \in \rho(e^{2\pi(A+B\Delta C)})$ for all $s \in \mathbb{R}$, and so $\sigma(e^{2\pi(A+B\Delta C)}) \cap \mathbb{T} = \emptyset$. This shows that $r \le \alpha$, and so $\beta \le \alpha$.

Now suppose $r < \alpha$. If $\|\Delta\| \le r$, then $\sigma(\{e^{t(A+B\Delta C)}\}) \cap \mathbb{T} = \emptyset$, and so $e^{i\xi t} \in \rho(e^{t(A+B\Delta C)})$ for all $\xi \in [0,1], t \in \mathbb{R}$. That is, $1 \in \rho(e^{t(A_\xi + B\Delta C)})$. This says $r \le \beta$ and so $\alpha \le \beta$. □

Under the additional assumption that the semigroup $\{e^{tA}\}_{t\ge 0}$ is exponentially stable (that is, hyperbolic with a trivial dichotomy projection $P = I$), Lemma 3.5 gives, in fact, a formula for the constant *stability* radius. Indeed, the following simple proposition holds.

PROPOSITION 3.6. *Let* $\{e^{tA}\}_{t\ge 0}$ *be an exponentially stable semigroup. Then*

$$rc_{dich}(\{e^{tA}\}, B, C) = rc_{stab}(\{e^{tA}\}, B, C).$$

Proof. Denote the left-hand side by α and the right-hand side by β. Take $r < \beta$ and any Δ with $\|\Delta\| \le r$. By definition of the constant stability radius, $\omega_0(\{e^{t(A+B\Delta C)}\}) < 0$. In particular, $\sigma(e^{t(A+B\Delta C)}) \cap \mathbb{T} = \emptyset$, and $r \le \alpha$ shows that $\beta \le \alpha$.

Suppose that $\beta < r < \alpha$ for some r. By the definition of the stability radius β, there exists a Δ with $\|\Delta\| \in (\beta, r)$ such that the semigroup $\{e^{t(A+B\Delta C)}\}_{t\ge 0}$ is not stable.

For any $\tau \in [0,1]$ one has $\|\tau\Delta\| \le r < \alpha$. By the definition of the dichotomy radius α it follows that the semigroup $\{e^{t(A+\tau B\Delta C)}\}_{t\ge 0}$ is hyperbolic for each $\tau \in [0,1]$. Now consider its dichotomy projection

$$P(\tau) = (2\pi i)^{-1} \int_{\mathbb{T}} \left(\lambda - e^{A+\tau B\Delta C}\right)^{-1} \, d\lambda,$$

which is the Riesz projection corresponding to the part of $\sigma(e^{A+\tau B\Delta C})$ located inside of the open unit disk. The function $\tau \mapsto P(\tau)$ is norm continuous. Indeed, since the bounded perturbation $\tau B\Delta C$ of the generator A is continuous in τ, the operators $e^{t(A+\tau B\Delta C)}$, $t \ge 0$, depend on τ continuously (see, e.g., [30, Corollary 3.1.3]); this implies the continuity of $P(\cdot)$ (see, e.g., [11, Theorem I.2.2]).

By assumption $\{e^{tA}\}_{t\ge 0}$ is exponentially stable, so $P(0) = I$. Also, $P(1) \ne I$ since the semigroup $\{e^{t(A+B\Delta C)}\}_{t\ge 0}$ with $\|\Delta\| \le r < \alpha$ is hyperbolic but not stable. Since either $\|I - P(\tau)\| = 0$ or $\|I - P(\tau)\| \ge 1$, this contradicts the continuity of $\|P(\cdot)\|$. □

A review of the above development shows that the inequality claimed in (3.7) of Theorem 3.1 can now be proved.

Proof. (*of Theorem 3.1*) Indeed, $r_{stab}(\{e^{tA}\}, B, C) \le rc_{stab}(\{e^{tA}\}, B, C)$, and so

$$\begin{aligned}
\frac{1}{\|\mathbb{L}\|} \le r_{stab}(\{e^{tA}\}, B, C) &\le rc_{stab}(\{e^{tA}\}, B, C) && \text{(Theorem 3.2)} \\
&\le rc_{dich}(\{e^{tA}\}, B, C) && \text{(Proposition 3.6)} \\
&\le \inf_{\xi\in[0,1]} rc^1_{stab}(e^{2\pi A_\xi}, B, C) && \text{(Lemma 3.5)} \\
&\le \inf_{\xi\in[0,1]} \frac{1}{\sup_{k\in\mathbb{Z}} \|C(A_\xi - ik)^{-1}B\|} && \text{(Theorem 3.4)} \\
&= \frac{1}{\sup_{s\in\mathbb{R}} \|C(A - is)^{-1}B\|}
\end{aligned}$$

□

We will need below the following simple corollary that holds for *bounded* generators A. (In fact, as shown in [13, Cor. 2.5], formula (3.16) below holds provided A generates a semigroup $\{e^{tA}\}_{t\ge0}$ that is *uniformly* continuous just for $t > 0$.)

COROLLARY 3.7. *Assume $A \in \mathcal{L}(X)$ generates a (uniformly continuous) stable semigroup on a Banach space X. Then*

$$rc_{stab}(\{e^{tA}\}, B, C) = \frac{1}{\sup_{s\in\mathbb{R}} \|C(A - is)^{-1}B\|}. \tag{3.16}$$

Proof. By Theorem 3.1, it remains to prove only the inequality "≥". Fix Δ with $\|\Delta\|$ strictly less than the right-hand side of (3.16). Since $A + B\Delta C \in \mathcal{L}(X)$, it suffices to show that $A + B\Delta C - \lambda = (A - \lambda)(I + (A - \lambda)^{-1}B\Delta C)$ is invertible for each λ with $\mathrm{Re}\,\lambda \ge 0$. By the analyticity of resolvent, $\sup_{\mathrm{Re}\,\lambda\ge0} \|C(A - \lambda)^{-1}B\| \le \sup_{s\in\mathbb{R}} \|C(A - is)^{-1}B\|$. Thus,

$$\begin{aligned}
\|\Delta\| &< \frac{1}{\sup_{s\in\mathbb{R}} \|C(A - is)^{-1}B\|} \\
&\le \frac{1}{\sup_{\mathrm{Re}\,\lambda\ge0} \|C(A - \lambda)^{-1}B\|} \le \frac{1}{\|C(A - \lambda)^{-1}B\|}, \quad \mathrm{Re}\,\lambda \ge 0
\end{aligned}$$

implies that $I + C(A-\lambda)^{-1}B\Delta$ is invertible. Therefore (cf. the proof of Theorem 3.2), $I + (A - \lambda)^{-1}B\Delta C$ is invertible. □

3.4. The norm of the input-output operator. Since the lower bound on the stability radius is given by the norm of the input-output operator, which is defined by way of the solution operators, it is of interest to express this quantity in terms of the operators A, B and C. In this subsection it is shown that for autonomous systems this quantity can, in fact, be expressed explicitly in terms of the transfer function:

$$\|\mathbb{L}\| = \sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\| \int_{\mathbb{R}} C(A - is)^{-1}Bu(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},Y)}}{\| \int_{\mathbb{R}} u(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},U)}}. \tag{3.17}$$

Here we use $\mathcal{S}(\mathbb{R}, X)$ to denote the Schwartz class of rapidly decreasing X-valued functions defined on $\mathbb{R}$: $\{v : \mathbb{R} \to X \big| \sup_{s\in\mathbb{R}} \|s^m v^{(n)}(s)\| < \infty; n, m \in \mathbb{N}\}$. As noted in (3.6), $\|\mathbb{L}\|$ equals $\sup_{s\in\mathbb{R}} \|C(A - is)^{-1}B\|$ if U and Y are Hilbert spaces and $p = 2$. The section concludes by providing a similar expression, involving sums, which serves as a lower bound for the constant stability radius.

The current focus is on autonomous systems so let $\{e^{tA}\}_{t\geq 0}$ be a strongly continuous semigroup generated by A and consider the evolution semigroups $\{E^t_{\mathbb{R}}\}_{t\geq 0}$ defined on functions on the entire real line as in (1.2), and $\{E^t\}_{t\geq 0}$ defined for functions on the half-line as in (2.4). As before, $\Gamma_{\mathbb{R}}$ and Γ will denote the generators of these semigroups on $L^p(\mathbb{R}, X)$ and $L^p(\mathbb{R}_+, X)$, respectively. Both semigroups will be used as we first show that $\|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\|$ equals the expression in (3.17) and then check that $\|\mathbb{L}\| \equiv \|\mathcal{C}\Gamma^{-1}\mathcal{B}\| = \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\|$.

Given $v \in \mathcal{S}(\mathbb{R}, X)$, let g_v denote the function

$$g_v(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} v(s)e^{i\tau s}\,ds, \quad \tau \in \mathbb{R},$$

and set $\mathfrak{G} = \{g_v : v \in \mathcal{S}(\mathbb{R}, X)\}$. Assuming $\sup_{s\in\mathbb{R}} \|(A - is)^{-1}\| < \infty$, define, for a given $v \in \mathcal{S}(\mathbb{R}, X)$, the function

$$f_v(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} (A - is)^{-1}v(s)e^{i\tau s}\,ds, \quad \tau \in \mathbb{R},$$

and set $\mathfrak{F} = \{f_v : v \in \mathcal{S}(\mathbb{R}, X)\}$.

Proposition 3.8. *Assume* $\sup_{s\in\mathbb{R}} \|(A - is)^{-1}\| < \infty$. *Then*

(i) $\mathfrak{G}$ *consists of differentiable functions, and is dense in* $L^p(\mathbb{R}, X)$*;*

(ii) $\mathfrak{F}$ *is dense in* $\mathrm{Dom}(\Gamma_{\mathbb{R}})$*;*

(iii) if $v \in \mathcal{S}(\mathbb{R}, X)$ *then* $\Gamma_{\mathbb{R}} f_v = g_v$.

Proof. For $g \in L^1(\mathbb{R}, X)$, denote the Fourier transform by

$$\hat{g}(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-is\tau}g(s)\,ds.$$

Note that $\mathfrak{G} = \{g : \mathbb{R} \to X : \exists\, v \in \mathcal{S}(\mathbb{R}, X) \text{ so that } \hat{g} = v\}$, and so $\mathfrak{G}$ contains the set $\{g \in L^1(\mathbb{R}, X) : \hat{g} \in \mathcal{S}(\mathbb{R}, X)\}$. Since the latter set is dense in $L^p(\mathbb{R}, X)$, property (i) follows.

$\mathfrak{G}$ consists of differentiable functions since for $v \in \mathcal{S}(\mathbb{R}, X)$, the integral defining g_v converges absolutely. Moreover, for $v \in \mathcal{S}(\mathbb{R}, X)$, the function $w(s) = (A - is)^{-1}v(s)$, $s \in \mathbb{R}$, is also in $\mathcal{S}(\mathbb{R}, X)$, since $\sup_{s\in\mathbb{R}} \|(A - is)^{-1}\| < \infty$. Hence f_v is differentiable with derivative

$$f_v'(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} is(A - is)^{-1}v(s)e^{i\tau s}\,ds = \frac{1}{2\pi}\int_{\mathbb{R}} isw(s)e^{i\tau s}\,ds.$$

So $f_v' \in L^p(\mathbb{R}, X)$, and hence $\mathfrak{F}$ is dense in $\mathrm{Dom}(-d/dt + \mathcal{A})$.

Property (iii) follows from the following calculation:

$$\begin{aligned}(\Gamma f_v)(\tau) &= \frac{1}{2\pi}\int_{\mathbb{R}} [-is(A - is)^{-1}v(s)e^{is\tau} + A(A - is)^{-1}v(s)e^{is\tau}]\,ds \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} (A - is)(A - is)^{-1}v(s)e^{is\tau}\,ds = g_v(\tau).\end{aligned}$$

□

Set $\Lambda_{\mathcal{S}} = \{v \in \mathcal{S}(\mathbb{R}, X) : v(s) \in \mathrm{Dom}(A) \text{ for } s \in \mathbb{R},\ Av \in \mathcal{S}(\mathbb{R}, X)\}$.

Proposition 3.9. *Let* $\{e^{tA}\}_{t\geq 0}$ *be a strongly continuous semigroup generated by* A*. Let* Γ *and* $\Gamma_{\mathbb{R}}$ *be the generators of the evolution semigroups on* $L^p(\mathbb{R}_+, X)$ *and* $L^p(\mathbb{R}, X)$*, as defined in* (2.4) *and* (1.2)*, respectively. Then the following assertions hold:*

(i) if $\sigma(A)\cap i\mathbb{R}=\emptyset$ and $\sup_{s\in\mathbb{R}}\|(A-is)^{-1}\|<\infty$ then

$$\|\Gamma_{\mathbb{R}}\|_{\bullet,L^p(\mathbb{R},X)}=\inf_{v\in\Lambda_{\mathcal{S}}}\frac{\|\int_{\mathbb{R}}(A-is)v(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},X)}}{\|\int_{\mathbb{R}}v(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},X)}};$$

(ii) if $\Gamma_{\mathbb{R}}$ is invertible on $L^p(\mathbb{R},X)$, then $\{e^{tA}\}_{t\ge0}$ is hyperbolic and

$$\|\Gamma_{\mathbb{R}}^{-1}\|_{\mathcal{L}(L^p(\mathbb{R},X))}=\sup_{v\in\mathcal{S}(\mathbb{R},X)}\frac{\|\int_{\mathbb{R}}(A-is)^{-1}v(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},X)}}{\|\int_{\mathbb{R}}v(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},X)}};$$

(iii) if Γ is invertible on $L^p(\mathbb{R}_+,X)$, then $\{e^{tA}\}_{t\ge0}$ is exponentially stable and

$$\|\Gamma^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}_+,X))}=\|\Gamma_{\mathbb{R}}^{-1}\|_{\mathcal{L}(L^p(\mathbb{R},X))}.$$

Proof. To show (i) let $v\in\mathcal{S}(\mathbb{R},X)$. Since $\sup_{s\in\mathbb{R}}\|(A-is)^{-1}\|<\infty$, the formula $w(s)=(A-is)^{-1}v(s), s\in\mathbb{R}$, defines a function, w, in $\Lambda_{\mathcal{S}}$. Now,

$$g_v(\tau)=\frac{1}{2\pi}\int_{\mathbb{R}}(A-is)(A-is)^{-1}v(s)e^{is\tau}\,ds=\frac{1}{2\pi}\int_{\mathbb{R}}(A-is)w(s)e^{is\tau}\,ds$$

and

$$f_v(\tau)=\frac{1}{2\pi}\int_{\mathbb{R}}w(s)e^{is\tau}\,ds.$$

However, from Proposition 3.8,

$$\|\Gamma_{\mathbb{R}}\|_{\bullet}=\inf_{f_v\in\mathfrak{F}}\frac{\|\Gamma_{\mathbb{R}}f_v\|}{\|f_v\|}=\inf_{v\in\mathcal{S}(\mathbb{R},X)}\frac{\|g_v\|}{\|f_v\|}=\inf_{w\in\Lambda_{\mathcal{S}}}\frac{\|\int_{\mathbb{R}}(A-is)w(s)e^{is(\cdot)}\,ds\|}{\|\int_{\mathbb{R}}w(s)e^{is(\cdot)}\,ds\|}.$$

To see (ii) note that

$$\|\Gamma_{\mathbb{R}}^{-1}\|=\|\Gamma_{\mathbb{R}}\|_{\bullet}^{-1}=\left[\inf_{v\in\mathcal{S}(\mathbb{R},X)}\frac{\|\Gamma_{\mathbb{R}}f_v\|}{\|f_v\|}\right]^{-1}=\sup_{v\in\mathcal{S}(\mathbb{R},X)}\frac{\|f_v\|}{\|g_v\|}.$$

For (iii) note that $\|\Gamma_{\mathbb{R}}\|_{\bullet,L^p(\mathbb{R},X)}\le\|\Gamma\|_{\bullet,L^p(\mathbb{R}_+,X)}$. Indeed, let $f\in L^p(\mathbb{R},X)$ with $\operatorname{supp}f\subseteq\mathbb{R}_+$. If $f\in\operatorname{Dom}(-d/dt+\mathcal{A})$, then $\operatorname{supp}\Gamma_{\mathbb{R}}f\subseteq\mathbb{R}_+$ and $\|\Gamma_{\mathbb{R}}f\|_{L^p(\mathbb{R},X)}=\|\Gamma f\|_{L^p(\mathbb{R}_+,X)}$. To see that $\|\Gamma_{\mathbb{R}}\|_{\bullet}\ge\|\Gamma\|_{\bullet}$, let $\epsilon>0$ and choose $f\in\operatorname{Dom}(-d/dt+\mathcal{A})$ with compact support such that $\|f\|_{L^p(\mathbb{R},X)}=1$ and $\|\Gamma_{\mathbb{R}}\|_{\bullet}\ge\|\Gamma_{\mathbb{R}}f\|-\epsilon$. Now choose $\tau\in\mathbb{R}$ such that $f_\tau(s):=f(s-\tau)$, $s\in\mathbb{R}$, defines a function, $f_\tau\in L^p(\mathbb{R},X)$, with $\operatorname{supp}f_\tau\subseteq\mathbb{R}_+$. Let $\bar f_\tau$ denote the element of $L^p(\mathbb{R}_+,X)$ which coincides with f_τ on $\mathbb{R}_+$. Then $\|f_\tau\|=\|\bar f_\tau\|$ and $\Gamma\bar f_\tau=-d/dt\,f(\cdot-\tau)+Af(\cdot-\tau)=(\Gamma_{\mathbb{R}}f)_\tau$. Therefore, $\|\Gamma_{\mathbb{R}}\|_{\bullet}\ge\|\Gamma_{\mathbb{R}}f\|-\epsilon=\|(\Gamma_{\mathbb{R}}f)_\tau\|-\epsilon=\|\Gamma_{\mathbb{R}}\bar f_\tau\|-\epsilon\ge\|\Gamma\|_{\bullet}-\epsilon$. ∎

Proposition 3.10. *The set $\mathfrak{G}_U=\{g_u:u\in\mathcal{S}(\mathbb{R},U)\}$ is dense in $L^p(\mathbb{R},U)$. If $u\in\mathcal{S}(\mathbb{R},U)$ and $B\in\mathcal{L}(U,X)$ then $Bu\in\mathcal{S}(\mathbb{R},X)$ and $\Gamma_{\mathbb{R}}f_{Bu}=\mathcal{B}g_u$.*

Proof. The first statement is clear, as in Proposition 3.8. The second follows from the properties of Schwartz functions, and from the calculation:

$$\Gamma_{\mathbb{R}}f_{Bu}=g_{Bu}(\tau)=\frac{1}{2\pi}\int_{\mathbb{R}}Bu(s)e^{is\tau}\,ds=B\frac{1}{2\pi}\int_{\mathbb{R}}u(s)e^{is\tau}\,ds.$$

∎

Recall, see Remarks 1.4 and Theorem 2.2, that $\{e^{tA}\}_{t\geq 0}$ is hyperbolic (resp., stable) if and only if $\Gamma_{\mathbb{R}}$ (resp., Γ) is invertible on $L^p(\mathbb{R},X)$ (resp., $L^p(\mathbb{R}_+,X)$).

THEOREM 3.11. *If $\Gamma_{\mathbb{R}}$ is invertible on $L^p(\mathbb{R},X)$, then*

$$\|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\| = \sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\|\int_{\mathbb{R}} C(A-is)^{-1}Bu(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},Y)}}{\|\int_{\mathbb{R}} u(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},U)}}. \tag{3.18}$$

If Γ is invertible on $L^p(\mathbb{R}_+,X)$, then the norm of $\mathbb{L}=\mathcal{C}\Gamma^{-1}\mathcal{B}$, as an operator from $L^p(\mathbb{R}_+,U)$ to $L^p(\mathbb{R}_+,Y)$, is given by the above formula:

$$\|\mathbb{L}\| = \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\|. \tag{3.19}$$

If, in addition, U and Y are Hilbert spaces and $p=2$, then

$$\|\mathbb{L}\| = \sup_{s\in\mathbb{R}} \|C(A-is)^{-1}B\|_{\mathcal{L}(U,Y)}. \tag{3.20}$$

Proof. For $u\in\mathcal{S}(\mathbb{R},U)$, consider functions f_{Bu} and g_u. Proposition 3.10 gives $f_{Bu}=\Gamma_{\mathbb{R}}^{-1}\mathcal{B}g_u$ and

$$\begin{aligned}\|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\| &= \sup_{g_u\in\mathfrak{G}_U}\frac{\|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}g_u\|_{L^p(\mathbb{R},Y)}}{\|g_u\|_{L^p(\mathbb{R},U)}} = \sup_{g_u\in\mathfrak{G}_U}\frac{\|\mathcal{C}f_{Bu}\|}{\|g_u\|}\\ &= \sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\|\int_{\mathbb{R}} C(A-is)^{-1}Bu(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},Y)}}{\|\int_{\mathbb{R}} u(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},U)}},\end{aligned}$$

which proves (3.18).

Now, if Γ is invertible on $L^p(\mathbb{R}_+,X)$, then $\{e^{tA}\}_{t\geq 0}$ is exponentially stable by Corollary 2.6. Hence, $\Gamma_{\mathbb{R}}$ is invertible on $L^p(\mathbb{R},X)$. Moreover, for the case of the *stable* semigroup $\{e^{tA}\}_{t\geq 0}$, the formula for $\Gamma_{\mathbb{R}}^{-1}$ (see, e.g., [24]) takes the form

$$(\Gamma_{\mathbb{R}}^{-1}f)(t) = \int_0^\infty e^{sA}f(t-s)\,ds = \int_{-\infty}^t e^{(t-s)A}f(s)\,ds.$$

If $\operatorname{supp} f\subseteq(0,\infty)$, then

$$(\Gamma_{\mathbb{R}}^{-1}f)(t) = \int_{-\infty}^t e^{(t-s)A}f(s)\,ds = \int_0^t e^{(t-s)A}f(s)\,ds. \tag{3.21}$$

For a function $h\in L^p(\mathbb{R}_+,X)$, define an extension $\tilde{h}\in L^p(\mathbb{R},X)$ by $\tilde{h}(t)=h(t)$ for $t\geq 0$ and $\tilde{h}(t)=0$ for $t<0$. Then (3.21) shows that $\Gamma_{\mathbb{R}}^{-1}\tilde{h}=(\Gamma^{-1}h)^{\sim}$. In particular, for $u\in L^p(\mathbb{R}_+,U)$, $\widetilde{\mathbb{L}u}=\widetilde{\mathcal{C}\Gamma^{-1}\mathcal{B}u}=\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\tilde{u}$. Therefore,

$$\begin{aligned}\|\mathbb{L}u\|_{L^p(\mathbb{R}_+,Y)} &= \|\widetilde{\mathbb{L}u}\|_{L^p(\mathbb{R},Y)} = \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\tilde{u}\|_{L^p(\mathbb{R},Y)}\\ &\leq \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\|\cdot\|\tilde{u}\|_{L^p(\mathbb{R},U)} = \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\|\cdot\|u\|_{L^p(\mathbb{R}_+,U)}.\end{aligned}$$

This shows that $\|\mathbb{L}\|\leq\|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\|$.

To prove that equality holds in (3.19), let $\epsilon>0$ and choose $u\in L^p(\mathbb{R},U)$, $\|u\|=1$, such that $\|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}u\|_{L^p(\mathbb{R},Y)}\geq\|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\|-\epsilon$. Without loss of generality, u may be assumed to have compact support. Now choose r such that $\operatorname{supp} u(\cdot-r)\subseteq(0,\infty)$

and set $w(\cdot) := u(\cdot - r)$. Then $w \in L^p(\mathbb{R}, U)$ with $\operatorname{supp} w \subseteq (0, \infty)$. Let $\bar{w}$ denote the element of $L^p(\mathbb{R}_+, U)$ that coincides with w on $\mathbb{R}_+$. As in (3.21) we have

$$\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}w(t) = C\int_0^t e^{(t-s)A}Bw(s)\,ds = C\int_{-\infty}^t e^{(t-s)A}Bw(s)\,ds.$$

Since $\|\bar{w}\|_{L^p(\mathbb{R}_+,U)} = \|w\|_{L^p(\mathbb{R},U)} = \|u\|_{L^p(\mathbb{R},U)} = 1$, it follows that

$$\begin{aligned}\|\mathbb{L}\| &\geq \|\mathbb{L}\bar{w}\|_{L^p(\mathbb{R}_+,Y)} = \|\widetilde{\mathbb{L}\bar{w}}\|_{L^p(\mathbb{R},Y)} \\ &= \|\mathbb{L}\tilde{\bar{w}}\|_{L^p(\mathbb{R},Y)} = \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}w\|_{L^p(\mathbb{R},Y)} \\ &= \|C\int_{-\infty}^{\cdot} e^{(\cdot-\tau)A}Bu(\tau)\,d\tau\|_{L^p(\mathbb{R},Y)} \\ &= \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}u\|_{L^p(\mathbb{R},Y)} \geq \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\| - \epsilon.\end{aligned}$$

This confirms (3.19). Parseval's formula and (3.7) give (3.20). ∎

If $\{e^{tA}\}_{t\geq 0}$ is exponentially stable then the inequalities in (3.7) give lower and upper bounds on the stability radius in terms of $\mathbb{L}$ and $C(A - is)^{-1}B$, respectively. The previous theorem shows that $\|\mathbb{L}\|$ can be explicitly expressed in terms of an integral involving $C(A - is)^{-1}B$. We conclude by observing that a lower bound for the constant stability radius can be expressed by a similar formula involving a sum. For this, let $\xi \in [0, 1]$ and set

$$S_\xi := \sup_{\{u_k\}\in\Lambda} \frac{\|\sum_k C(A - i\xi - ik)^{-1}Bu_k e^{ik(\cdot)}\|_{L^p([0,2\pi],Y)}}{\|\sum_k u_k e^{ik(\cdot)}\|_{L^p([0,2\pi],U)}}.$$

We note that S_ξ is computed as in equation (3.11) with A replaced by $A_\xi = A - i\xi$.

COROLLARY 3.12. *Let $\{e^{tA}\}_{t\geq 0}$ be an exponentially stable semigroup generated by A. Then*

$$\frac{1}{\sup_{\xi\in[0,1]} S_\xi} \leq rc_{stab}(\{e^{tA}\}, B, C) \leq \frac{1}{\sup_{s\in\mathbb{R}} \|C(A - is)^{-1}B\|}.$$

Proof. Fix $\xi \in [0, 1]$, and let $\Gamma_{per,\xi}$ denote the generator on $L^p([0, 2\pi], X)$ of the evolution semigroup induced by $\{e^{tA_\xi}\}_{t\geq 0}$. By Theorem 3.3, $\|\mathcal{C}\Gamma_{per,\xi}^{-1}\mathcal{B}\| = S_\xi$, and so by Theorem 3.4,

$$\frac{1}{S_\xi} \leq rc^1_{stab}(e^{2\pi A_\xi}, B, C) \leq \frac{1}{\sup_{k\in\mathbb{Z}} \|C(A_\xi - ik)^{-1}B\|}.$$

By Proposition 3.6, taking the infimum over $\xi \in [0, 1]$ gives

$$\begin{aligned}\frac{1}{\sup_{\xi\in[0,1]} S_\xi} &\leq \inf_{\xi\in[0,1]} rc^1_{stab}(e^{2\pi A_\xi}, B, C) = rc_{stab}(\{e^{tA}\}, B, C) \\ &\leq \inf_{\xi\in[0,1]} \frac{1}{\sup_{k\in\mathbb{Z}} \|C(A_\xi - ik)^{-1}B\|} \\ &= \frac{1}{\sup_{s\in\mathbb{R}} \|C(A - is)^{-1}B\|}.\end{aligned}$$

∎

3.5. Two counterexamples. In contrast to the Hilbert space setting, the following Banach space examples show that either inequality in (3.7) may be strict. We start with the example where the second inequality in (3.7) is strict.

EXAMPLE 3.13. An example due to W. Arendt (see, e.g., [29], Example 1.4.5) exhibits a (positive) strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ on a Banach space X with the property that $s_0(A) < \omega_0(A) < 0$ for the abscissa of uniform boundedness of the resolvent and the growth bound. Now, for α such that $0 \leq \alpha \leq -\omega_0(A)$, consider a rescaled semigroup generated by $A + \alpha$, and denote by $\Gamma_{A+\alpha}$ the generator of the induced evolution semigroup on $L^p(\mathbb{R}_+, X)$. The following relationships hold:

$$\text{for } 0 \leq \alpha < -\omega_0(A), \quad s_0(A+\alpha) = s_0(A) + \alpha < \omega_0(A) + \alpha = \omega_0(A+\alpha) < 0;$$

$$\text{for } \alpha_0 := -\omega_0(A), \quad s_0(A+\alpha_0) < \omega_0(A+\alpha_0) = 0.$$

This says that $s_0(A+\alpha) < 0$ for all $\alpha \in [0, \alpha_0]$ and hence

$$M := \sup_{\alpha\in[0,\alpha_0]} \sup_{s\in\mathbb{R}} \|(A+\alpha - is)^{-1}\| < \infty.$$

Now note (see Corollary 2.6) that $\omega_0(A+\alpha) < 0$ if and only if $\|\Gamma^{-1}_{A+\alpha}\| < \infty$. Since $\omega_0(A+\alpha) \to 0$ as $\alpha \to \alpha_0$, we conclude that $\|\Gamma^{-1}_{A+\alpha}\| \to \infty$ as $\alpha \to \alpha_0$. Since $\alpha \mapsto \|\Gamma^{-1}_{A+\alpha}\|$ is a continuous function of α on $[0, \alpha_0)$, there exists $\alpha_1 \in [0, \alpha_0)$ such that $\|\Gamma^{-1}_{A+\alpha_1}\| > M$, and so the following inequality is strict:

$$\frac{1}{\|\Gamma^{-1}_{A+\alpha_1}\|} < \frac{1}{\sup_{s\in\mathbb{R}} \|(A+\alpha_1 - is)^{-1}\|}.$$

Also, we claim that there exists $\alpha_2 \in [0, \alpha_0)$ such that the following inequality is strict:

$$rc_{stab}(\{e^{t(A+\alpha_2)}\}, I, I) < \frac{1}{\sup_{s\in\mathbb{R}} \|(A+\alpha_2 - is)^{-1}\|}.$$

To see this, let us suppose that for each $\alpha \in [0, \alpha_0)$ one has $rc_{stab}(\{e^{t(A+\alpha)}\}, I, I) \geq 1/(2M)$. Again, using that $\omega_0(A+\alpha) \to 0$ as $\alpha \to \alpha_0$, find $\alpha \in [0, \alpha_0)$ such that $|\omega_0(A+\alpha)| < 1/(2M)$. Let $\Delta = \omega_0(A+\alpha)I$. Since $\|\Delta\| = |\omega_0(A+\alpha)|$, by the definition of stability radius one has:

$$0 > \omega_0(A+\alpha+\Delta) = \omega_0(A+\alpha) - \omega_0(A+\alpha) = 0,$$

a contradiction. Thus, there exists $\alpha_2 \in [0, \alpha_0)$ such that

$$rc_{stab}(\{e^{t(A+\alpha_2)}\}, I, I) \leq \frac{1}{2M} < \frac{1}{M} \leq \frac{1}{\sup_{s\in\mathbb{R}} \|(A+\alpha_2 - is)^{-1}\|},$$

as claimed. ◊

This example shows that the second inequality in (3.7) can be strict due to the Banach-space pathologies related to the failure of Gearhart's Theorem 1.1. Another example, given below, shows that the first inequality in (3.7) could be strict due to the lack of Parseval's formula (see (3.15) in the proof of Theorem 3.4): That is, the choice of $p = 2$ in (3.6) is as important as the fact that X in (3.6) is a *Hilbert* space. First, we need a formula for the norm of the input-output operator on $L^1(\mathbb{R}_+, X)$.

PROPOSITION 3.14. *Assume $\{e^{tA}\}_{t\geq 0}$ is an exponentially stable C_0 semigroup on a Banach space X. The norm of the operator $\mathbb{L} = \Gamma^{-1}$ on $L^1(\mathbb{R}_+, X)$ is*

$$\|\Gamma^{-1}\|_{\mathcal{L}(L^1(\mathbb{R}_+,X))} = \sup_{\|x\|=1} \int_0^\infty \|e^{tA}x\|\,dt. \tag{3.22}$$

Proof. Recall, see (2.5), that

$$\Gamma^{-1}f(t) = -\int_0^t e^{\tau A} f(t-\tau)\,d\tau, \quad t \in \mathbb{R}_+, \quad f \in L^1(\mathbb{R}_+, X)$$

is the convolution operator. Choose positive $\delta_n \in L^1(\mathbb{R}_+, \mathbb{R})$ with $\|\delta_n\|_{L^1} = 1$ such that

$$\|g * \delta_n - g\|_{L^1(\mathbb{R}_+,X)} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for each} \quad g \in L^1(\mathbb{R}_+, X).$$

Fix $x \in X$, $\|x\| = 1$, let $f = \delta_n x \in L^1(\mathbb{R}_+, X)$ and note that

$$\Gamma^{-1}f(t) = -\int_0^t e^{\tau A} x\, \delta_n(t-\tau)\,d\tau = -(g * \delta_n)(t) \quad \text{for} \quad g(t) = e^{tA}x, \quad t \in \mathbb{R}_+.$$

This implies "$\geq$" in (3.22). To see "$\leq$", take $f = \sum_{i=1}^N \alpha_i x_i$ with $\alpha_i \in L^1(\mathbb{R}_+, \mathbb{R})$ having disjoint supports and $\|x_i\| = 1$, $i = 1, \dots, N$. Now $\|f\|_{L^1(\mathbb{R}_+,X)} = \sum_i \|\alpha_i\|_{L^1}$ and for $f_i(t) = e^{tA}x_i$ one has

$$\Gamma^{-1}f(t) = -\int_0^t \sum_i e^{\tau A} x_i \alpha_i(t-\tau)\,d\tau = -\sum_i (f_i * \alpha_i)(t).$$

Using Young's inequality,

$$\begin{aligned}\|\Gamma^{-1}f\|_{L^1(\mathbb{R}_+,X)} &\leq \sum_i \|f_i * \alpha_i\|_{L^1(\mathbb{R}_+,X)} \\ &\leq \sum_i \|f_i\|_{L^1(\mathbb{R}_+,X)} \|\alpha_i\|_{L^1} \leq \sup_{\|x\|=1} \int_0^\infty \|e^{tA}x\|\,dt \sum_i \|\alpha_i\|_{L^1}.\end{aligned}$$

□

EXAMPLE 3.15. Take $X = \mathbb{C}^2$ with the ℓ_1 norm. Let

$$A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{so that} \quad e^{tA} = \begin{pmatrix} e^{-t}\cos(t) & e^{-t}\sin(t) \\ -e^{-t}\sin(t) & e^{-t}\cos(t) \end{pmatrix},$$

and

$$(A - is)^{-1} = \frac{1}{(1+is)^2 + 1} \begin{pmatrix} -1 - is & -1 \\ 1 & -1 - is \end{pmatrix}.$$

Since the extreme points of X are $e^{i\theta}e_1$ and $e^{i\theta}e_2$ ($\theta \in \mathbb{R}$), where e_1 and e_2 are the unit vectors of $\mathbb{C}^2$, we see that

$$\|(A - is)^{-1}\| = \frac{|1 + is| + 1}{|(1+is)^2 + 1|}.$$

It may be numerically established that

$$\sup_{s\in\mathbb{R}} \|(A-isI)^{-1}\| \approx 1.087494476.$$

By Corollary 3.7, the reciprocal to the last expression is equal to $rc_{stab}(\{e^{tA}\}, I, I)$. On the other hand, using Proposition 3.14,

$$\begin{aligned}\|\mathbb{L}\| = \|\Gamma_A^{-1}\| &= \sup_{\|x\|=1} \int_0^\infty \|e^{tA}x\|\, dt \\ &= \int_0^\infty |e^{-t}\cos(t)| + |e^{-t}\sin(t)|\, dt \approx 1.262434309.\end{aligned}$$

Therefore, the first inequality in (3.7) may be strict. ♢

The following example shows that the norm of the input-output operator depends on p.

Example 3.16. Let

$$A = \begin{pmatrix} 9/2 & -5/2 \\ 25/2 & -13/2 \end{pmatrix},$$

acting on $\mathbb{C}^2$ with the Euclidean norm. Thus

$$e^{tA} = e^{-t}\begin{pmatrix} \cos t + (11/2)\sin t & -(5/2)\sin t \\ (25/2)\sin t & \cos t - (11/2)\sin t \end{pmatrix}.$$

Then

$$\|\Gamma^{-1}\|_{L_1\to L_1} \geq \int_0^\infty \|e^{tA}e_1\|\, dt \approx 7.748310791,$$

whereas

$$\|\Gamma^{-1}\|_{L_2\to L_2} = \sup_{s\in R} \|(A-is)^{-1}\| \approx 2.732492852.$$

♢

4. Internal and External Stability. Work aimed at properties of stability and robustness of linear time-invariant systems is often based on transform techniques. More specifically, if the transfer function $H(\lambda) = C(A-\lambda)^{-1}B$ is a bounded analytic function of λ in the right half-plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Re}\ \lambda > 0\}$, then the autonomous system (3.4) is said to be *externally stable*. This property is often used to deduce *internal stability* of the system, i.e., the uniform exponential stability of the nominal system $\dot{x} = Ax$. The relationship between internal and external stability has been studied extensively; see, e.g., [2, 8, 7, 20, 25, 39, 40] and the references therein. In this section we examine the extent to which these techniques apply to Banach-space settings and time-varying systems. For this, *input-output stability* of the system (3.3) will refer to the property that the input-output operator $\mathbb{L}$ is bounded from $L^p(\mathbb{R}_+, U)$ to $L^p(\mathbb{R}_+, Y)$. If internal stability is assumed initially, then the inequalities in (3.7) exhibit a relationship between these concepts of stability. The next two theorems look at these relationships more closely and show, in particular, when internal stability may be deduced from one of the "external" stability conditions. Therefore, throughout this section $\{U(t,\tau)\}_{t\geq\tau}$ will denote a strongly continuous exponentially bounded evolution family that is *not* assumed to be exponentially stable.

4.1. The nonautonomous case. In this subsection we give a very short proof of the fact that for general nonautonomous systems on Banach spaces, internal stability is equivalent to stabilizability, detectability and input-output stability. Before proceeding, it is worth reviewing some properties of time-invariant systems. For this, let $\{e^{tA}\}_{t\geq 0}$ be a strongly continuous semigroup generated by A on X, and let $H^\infty_+(\mathcal{L}(X))$ denote the space of operator-valued functions $G : \mathbb{C} \to \mathcal{L}(X)$ which are analytic on $\mathbb{C}_+$ and $\sup_{\lambda\in\mathbb{C}_+} \|G(\lambda)\| < \infty$. If X is a Hilbert space, it is well known that $\{e^{tA}\}_{t\geq 0}$ is exponentially stable if and only if $\lambda \mapsto (\lambda - A)^{-1}$ is an element of $H^\infty_+(\mathcal{L}(X))$; see, e.g., [10], Theorem 5.1.5. This is a consequence of the fact that when X is a Hilbert space, $s_0(A) = \omega_0(e^{tA})$ (see [29] or Theorem 1.1). If X is a Banach space, then strict inequality $s_0(A) < \omega_0(e^{tA})$ can hold, and so exponential stability is no longer determined by the operator $G(\lambda) = (\lambda - A)^{-1}$. Extending these ideas to address systems (3.4), one considers $H(\lambda) = C(\lambda - A)^{-1}B$: it can be shown that if U and Y are Hilbert spaces, then (3.4) *is internally stable if and only if it is stabilizable, detectable and externally stable* (i.e., $H(\cdot) \in H^\infty_+(\mathcal{L}(U, Y))$). See R. Rebarber [39] for a general result of this type. It should be pointed out that this work of Rebarber and others more recently allows for a certain degree of unboundedness of the operators B and C. Such "regular" systems (see [44]), and their time-varying generalizations, might be addressed by combining the techniques of the present paper (including the characterization of generation of evolution semigroups as found in [36]) along and with those of [17] and [19]. This will not be done here.

If one allows for Banach spaces, the conditions of stabilizability and detectability are *not* sufficient to ensure that external stability implies internal stability. Indeed, let A generate a semigroup for which $s_0(A) < \omega_0(e^{tA}) = 0$ (see Example 3.13). Then the system (3.4) with $B = I$ and $C = I$ is trivially stabilizable and detectable and externally stable. But since $\omega_0(e^{tA}) = 0$, it is not internally stable.

Since the above italicized statement concerning external stability fails for Banach-space systems (3.4) and does not apply to time-varying systems (3.3), we aim to prove the following extension of this.

THEOREM 4.1. *The system* (3.3) *is internally stable if and only if it is stabilizable, detectable and input-output stable.*

This theorem appears as part of Theorem 4.3 below. A version of it for finite-dimensional time-varying systems was proven by B. D. O. Anderson in [2]. The fact that Theorem 4.1 actually *extends* the Hilbert-space statement above follows from the fact that the Banach-space inequality $\sup_{\lambda\in\mathbb{C}_+} \|H(\lambda)\| \leq \|\mathbb{L}\|$ (see [45]) which relates the operators that define external and input-output stability is actually an *equality* for Hilbert-space systems (see also [44]).

In Theorems 4.1 and 4.3, below, the following definitions are used.

DEFINITION 4.2. *The nonautonomous system* (3.3) *is said to be*

(a) stabilizable *if there exists* $F(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(X, U))$ *and a corresponding exponentially stable evolution family* $\{U_{BF}(t, \tau)\}_{t\geq\tau}$ *such that, for* $t \geq s$ *and* $x \in X$*, one has:*

$$U_{BF}(t, s)x = U(t, s)x + \int_s^t U(t, \tau)B(\tau)F(\tau)U_{BF}(\tau, s)x\, d\tau; \tag{4.1}$$

(b) detectable *if there exists* $K(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y, X))$ *and a corresponding exponentially stable evolution family* $\{U_{KC}(t, \tau)\}_{t\geq\tau}$ *such that, for* $t \geq s$ *and*

$x \in X$, one has:

$$U_{KC}(t,s)x = U(t,s)x + \int_s^t U_{KC}(t,\tau)K(\tau)C(\tau)U(\tau,s)x\,d\tau. \tag{4.2}$$

An autonomous control system is called stabilizable *if there is an operator $F \in \mathcal{L}(X,U)$ such that $A+BF$ generates a uniformly exponentially stable semigroup; that is, $\omega_0(A+BF) < 0$. Such a system is* detectable *if there is an operator $K \in \mathcal{L}(Y,X)$ such that $A+KC$ generates a uniformly exponentially stable semigroup.*

Using Theorem 2.5 to characterize exponential stability in terms of the operator $\mathbb{G}$ as in (2.6) makes the proof of the following theorem a straightforward manipulation of the appropriate operators.

THEOREM 4.3. *The following are equivalent for a strongly continuous exponentially bounded evolution family of operators $\mathcal{U} = \{U(t,\tau)\}_{t\geq\tau}$ on a Banach space X.*

(i) $\mathcal{U}$ is exponentially stable on X;
(ii) $\mathbb{G}$ is a bounded operator on $L^p(\mathbb{R}_+,X)$;
(iii) system (3.3) *is stabilizable and $\mathbb{G}\mathcal{B}$ is a bounded operator from $L^p(\mathbb{R}_+,U)$ to $L^p(\mathbb{R}_+,X)$;*
(iv) system (3.3) *is detectable and $\mathcal{C}\mathbb{G}$ is a bounded operator from $L^p(\mathbb{R}_+,X)$ to $L^p(\mathbb{R}_+,Y)$;*
(v) system (3.3) *is stabilizable and detectable and $\mathbb{L} = \mathcal{C}\mathbb{G}\mathcal{B}$ is a bounded operator from $L^p(\mathbb{R}_+,U)$ to $L^p(\mathbb{R}_+,Y)$.*

Proof. The equivalence of (i) and (ii) is the equivalence of (i) and (ii) in Theorem 2.5.

To see that (ii) implies (iii), (iv), and (v), note that $\mathcal{B}$ and $\mathcal{C}$ are bounded, and thus $\mathbb{L}$ is bounded when $\mathbb{G}$ is bounded. So when (ii) holds, the exponential stability of $\mathcal{U}$, together with boundedness of $B(\cdot)$, $C(\cdot)$, $F(\cdot)$ and $K(\cdot)$, assure the existence of the evolution families $\{U_{BF}(t,\tau)\}_{t\geq\tau}$ and $\{U_{KC}(t,\tau)\}_{t\geq\tau}$ as solutions of the integral equations in Definition 4.2; thereby showing that (iii), (iv), and (v) hold.

To see that (iii) $\Rightarrow$ (ii), first note that the assumption of stabilizability assures the existence of an exponentially stable evolution family $\mathcal{U}_{BF} = \{U_{BF}(t,\tau)\}_{t\geq\tau}$ satisfying equation (4.1) for some $F(\cdot) \in L^\infty(\mathbb{R}_+,\mathcal{L}_s(X,U))$. Given this exponentially stable family, we define the operator $\mathbb{G}_{BF}$ by

$$\mathbb{G}_{BF}f(s) := \int_0^s U_{BF}(s,\tau)f(\tau)\,d\tau = \int_0^\infty (E^\tau_{BF}f)(s)\,d\tau \tag{4.3}$$

where $\{E^\tau_{BF}f\}_{t\geq 0}$ is the semigroup induced by the evolution family $\mathcal{U}_{BF}$ as described in equation (2.1). $\mathbb{G}_{BF}$ is a bounded operator on $L^p(\mathbb{R}_+,X)$ by the equivalence of (i) and (ii).

For $f(\cdot) \in L^p(\mathbb{R}_+,X)$ and $s \in \mathbb{R}_+$, take $x = f(s)$ in equation (4.1). Then, let $\xi = \tau - s$, to obtain

$$U_{BF}(t,s)f(s) = U(t,s)f(s) + \int_0^{t-s} U(t,\xi+s)B(\xi+s)F(\xi+s)U_{BF}(\xi+s,s)f(s)\,d\xi.$$

From this equation and from the definition of the semigroups $\{E^t\}_{t\geq 0}$, and $\{E^t_{BF}\}_{t\geq 0}$ we obtain

$$(E^{t-s}_{BF}f)(t) = (E^{t-s}f)(t) + \int_0^{t-s} (E^{t-s-\xi}\mathcal{B}\mathcal{F}E^\xi_{BF}f)(t)\,d\xi$$

and hence for $0 \le r$ and $0 \le \sigma$ that

$$(E^r_{BF} f)(\sigma) = (E^r f)(\sigma) + \int_0^r (E^{r-\xi}\mathcal{B}\mathcal{F}E^{\xi}_{BF} f)(\sigma)\, d\xi.$$

Integrate from 0 to ∞ to obtain

$$(\mathbb{G}_{BF} f)(\sigma) = (\mathbb{G} f)(\sigma) + \int_0^\infty \int_0^r (E^{r-\xi}\mathcal{B}\mathcal{F}E^{\xi}_{BF} f)(\sigma)\, d\xi\, dr.$$

Let $r = \zeta + \eta$ and $\xi = \eta$ to obtain

$$\begin{aligned}(\mathbb{G}_{BF} f)(\sigma) &= (\mathbb{G} f)(\sigma) + \int_0^\infty \int_0^\infty (E^{\zeta}\mathcal{B}\mathcal{F}E^{\eta}_{BF} f)(\sigma)\, d\eta\, d\zeta \\ &= (\mathbb{G} f)(\sigma) + (\mathbb{G}\mathcal{B}\mathcal{F}\mathbb{G}_{BF} f)(\sigma).\end{aligned} \tag{4.4}$$

That $\mathbb{G}$ is bounded now follows from equation (4.4), the boundedness of $\mathbb{G}\mathcal{B}$, and the boundedness of $\mathbb{G}_{BF}$ and $\mathcal{F}$.

To see that (iv) $\Rightarrow$ (ii), first note that the assumption of detectability assures the existence of an exponentially stable evolution family $\mathcal{U}_{KC} = \{U_{KC}(t,\tau)\}_{t\ge\tau}$ satisfying equation (4.2) for some $K(\cdot) \in L^\infty(\mathbb{R}_+, \mathcal{L}_s(Y,X))$. Given this exponentially stable family, the operator $\mathbb{G}_{KC}$, defined in a manner analogous to $\mathbb{G}_{BF}$ in equation (4.3), is a bounded operator on $L^p(\mathbb{R}_+, X)$. A derivation beginning with equation (4.2), and similar to that which gave equation (4.4), now gives $\mathbb{G}_{KC} = \mathbb{G} + \mathbb{G}_{KC}\mathcal{K}\mathcal{C}\mathbb{G}$. This equation, together with the assumed boundedness of $\mathbb{G}_{KC}$, $\mathcal{K}$, and $\mathcal{C}\mathbb{G}$, gives the boundedness of $\mathbb{G}$.

Finally, to see that (v) $\Rightarrow$ (ii), again note that the assumption of detectability yields an exponentially stable evolution family $\mathcal{U}_{KC}$ and an associated bounded operator $\mathbb{G}_{KC}$. For $u(\cdot) \in L^p(\mathbb{R}_+, U)$, and $s \in \mathbb{R}_+$ take $x = B(s)u(s)$ in equation (4.2). A calculation similar to that which gave equation (4.4) now gives $\mathbb{G}_{KC}\mathcal{B} = \mathbb{G}\mathcal{B} + \mathbb{G}_{KC}\mathcal{K}\mathcal{C}\mathbb{G}\mathcal{B}$. The assumed boundedness of $\mathbb{L} = \mathcal{C}\mathbb{G}\mathcal{B}$, $\mathcal{K}$, and $\mathbb{G}_{KC}$, now yields the boundedness of $\mathbb{G}\mathcal{B}$. The boundedness of $\mathbb{G}\mathcal{B}$ together with the assumption of stabilizability implies that $\mathbb{G}$ is bounded by the equivalence of (iii) and (ii). ∎

4.2. The autonomous case. The main result of this subsection is Theorem 4.4 which builds on Theorem 3.11 and parallels Theorem 4.3 for autonomous systems of the form (3.4). The main point is to provide explicit conditions, in terms of the operators A, B and C, which imply internal stability.

Let $A_\alpha := A - \alpha I$ denote the generator of the rescaled semigroup $\{e^{-\alpha t}e^{tA}\}_{t\ge0}$.

THEOREM 4.4. *Let $\{e^{tA}\}_{t\ge0}$ be a strongly continuous semigroup on a Banach space X generated by A. Let U and Y be Banach spaces and assume $B \in \mathcal{L}(U,X)$ and $C \in \mathcal{L}(X,Y)$. Then the following are equivalent.*

(i) $\{e^{tA}\}_{t\ge0}$ *is exponentially stable;*

(ii) $\mathbb{G}$ *is a bounded operator on* $L^p(\mathbb{R}_+, X)$*;*

(iii) $\sigma(A) \cap \overline{\mathbb{C}}_+ = \emptyset$ *and* $\displaystyle\sup_{v\in\mathcal{S}(\mathbb{R},X)} \frac{\|\int_{\mathbb{R}}(A_\alpha - is)^{-1}v(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},X)}}{\|\int_{\mathbb{R}} v(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},X)}} < \infty$ *for all* $\alpha \ge 0$*;*

(iv) $\sigma(A) \cap \overline{\mathbb{C}}_+ = \emptyset$, $\displaystyle\sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\|\int_{\mathbb{R}}(A_\alpha - is)^{-1}Bu(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},X)}}{\|\int_{\mathbb{R}} u(s)e^{is(\cdot)}\,ds\|_{L^p(\mathbb{R},U)}} < \infty$

for all $\alpha \geq 0$*, and* (3.4) *is stabilizable;*

(v) $\sigma(A) \cap \overline{\mathbb{C}}_+ = \emptyset$, $\displaystyle\sup_{v \in \mathcal{S}(\mathbb{R},X)} \frac{\| \int_{\mathbb{R}} C(A_\alpha - is)^{-1} v(s) e^{is(\cdot)}\, ds\|_{L^p(\mathbb{R},Y)}}{\| \int_{\mathbb{R}} v(s) e^{is(\cdot)}\, ds\|_{L^p(\mathbb{R},X)}} < \infty$

for all $\alpha \geq 0$*, and* (3.4) *is detectable;*

(vi) $\sigma(A) \cap \overline{\mathbb{C}}_+ = \emptyset$, $\displaystyle\sup_{u \in \mathcal{S}(\mathbb{R},U)} \frac{\| \int_{\mathbb{R}} C(A_\alpha - is)^{-1} Bu(s) e^{is(\cdot)}\, ds\|_{L^p(\mathbb{R},Y)}}{\| \int_{\mathbb{R}} u(s) e^{is(\cdot)}\, ds\|_{L^p(\mathbb{R},U)}} < \infty$

for all $\alpha \geq 0$*, and* (3.4) *is both stabilizable and detectable.*

Moreover, if $\{e^{tA}\}_{t\geq 0}$ *is exponentially stable, then the norm of the input-output operator,* $\mathbb{L} = \mathcal{C}\mathbb{G}\mathcal{B}$*, is equal to*

$$\sup_{u \in \mathcal{S}(\mathbb{R},U)} \frac{\| \int_{\mathbb{R}} C(A - is)^{-1} Bu(s) e^{is(\cdot)}\, ds\|_{L^p(\mathbb{R},Y)}}{\| \int_{\mathbb{R}} u(s) e^{is(\cdot)}\, ds\|_{L^p(\mathbb{R},U)}}.$$

Proof. First note the equivalence of statements (i) and (ii) follows from Remarks 2.3. Also, the implication (i)⇒(vi), as well as the last statement of the theorem, follow from Theorem 3.11. The exponential stability of $\{e^{tA}\}_{t\geq 0}$ is equivalent to the invertibility of $\Gamma_{\mathbb{R}}$ (Corollary 2.6), and so (iii) follows from (i) by Proposition 3.9.

To see that (iii) implies (i), begin by setting $\alpha = 0$. We wish to use properties of $\mathfrak{F}$ as in Proposition 3.8. We begin by observing that if the expression in (iii) is finite, then $\sup_{s\in\mathbb{R}} \|(A - is)^{-1}\| < \infty$. Indeed, if this were not the case, then there would exist $s_n \in \mathbb{R}$ and $x_n \in \mathrm{Dom}(A)$ with $\|x_n\| = 1$ such that $\|(A - is_n)x_n\| \to 0$ as $n \to \infty$. Choose functions $\beta_n \in \mathcal{S}(\mathbb{R})$ with the property that

$$\lim_{n\to\infty} \frac{\| \int_{\mathbb{R}} \beta_n(s)(is_n - is) e^{is(\cdot)}\, ds\|_{L^p(\mathbb{R})}}{\| \int_{\mathbb{R}} \beta_n(s) e^{is(\cdot)}\, ds\|_{L^p(\mathbb{R})}} = 0. \tag{4.5}$$

Note: to construct such a sequence of functions β_n, one takes, without loss of generality, $s_n = 0$ in (4.5) and chooses a "bump" function $\beta_0(s)$ where $\beta_0(0) = 1$ and β_0 has support in $(-1, 1)$. Then set $\beta_n(s) = n\beta_0(ns)$. If $\check{}$ denotes the inverse Fourier transform, then $\check{\beta}_n(\tau) = \check{\beta}_0(\tau/n)$. Also, for $\alpha_n(s) = s\beta_n(s)$, one has $\check{\alpha}_n(\tau) = \frac{1}{n}\check{\alpha}_0(\tau/n)$, and so

$$\frac{\| \int_{\mathbb{R}} \beta_n(s) s e^{is\cdot}\, ds\|^p_{L^p(\mathbb{R})}}{\| \int_{\mathbb{R}} \beta_n(s) e^{is\cdot}\, ds\|^p_{L^p(\mathbb{R})}} = \frac{\|\check{\alpha}_n\|^p}{\|\check{\beta}_n\|^p} = \frac{\left(\frac{1}{n}\right)^p \|\check{\alpha}_0\|^p}{\|\check{\beta}_0\|^p} \to 0, \quad \text{as } n \to \infty.$$

Now, setting $v_n(s) := \beta_n(s)(A - is)x_n$ gives a function v_n in $\mathcal{S}(\mathbb{R}, X)$ with the properties that $(A - is)^{-1} v_n(s) = \beta_n(s)x_n$. Thus,

$$\begin{aligned}
&\frac{\| \int_{\mathbb{R}} (A - is)^{-1} v_n(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} v_n(s) e^{is(\cdot)}\, ds\|} = \frac{\| \int_{\mathbb{R}} \beta_n(s) x_n e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} \beta_n(s)(A - is) x_n e^{is(\cdot)}\, ds\|} \\
&= \frac{\| \int_{\mathbb{R}} \beta_n(s) x_n e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} \beta_n(s)(A - is_n) x_n e^{is(\cdot)}\, ds + \beta_n(s)(is_n - is) x_n e^{is(\cdot)}\, ds\|} \\
&\geq \frac{\| \int_{\mathbb{R}} \beta_n(s) x_n e^{is(\cdot)}\, ds\|}{\|(A - is_n)x_n\| \, \| \int_{\mathbb{R}} \beta_n(s) e^{is(\cdot)}\, ds\| + \| \int_{\mathbb{R}} \beta_n(s)(is_n - is) x_n e^{is(\cdot)}\, ds\|} \\
&= \left(\|(A - is_n)x_n\| + \frac{\| \int_{\mathbb{R}} \beta_n(s)(is_n - is) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} \beta_n(s) e^{is(\cdot)}\, ds\|} \right)^{-1}.
\end{aligned}$$

By the choice of s_n, x_n and β_n, this last expression goes to ∞ as $n \to \infty$, contradicting (iii). Hence if the expression in (iii) is finite for $\alpha = 0$, then $\sup_{s\in\mathbb{R}} \|(A-is)^{-1}\| < \infty$.

Now we may apply Proposition 3.8 (ii) and Proposition 3.9, to obtain

$$\|\Gamma_{\mathbb{R}}\|_{\bullet} = \inf_{v\in\mathcal{S}(\mathbb{R},X)} \frac{\|\Gamma_{\mathbb{R}} f_v\|}{\|f_v\|} = \inf_{v\in\mathcal{S}(\mathbb{R},X)} \frac{\|g_v\|}{\|f_v\|} = \left(\sup_{v\in\mathcal{S}(\mathbb{R},X)} \frac{\|f_v\|}{\|g_v\|} \right)^{-1} > 0.$$

This shows that $0 \notin \sigma_{ap}(\Gamma_{\mathbb{R}})$ and so, by [22], it follows that $\sigma_{ap}(e^{tA}) \cap \mathbb{T} = \emptyset$. On the other hand, since $\sigma(A) \cap i\mathbb{R} = \emptyset$, it follows from the spectral mapping theorem for the residual spectrum, $\sigma_r(e^{tA})$, that

$$\sigma(e^{tA}) \cap \mathbb{T} = \left[\sigma_{ap}(e^{tA}) \cup \sigma_r(e^{tA})\right] \cap \mathbb{T} = \emptyset.$$

The same argument holds for any $\alpha \geq 0$. As a result, $\{e^{tA_\alpha}\}_{t\geq 0}$ is hyperbolic for each $\alpha \geq 0$, and thus $\{e^{tA}\}_{t\geq 0}$ is exponentially stable.

So far it has been shown that the statements (i)–(iii) are equivalent, and that statement (i) implies (vi). By showing that (vi)$\Rightarrow$(iv)$\Rightarrow$(iii) and (vi)$\Rightarrow$(v)$\Rightarrow$(iii), we complete the proof.

To see that (vi) implies (iv), begin by setting $\alpha = 0$. Since (3.4) is detectable, there exists $K \in \mathcal{L}(Y, X)$ such that $A + KC$ generates an exponentially stable semigroup. By the implication (i)$\Rightarrow$(iii) for the semigroup $\{e^{t(A+KC)}\}$, it follows that

$$M_1 := \sup_{v\in\mathcal{S}(\mathbb{R},X)} \frac{\| \int_{\mathbb{R}} (A + KC - is)^{-1} v(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} v(s) e^{is(\cdot)}\, ds\|}$$

is finite. So,

$$\begin{aligned} &\sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\| \int_{\mathbb{R}} (A + KC - is)^{-1} Bu(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} u(s) e^{is(\cdot)}\, ds\|} \\ &= \sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\| \int_{\mathbb{R}} (A + KC - is)^{-1} Bu(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} Bu(s) e^{is(\cdot)}\, ds\|} \cdot \frac{\|B \int_{\mathbb{R}} u(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} u(s) e^{is(\cdot)}\, ds\|} \leq M_1 \|B\|. \end{aligned} \tag{4.6}$$

By hypothesis in (vi),

$$M_2 := \sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\| \int_{\mathbb{R}} C(A - is)^{-1} Bu(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} u(s) e^{is(\cdot)}\, ds\|}$$

is finite. For $u \in \mathcal{S}(\mathbb{R}, U)$, let $w(s) = KC(A - is)^{-1} Bu(s), \quad s \in \mathbb{R}$. Then,

$$\frac{\| \int_{\mathbb{R}} (A + KC - is)^{-1} KC(A - is)^{-1} Bu(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} u(s) e^{is(\cdot)}\, ds\|} \tag{4.7}$$

$$\begin{aligned} &= \frac{\| \int_{\mathbb{R}} (A + KC - is)^{-1} w(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} w(s) e^{is(\cdot)}\, ds\|} \cdot \frac{\|K \int_{\mathbb{R}} C(A - is)^{-1} Bu(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} u(s) e^{is(\cdot)}\, ds\|} \\ &\leq M_1 \|K\| M_2. \end{aligned}$$

Finally, since

$$(A - is)^{-1}B = (A + KC - is)^{-1}B + (A + KC - is)^{-1}KC(A - is)^{-1}B,$$

it follows from (4.6) and (4.7) that

$$\sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\| \int_{\mathbb{R}} (A - is)^{-1}Bu(s)e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} u(s)e^{is(\cdot)}\, ds\|} \le M_1\|B\| + M_1\|K\|M_2.$$

This argument holds for all $\alpha \ge 0$, so the implication (vi)⇒(iv) follows.

To see that (iv) implies (iii) we again argue only in the case $\alpha = 0$. Since (3.4) is stabilizable, there exists $F \in B(X, U)$ such that $A + BF$ generates an exponentially stable semigroup. By the implication (i)⇒(iii) for the semigroup $\{e^{t(A+BF)}\}$, it follows that

$$M_3 := \sup_{v\in\mathcal{S}(\mathbb{R},X)} \frac{\| \int_{\mathbb{R}} (A + BF - is)^{-1}v(s)e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} v(s)e^{is(\cdot)}\, ds\|}$$

is finite. By hypotheses in (iv),

$$M_4 := \sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\| \int_{\mathbb{R}} (A - is)^{-1}Bu(s)e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} u(s)e^{is(\cdot)}\, ds\|}$$

is finite. For $v \in \mathcal{S}(\mathbb{R}, X)$, set $w(s) = F(A + BF - is)^{-1}v(s)$, $s \in \mathbb{R}$. Then,

$$\sup_{v\in\mathcal{S}(\mathbb{R},X)} \frac{\| \int_{\mathbb{R}} (A - is)^{-1}BF(A + BF - is)^{-1}v(s)e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} v(s)e^{is(\cdot)}\, ds\|} \tag{4.8}$$

$$= \frac{\| \int_{\mathbb{R}} (A - is)^{-1}Bw(s)e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} w(s)e^{is(\cdot)}\, ds\|} \cdot \frac{\|F \int_{\mathbb{R}} (A + BF - is)^{-1}v(s)e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} v(s)e^{is(\cdot)}\, ds\|}$$
$$\le M_4\|F\|M_3.$$

Since

$$(A - is)^{-1} = (A + BF - is)^{-1} + (A - is)^{-1}BF(A + BF - is)^{-1},$$

it follows from (4.8) that

$$\sup_{v\in\mathcal{S}(\mathbb{R},X)} \frac{\| \int_{\mathbb{R}} (A - is)^{-1}v(s)e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} v(s)e^{is(\cdot)}\, ds\|} \le M_3 + M_4\|F\|M_3.$$

Thus (iii) follows from (iv). Similar arguments show that (vi)⇒(v) and (v)⇒(iii). □

From the equivalence of statements (i) and (iii), it follows that the growth bound of a semigroup on a Banach space is given by

$$\omega_0(e^{tA}) = \inf\left\{\alpha \in \mathbb{R} : \sup_{v\in\mathcal{S}(\mathbb{R},X)} \frac{\| \int_{\mathbb{R}} (A_\alpha - is)^{-1}v(s)e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} v(s)e^{is(\cdot)}\, ds\|} < \infty\right\}.$$

This is a natural generalization of the formula for the growth bound for a semigroup on a Hilbert space as provided by Gearhart's Theorem, see [18, 27, 29, 33] and cf. Theorem 1.1:

$$\omega_0(e^{tA}) = s_0(A) = \inf\left\{\alpha \in \mathbb{R} : \sup_{\mathrm{Re}\lambda \geq \alpha} \|(A-\lambda)^{-1}\| < \infty\right\}.$$

THEOREM 4.5. *Let $\{e^{tA}\}_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with the property that $s_0(A) = \omega_0(e^{tA})$. Assume* (3.4) *is stabilizable and detectable. If $\overline{\mathbb{C}}_+ \subset \rho(A)$ and $M := \sup_{s\in\mathbb{R}} \|C(A-is)^{-1}B\| < \infty$, then $\{e^{tA}\}_{t\geq 0}$ is exponentially stable.*

Proof. Choose operators $F \in \mathcal{L}(X, U)$ and $K \in \mathcal{L}(Y, X)$ such that the semigroups generated by $A + BF$ and $A + KC$ are exponentially stable. Then $s_0(A + BF) < 0$ and $s_0(A + KC) < 0$, and so

$$M_1 := \sup_{s\in\mathbb{R}} \|(A + BF - is)^{-1}\|, \quad \text{and} \quad M_2 := \sup_{s\in\mathbb{R}} \|(A + KC - is)^{-1}\|.$$

are both finite. Since

$$(A - is)^{-1}B = (A + KC - is)^{-1}B + (A + KC - is)^{-1}KC(A - is)^{-1}B,$$

it follows that

$$M_3 := \sup_{s\in\mathbb{R}} \|(A - is)^{-1}B\| \leq M_2\|B\| + M_2\|K\|M.$$

Also,

$$(A - is)^{-1} = (A + BF - is)^{-1} + (A - is)^{-1}BF(A + BF - is)^{-1},$$

and so

$$\sup_{s\in\mathbb{R}} \|(A - is)^{-1}\| \leq M_1 + M_3\|F\|M_1.$$

Therefore, $\omega_0(e^{tA}) = s_0(A) < 0$. □

The following result, based on [21], describes a particular situation in which $s_0(A) = \omega_0(e^{tA})$.

COROLLARY 4.6. *Assume that for the generator A of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ on a Banach space X there exists an $\omega > \omega_0(e^{tA})$ such that*

$$\int_{-\infty}^{\infty} \|(\omega + i\tau - A)^{-1}x\|_X^2 \, d\tau < \infty \quad \textit{for all} \quad x \in X, \tag{4.9}$$

and

$$\int_{-\infty}^{\infty} \|(\omega + i\tau - A^*)^{-1}x^*\|_{X^*}^2 \, d\tau < \infty \quad \textit{for all} \quad x^* \in X^*, \tag{4.10}$$

where X^* is the adjoint space. Then system (3.4) *is internally stable if and only if it is stabilizable, detectable and externally stable.*

Proof. According to [21] (see also [29, Corollary 4.6.12]), conditions (4.9)–(4.10) imply $s_0(A) = \omega_0(e^{tA})$. Now Theorem 4.5 gives the result. □

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