EVOLUTIONARY SEMIGROUPS AND DICHOTOMY OF LINEAR SKEW-PRODUCT FLOWS ON LOCALLY COMPACT SPACES WITH BANACH FIBERS

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Abstract. We study evolutionary semigroups generated by a strongly continuous semi-cocycle over a locally compact metric space acting on Banach fibers. This setting simultaneously covers evolutionary semigroups arising from nonautonomuous abstract Cauchy problems and C_0-semigroups, and linear skew-product flows.

The spectral mapping theorem for these semigroups is proved. The hyperbolicity of the semigroup is related to the exponential dichotomy of the corresponding linear skew-product flow. To this end a Banach algebra of weighted composition operators is studied. The results are applied in the study of: "roughness" of the dichotomy, dichotomy and solutions of nonhomogeneous equations, Green's function for a linear skew-product flow, "pointwise" dichotomy versus "global" dichotomy, and evolutionary semigroups along trajectories of the flow.

1. Introduction

The spectral theory of linear skew-product flows (or processes) with finite dimensional fibers is, by now, a well-developed area in asymptotics theory of differential equations (see, e.g., [14, 15, 20, 38, 46, 47, 49, 50]). J. Hale in [14, p.60] stressed that this theory should be extended to the infinite dimensional setting. Indeed, in recent years significant progress has been made in the study of linear skew-product flows (LSPFs) with Banach fibers ([8, 9, 10, 28, 48]) over a compact

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metric space (see also [6, 7, 26]). An impressive list of possible applications can be found in [8, 14, 15, 16, 48].

In this paper we consider strongly continuous linear skew-product semiflows on bundles with Banach fibers over a locally compact metric space. Our philosophy is *not* to start with a pointwise construction of stable and unstable foliations (as in [50, 8, 9, 28, 46, 48]), but instead to begin by associating with the linear skew-product flow an *evolutionary semigroup* of operators, $\{T^t\}_{t\geq 0}$, on the space of continuous sections of the bundle. We prove two facts concerning the spectrum of this semigroup that relate its spectral properties to the asymptotic properties of the linear skew-product flow. The first of these facts is the Spectral Mapping Theorem (Theorem 3.3) for $\{T^t\}_{t\geq 0}$ and its generator Γ,

$$e^{t\sigma(\Gamma)} = \sigma(T^t)\backslash\{0\}, \quad t > 0,$$

while the second concerns the description of the spectral projections (Theorem 4.1).

As a result of this approach, we not only answer questions concerning LSPFs, but also address the theory of evolutionary semigroups associated with strongly continuous evolutionary families as studied in the theory of nonautonomous Cauchy problems and the general theory of C_0 semigroups (see [12, 13, 18, 27, 33, 36, 42]). Consequently, classical theorems on the exponential dichotomy of differential equations (see, e.g., [11, 31]) can easily be extended to the case concerning differential equations with unbounded coefficients. Some of the results appearing here were announced in [23].

The connection between evolutionary semigroups and the dichotomy of LSPFs is not new. In one of the first papers in this direction, J. Mather [30] proved that the LSPF generated by the differential of a diffeomorphism of a smooth manifold is Anosov (i.e., is hyperbolic or, in the terminology of the present paper, exponentially dichotomic) if and only if the corresponding evolutionary operator is hyperbolic (it's spectrum does not intersect the unit circle). This led to the notion of the Mather spectrum $\mathcal{M} := \sigma(T)$. Here $T = T^1$ and $\sigma(\cdot)$ denotes the usual spectrum of an operator. This notion is widely used in the theory of hyperbolic dynamical systems (see [40] for further references). The articles [4, 19] proved the spectral mapping theorem for evolutionary semigroups in the finite dimensional setting, and they described the spectral subbundles of these operators via the spectral subbundles of the corresponding LSPF. Moreover, they related the spectrum of Γ to the dynamical spectrum, Σ, of the LSPF, as defined by R. Sacker and G. Sell in [47].

In the present paper we proceed in the opposite direction: we derive the existence of spectral subbundles from the existence, and description, of the Riesz projections corresponding to T; the properties of Σ are described via the spectrum of the generator, $\sigma(\Gamma)$. These techniques build upon those used in [1] which addresses the finite dimensional setting and uses some C^*-algebraic methods. These methods were also used in [25] to prove the results for a uniformly continuous cocycle on a Hilbert space. The results have recently been generalized in [43] to the case involving a strongly continuous cocycle on a compact metric space with Banach fibers. The important articles [43, 44, 45] linked this theory with the theory of evolutionary semigroups associated with strongly continuous evolutionary families (as initiated and developed in [12, 13, 18, 27, 36]). As we will see below, the latter situation corresponds to a LSPF over the translations of $\mathbb{R}$; the evolutionary family can be thought of as the propagator of a well-posed differential equation on a Banach space [33, 34, 51]. For this situation, in the Banach space setting, the Spectral Mapping Theorem has been proven in [22, 23] (see [17],[33] and [39] for a general discussion on this theorem) and the Spectral Projection Theorem in [22, 23, 24, 42, 44, 45].

The Spectral Mapping Theorem appearing in the present paper applies to *any* evolutionary semigroup associated with a LSPF over a nonperiodic flow on a locally compact space, Θ, but it also provides a new proof for semigroups arising from evolutionary families, i.e., when $\Theta = \mathbb{R}$ and the flow is translation on $\mathbb{R}$. In fact, the same idea applies to evolutionary semigroups in the space of divergence-free sections of a bundle—a situation which is important for applications to hydromagnetodynamics [5]. Our proof of the Spectral Projection Theorem for LSPFs also differs from that in [43]. Here, we extend the algebraic technique used in [1, 25] (for Hilbert spaces) so that it applies in the general setting of Banach spaces. One advantage to this approach is that we are able to discuss the case of "discrete" time $t \in \mathbb{Z}$ and "pointwise dichotomies" (see Section 5; cf. [8]).

We now make the previous discussion more explicit by establishing some notation and outlining the main results.

For $t \in \mathbb{R}$, let φ^t be a continuous flow on a locally compact metric space Θ. Throughout the paper, the aperiodic trajectories of the flow are assumed to be dense in Θ. Let X be a Banach space and let $\mathcal{L}_s(X)$ be the space of bounded linear operators on X endowed strong-operator topology. Let $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}_s(X)$ be a continuous (semi)cocycle over φ^t. Since Θ is locally compact, we will allow Φ to grow at infinity not faster then exponentially. The linear skew-product (semi)flow $\hat{\varphi}^t$ is defined by the following formula (see Section 2)

$$\hat{\varphi}^t : \Theta \times X \to \Theta \times X, \quad (\theta, x) \mapsto (\varphi^t\theta, \Phi(\theta, t)x).$$

Example 1.1. As a very particular example, which will be referred as the "norm continuous compact setting," assume Θ is compact and $A \colon \Theta \to \mathcal{L}(X)$ is a continuous operator-valued function. Then, for each $\theta \in \Theta$, the operator $\Phi(\theta, t)$ can be thought of as a solving operator for the variational equation

$$\frac{dx}{dt} = A(\varphi^t\theta)x(t), \quad \theta \in \Theta,\ t \in \mathbb{R}. \tag{1.1}$$

Here, Φ is uniformly continuous and takes invertible values in $\mathcal{L}(X)$.

Example 1.2. As another example (see also [10]), which will be referred to as the "norm continuous line setting," assume $\Theta = \mathbb{R}$, and let $A \colon \mathbb{R} \to \mathcal{L}(X)$ be a bounded, continuous operator-valued function. Let $\{U(\tau, s)\}_{\tau \geq s}$ be the propagator of the nonautonomous differential equation

$$\frac{dx}{dt} = A(t)x(t), \quad t \in \mathbb{R}; \tag{1.2}$$

that is, the solution $x(\cdot)$ of (1.2) satisfies $x(\tau) = U(\tau, s)x(s)$. This can be related to the previous example by defining $\Phi(\theta, t) = U(\theta + t, \theta)$ and identifying the flow as translation, $\varphi^t\theta = \theta + t$ for $\theta \in \Theta = \mathbb{R}$ and $t \in \mathbb{R}$. Since the propagator U satisfies [11] the identity $U(\tau, s) = U(\tau, r)U(r, s)$, for $\tau \geq r \geq s$, and $U(s, s) = I$, Φ is a cocycle over φ^t.

These two examples show (see also [10]) how both the variational equation (1.1) and the nonautonomous equation (1.2) can be addressed in terms of linear skew-product flows on a locally compact metric space. We stress that the *strongly* continuous setting considered in the present paper (as opposed to the norm continuous situation described above) allows one to study the equations (1.1) and (1.2) even in the situation where the operators given by $A(\cdot)$ are *un*bounded.

Given a LSPF $\hat{\varphi}^t$, we associate to it a semigroup $\{T^t\}_{t\geq 0}$ of *evolutionary operators* defined on $C_0(\Theta, X)$, the space of strongly continuous functions vanishing at infinity (with the sup-norm), by

$$(T^t f)(\theta) = \Phi(\varphi^{-t}\theta, t) f(\varphi^{-t}\theta), \quad \theta \in \Theta, \quad f \in C_0(\Theta, X).$$

The generator of such a semigroup will be denoted by Γ. For the line setting this semigroup becomes a well-known [12, 13, 18, 27, 34, 36, 44, 45] evolutionary semigroup:

$$(T^t f)(\tau) = U(\tau, \tau - t) f(\tau - t), \quad \tau \in \mathbb{R}, \quad f \in C_0(\mathbb{R}, X).$$

The generator, Γ, of the evolutionary semigroups arising in the norm continuous setting of examples 1.1 and 1.2 can be expressed, respectively, as

$$(\Gamma f)(\theta) = -\frac{d}{dt} f \circ \varphi^t(\theta)\Big|_{t=0} + A(\theta) f(\theta), \text{ and } (\Gamma f)(\tau) = -\frac{df}{d\tau} + A(\tau) f(\tau).$$

A discussion on the delicate question about when these formulas hold for the strongly continuous setting can be found in [34, 35]. If, for the line setting, $A_0 \equiv A(\tau)$ generates a C_0 semigroup $\{e^{tA_0}\}_{t\geq 0}$, then $(T^t f)(\tau) = e^{tA_0} f(\tau - t)$ and Γ is the closure of $-d/dt + A_0$. If Φ is the differential of a diffeomorphism on a smooth manifold Θ, then T^t is the "push-forward" operator and Γ is the Lie derivative.

In Section 3, for the strongly continuous locally compact setting, we prove the following theorem.

Spectral Mapping Theorem . *The spectrum $\sigma(\Gamma)$ is invariant with respect to translations along the imaginary axis, and the spectrum $\sigma(T^t)$, $t > 0$, is invariant with respect to rotations centered at origin. Moreover, $\sigma(T^t)\backslash\{0\} = e^{t\sigma(\Gamma)}$.*

The proof of this theorem develops some ideas from [30] on "localization" of almost-eigenfunctions for evolutionary operators.

In Section 4 we discuss exponentially dichotomic LSPFs. The terminology "exponential dichotomy," as in [8] (see also [15, 16, 28, 48]), refers to the existence of a strongly continuous, bounded projection-valued function $P\colon \Theta \to \mathcal{L}_s(X)$. This $P(\cdot)$ defines a $\hat{\varphi}^t$-invariant splitting $\Theta \times X = \mathcal{X}_P + \mathcal{X}_Q$ with uniform exponential decay of Φ_P on $\mathcal{X}_P$

and of Φ_Q^{-1} on $\mathcal{X}_Q$, $Q := I - P$. The subscripts here denote the restrictions of the LSPF; since we do not assume that Φ is invertible, the existence of Φ_Q^{-1} is part of the definition of exponential dichotomy.

The main result in Section 4 is the following theorem where an operator T is called *hyperbolic* if $\sigma(T) \cap \mathbb{T} = \emptyset$. If T is hyperbolic, its Riesz projection corresponding to $\sigma(T) \cap \mathbb{D}$ is denoted by $\mathcal{P}$. (Here, $\mathbb{T}$ and $\mathbb{D}$ denote the unit circle and unit disk in $\mathbb{C}$.)

Spectral Projection Theorem . *Assume T is hyperbolic. Then its Riesz projection, $\mathcal{P}$, is an operator of multiplication. That is, if $f \in C_0(\Theta, X)$, then $(\mathcal{P}f)(\theta) = P(\theta)f(\theta)$ for some strongly continuous bounded projection-valued function $P(\cdot)\colon \Theta \to \mathcal{L}_s(X)$.*

We use this theorem to show that the LSPF $\hat{\varphi}^t$ has exponential dichotomy with $P(\cdot)$ if and only if T is hyperbolic, and we relate $P(\cdot)$ with the Riesz projection for T, as above. The related result for compact Θ was proved in [43] (see also [42, 44, 45] for $\Theta = \mathbb{R}$), where a quite different method is used. We stress that our method also works for "discrete" time $t \in \mathbb{Z}$, that is, when φ is a homeomorphism, and $\Phi : \Theta \times \mathbb{Z} \to \mathcal{L}_s(X)$. For "continuous" time, $t \in \mathbb{R}$, together with the Spectral Mapping Theorem, this shows that the LSPF $\hat{\varphi}^t$ has exponential dichotomy with $P(\cdot)$ if and only if Γ is invertible. This generalizes some results in [3, 37]. In particular (see [22, 23]), for $\Theta = \mathbb{R}$ and $A(\tau) \equiv A_0$ this shows that the growth bound for any C_0 semigroup $\{e^{tA_0}\}_{t\geq 0}$ on X coincides with the spectral bound for $-d/dt + A_0$ on $C_0(\mathbb{R}, X)$. A similar fact (see [22]) for L_p-spaces can be used in a variety of contexts (cf. [32, 52]).

Our method of proof exploits the invertibility properties of a particular Banach algebra of weighted translation operators on $C_0(\Theta, X)$. To prove the Spectral Projection Theorem, we consider for each $\theta \in \Theta$ a weighted shift operator, $\pi_\theta(T)$, acting on the space $\ell_\infty(\mathbb{Z}, X)$ and given by the diagonal matrix:

$$\pi_\theta(T) = \operatorname{diag}\{\Phi(\varphi^{n-1}\theta, 1)\}_{n\in\mathbb{Z}} S;$$

here, S denotes the shift operator on $\ell_\infty(\mathbb{Z}, X)$: $(x_n)_{n\in\mathbb{Z}} \mapsto (x_{n-1})_{n\in\mathbb{Z}}$. The hyperbolicity of $\pi_\theta(T)$ is equivalent to the existence of the discrete exponential dichotomy for $\hat{\varphi}^t$ along the trajectory through θ. These operators were, in fact, introduced in [16] for the line setting, and in

[1, 8, 25] for the compact setting. A byproduct of our approach is the following theorem.

Pointwise Dichotomy Theorem . *The LSPF $\hat{\varphi}^t$ has exponential dichotomy on Θ if and only if $\pi_\theta(T)$ is hyperbolic for every $\theta \in \Theta$, and* $\sup\{\|(\lambda - \pi_\theta(T))^{-1}\| : \lambda \in \mathbb{T}, \theta \in \Theta\} < \infty$.

The theorems listed above help to unify a number of ideas concerning "spectral properties" of dynamical systems. Recall that the dynamical spectrum, Σ, for the LSPF $\hat{\varphi}^t$ (see [47]) is the set of all $\lambda \in \mathbb{R}$ such that the LSPF, $\hat{\varphi}^t_\lambda$, corresponding to the cocycle $e^{-\lambda t}\Phi(\theta, t)$ does not have exponential dichotomy. The Bohl spectrum, $\mathcal{B}$, for equation (1.1) is the set of $\lambda \in \mathbb{R}$ such that the equation $x' = [A(\tau) - \lambda]x$ does not have exponential dichotomy (see [11]). Using the above theorems, one can relate all of these sets (in the general strongly continuous, locally compact setting) as follows:

$$\Sigma = \mathcal{B} = \ln \mathcal{M} = \sigma(\Gamma) \cap \mathbb{R} = \ln |\sigma(T) \setminus \{0\}|. \tag{1.3}$$

Finally, consider an autonomous equation $x'(t) = A_0 x(t)$, $t \in \mathbb{R}$, where A_0 generates a C_0 semigroup, $\{e^{tA_0}\}_{t\geq 0}$. Assume, in addition, that the spectral mapping theorem is valid for $\{e^{tA_0}\}_{t\geq 0}$. This happens, e.g., if A_0 is a sectorial operator (see [33] for a detailed discussion). Then each of the sets in (1.3) coincides with $\{z + i\xi : z \in \sigma(A_0), \xi \in \mathbb{R}\} \cap \mathbb{R}$. If X is a Hilbert space and A_0 is self-adjoint, then the last set is just $\sigma(A_0)$. This shows that for abstract Cauchy problems for nonautonomous differential equations and for LSPFs in Banach spaces, the spectrum of Γ should play the same role as the spectrum of A_0 for autonomous equations.

In Section 5 we give several consequences of the results mentioned above. The list of these topics includes: "roughness" of the dichotomy, dichotomy and solutions of nonhomogenious equations (1.1)–(1.2), Green's function for a LSPF, "pointwise" dichotomy versus "global" dichotomy, and evolutionary semigroups along trajectories of the flow. The paper concludes by listing several open problems.

The following notation is used throughout the paper: Θ denotes a locally compact metric space; for $\theta \in \Theta$ and $\delta > 0$, $B(\theta, \delta)$ denotes an open ball of radius δ centered at θ. For any Banach space X, $\mathcal{L}(X)$ denotes the space of bounded linear operators on X, and $\mathcal{L}_s(X)$ denotes this space endowed with the strong-operator topology. For any operator A on X, its domain is denoted by $\mathcal{D}(A)$ and its spectrum by $\sigma(A)$. In $\mathbb{C}$, let $\mathbb{T} = \{z : |z| = 1\}$, and $\mathbb{D} = \{z : |z| \le 1\}$. For a set of operators $d^{(n)}$, $n \in \mathbb{Z}$, in $\mathcal{L}(X)$, $\operatorname{diag}\{d^{(n)}\}_{n\in\mathbb{Z}}$ acts on $\ell_\infty(\mathbb{Z}, X)$ as the infinite matrix consisting of diagonal entries $d^{(n)}$. For a projection P (or $\mathcal{P}$, or a projection-valued function $P(\cdot)$), we use the letter Q (respectively, $\mathcal{Q}$, $Q(\cdot)$) to denote the complementary projection: $Q = I - P$.

2. Evolutionary Semigroups: Preliminaries

2.1. **Evolutionary semigroups.** Let φ^t be a continuous flow on Θ, and let $\Phi\colon \Theta \times \mathbb{R}_+ \to \mathcal{L}(X)$ be a strongly continuous, exponentially bounded cocycle over φ^t. That is, we assume the function

$$(\theta, t) \mapsto \Phi(\theta, t)x$$

to be continuous from $\Theta \times \mathbb{R}_+$ to X for each $x \in X$, and there exist $C, \omega > 0$ such that for every $\theta \in \Theta$,

a) $\Phi(\theta, t+s) = \Phi(\varphi^t\theta, s)\Phi(\theta, t) \quad$ for $t, s >\geq 0$;
b) $\Phi(\theta, 0) = I$;
c) $\|\Phi(\theta, t)\|_{\mathcal{L}(X)} \le Ce^{\omega t}$ for all $t \in \mathbb{R}_+$.

Note that the operators $\Phi(\theta, t)$ are not assumed to be invertible. The last inequality holds automatically if Θ is compact.

The *linear skew-product flow* (LSPF) associated with Φ is the map

$$\hat{\varphi}^t\colon \Theta \times X \to \Theta \times X, \qquad \hat{\varphi}^t(\theta, x) = (\varphi^t\theta, \Phi(\theta, t)x), \qquad t \ge 0. \tag{2.1}$$

We note that for exponentially bounded cocycles, the map $\hat{\varphi}^t$ is continuous if and only if the corresponding cocycle is strongly continuous.

To a cocycle Φ over a flow φ^t, one can associate the family of operators $\{T^t\}_{t\ge 0}$ in $B(C_0(\Theta, X))$ defined by

$$(T^t f)(\theta) = \Phi(\varphi^{-t}\theta, t) f(\varphi^{-t}\theta), \quad \theta \in \Theta, \quad f \in C_0(\Theta, X). \tag{2.2}$$

It's easy to check that this defines a semigroup of operators: $T^{t+s} = T^t T^s$, $T^0 = I$. As noted below, this *evolutionary semigroup* is a strongly continuous semigroup on $C_0(\Theta, X)$ if and only if Φ is a strongly continuous, exponentially bounded cocycle. The generator will be denoted by Γ.

Theorem 2.1. *$\{T^t\}_{t\geq 0}$, as defined in (2.2), is a C_0 semigroup on $C_0(\Theta, X)$ if and only if Φ is an exponentially bounded, strongly continuous cocycle.*

Proof. If Φ is a strongly continuous cocycle with $\|\Phi(\theta,t)\| \leq Ce^{\omega t}$ for some $C, \omega > 0$, then $\{T^t\}_{t\geq 0}$ can be seen to be strongly continuous as in [25] or [43].

Conversely, assume $\{T^t\}_{t\geq 0}$ is a C_0 semigroup. Clearly, Φ is exponentially bounded. Fix $\theta_0 \in \Theta$, $t_0 \in \mathbb{R}$, and $x \in X$. Let $\epsilon > 0$. For $\theta \in \Theta$, and $t \in \mathbb{R}$, consider

$$\begin{aligned}\|\Phi(\theta,t)x - \Phi(\theta_0,t_0)x\| \leq \|\Phi(\theta_0,t)x - \Phi(\theta_0,t)x\| + \|\Phi(\theta_0,t)x - \Phi(\varphi^{t-t_0}\theta_0,t_0)x\| \\ + \|\Phi(\varphi^{t-t_0}\theta_0,t_0)x - \Phi(\theta_0,t_0)x\|.\end{aligned} \tag{2.3}$$

Let D be a compact set in Θ containing θ_0 in its interior. Choose $\alpha\colon \Theta \to [0,1]$ with compact support such that $\alpha(\theta) = 1$ for all $\theta \in D$. Define $f \in C_0(\Theta, X)$ as $f(\theta) = \alpha(\theta)x$, and note that for $\theta \in D$ and $t \in \mathbb{R}$,

$$\Phi(\theta,t)x = (T^t f)(\varphi^t\theta).$$

First consider the middle term on the right-hand side of (2.3) and choose $\delta_1 > 0$ such that $\varphi^{t-t_0}\theta_0 \in D$ for all $|t - t_0| < \delta_1$. Choose $\delta_2 \in (0, \delta_1]$ such that $|t-t_0| < \delta_2$ implies $\|T^t f - T^{t_0} f\| \leq \epsilon/3$. Then for $|t - t_0| < \delta_2$,

$$\begin{aligned}\|\Phi(\theta_0,t)x - \Phi(\varphi^{t-t_0}\theta_0,t_0)x\| &= \|(T^t f)(\varphi^t\theta_0) - (T^{t_0} f)(\varphi^{t_0}(\varphi^{t-t_0}\theta_0))\| \\ &= \|(T^t f)(\varphi^t\theta_0) - (T^{t_0} f)(\varphi^t\theta_0)\| \\ &\leq \sup_{\theta\in\Theta} \|T^t f(\theta) - T^{t_0} f(\theta)\| < \epsilon/3.\end{aligned}$$

Secondly, since $T^t f$ is continuous, there exists $\delta' > 0$ such that $d(\theta_1,\theta_2) < \delta'$ implies $\|T^{t_0} f(\theta_1) - T^{t_0} f(\theta_2)\| < \epsilon/3$. Since $t \mapsto \varphi^t\theta_0$ is continuous, there exists $\delta_3 \in (0, \delta_2]$ such that $|t - t_0| < \delta_3$ implies $d(\varphi^t\theta_0, \varphi^{t_0}\theta_0) < \delta'$. Then, $|t - t_0| < \delta_3$ implies

$$\begin{aligned}\|\Phi(\varphi^{t-t_0}\theta_0,t_0)x - \Phi(\theta_0,t_0)x\| &= \|T^{t_0} f(\varphi^{t_0}(\varphi^{t-t_0}\theta_0)) - T^{t_0} f(\varphi^{t_0}\theta_0)\| \\ &= \|T^{t_0} f(\varphi^t\theta_0) - T^{t_0} f(\varphi^{t_0}\theta_0)\| < \epsilon/3.\end{aligned}$$

Finally, choose $\delta'' > 0$ so that $B(\theta_0, \delta'') \subset D$, and $\theta \in B(\theta_0, \delta'')$ implies

$$\|(T^t f)(\varphi^t\theta) - (T^t f)(\varphi^t\theta_0)\| < \epsilon/3.$$

We note that $\delta'' = \delta''(t)$ depends on t, but on the compact interval $[t_0 - \delta_3, t_0 + \delta_3]$, the map $t \mapsto T^t f(\varphi^t \cdot\,)$ is uniformly continuous, so δ'' may be chosen independent of t. Therefore, if $|t - t_0| < \delta_3$ and $\theta \in B(\theta_0, \delta'')$, then (2.3) shows that $\|\Phi(\theta, t)x - \Phi(\theta_0, t_0)x\| < \epsilon$. □

2.2. **Examples.** We give several examples of cocycles and corresponding evolutionary semigroups.

Example 2.2. *Norm continuous compact setting.* Let φ^t be a continuous flow on a compact metric space Θ, and let $A\colon \Theta \to \mathcal{L}(X)$ be (norm) continuous. For each $\theta \in \Theta$ consider the equation

$$\frac{dx}{dt} = A(\varphi^t\theta)x(t), \quad \theta \in \Theta,\ t \in \mathbb{R}. \tag{2.4}$$

Let $\Phi(\theta, t)$, $t \in \mathbb{R}$, be the solving operator for (2.4): $x(t) = \Phi(\theta, t)x(0)$. Then Φ is a cocycle and $A(\theta) = \frac{d}{dt}\Phi(\theta, t)\Big|_{t=0}$. Denote $(\mathbf{d}f)(\theta) = \frac{d}{dt} f \circ \varphi^t(\theta)\Big|_{t=0}$. The generator Γ of the group (2.2) is given as follows:

$$(\Gamma f)(\theta) = -(\mathbf{d}f)(\theta) + A(\theta)f(\theta), \tag{2.5}$$

and its domain $\mathcal{D}(\Gamma)$ is, in this case, $\mathcal{D}(\Gamma) = \{f \in C_0(\Theta, X) : \mathbf{d}f \in C_0(\Theta, X)\}$.

Equations of the type (2.4) arise from two sources. Firstly, they can be thought of as a linearization of a nonlinear equation in X in the vicinity of a compact invariant set Θ for the nonlinear equation. Secondly, Θ might be a compact hull, $\Theta = \text{closure}\{a(\,\cdot\, + \tau) : \tau \in \mathbb{R}\}$, of a given function $a\colon \mathbb{R} \to \mathcal{L}(X)$ (see [14, 46]).

Example 2.3. *Strongly continuous compact setting.* We now allow the operators $A(\theta)$ in (2.4) to be unbounded. A strongly continuous (semi)cocycle Φ is said to solve (2.4) if: For every $\theta \in \Theta$ and every $x_\theta \in \mathcal{D}(A(\theta))$ the function $t \mapsto x(t) := \Phi(\theta, t)x_\theta$ is differentiable for $t > 0$, $x(t) \in \mathcal{D}(A(\varphi^t\theta))$, and $x(\cdot)$ satisfies (2.4). By Theorem 2.1 the cocycle generates an evolutionary semigroup given by (2.2).

This setting might occur if, after the linearization of a nonlinear equation in X, the Frechet derivative $A(\theta)$ is an unbounded operator. Numerous examples of this strongly continuous setting can be found in [8].

Example 2.4. To be more specific, in the setting of Example 2.3, assume $A(\theta) \equiv A_0$, $\theta \in \Theta$, where A_0 generates a C_0 semigroup $\{e^{tA_0}\}_{t\geq 0}$ on X. The cocycle Φ for (2.4) is $\Phi(\theta, t) = e^{tA_0}$. In the tensor product $C_0(\Theta, X) = C_0(\Theta, \mathbb{R}) \bigotimes X$ one can express the evolutionary semigroup defined in (2.2) by $T^t = V^t \bigotimes e^{tA_0}$, where $V^t f := f \circ \varphi^{-t}$. Then (see, e.g., [33, p. 23]) the generator Γ_0 of $\{T^t\}_{t\geq 0}$ is the closure of the operator $\Gamma'_0 f = -\mathbf{d}f + A_0 f$ with

$$\mathcal{D}(\Gamma'_0) = \{f \in C_0(\Theta, X) : \mathbf{d}f \in C_0(\Theta, X),\ f : \Theta \to \mathcal{D}(A_0),\ \mathbf{d}f - A_0 f \in C_0(\Theta, X)\}.$$

Example 2.5. Suppose, in the setting of Example 2.3, that

$$A(\theta) = A_0 + A_1(\theta), \quad \theta \in \Theta, \tag{2.6}$$

where A_0 is a generator of a C_0 semigroup $\{e^{tA_0}\}_{t\geq 0}$ on X and $A_1 : \Theta \to \mathcal{L}_s(X)$ is (strongly) continuous and bounded on Θ. Note that $A_1(\cdot)$ defines an operator $\mathcal{A}_1 \in B(C_0(\Theta, X))$ by the rule $(\mathcal{A}_1 f)(\theta) = A_1(\theta) f(\theta)$. For Γ_0 as in Example 2.4 we have:

Proposition 2.6. *Assume that there exists a strongly continuous cocycle Φ that solves* (2.4) *with $A(\cdot)$ as in* (2.6). *Then the generator Γ of the evolutionary semigroup* (2.2) *is given by*

$$(\Gamma f)(\theta) = -(\mathbf{d}f)(\theta) + A_0 f(\theta) + A_1(\theta) f(\theta), \quad \mathcal{D}(\Gamma) = \mathcal{D}(\Gamma_0). \tag{2.7}$$

Proof. Since Φ defines a classical solution of (2.4)–(2.6), Φ also is a mild solution of (2.4), that is, Φ satisfies

$$\Phi(\theta, t)x = e^{tA_0}x + \int_0^t e^{(t-\tau)A_0} A_1(\varphi^\tau \theta) \Phi(\theta, \tau) x \, d\tau, \tag{2.8}$$

for $x \in X$, $\theta \in \Theta$, $t \geq 0$. (We point out the interesting Theorem 5.1 in [10] where the existence of a mild solution Φ was proved for any equations (2.4)–(2.6).)

Since Γ_0 generates a C_0 semigroup, and $\mathcal{A}_1 \in B(C_0(\Theta, X))$, the operator $\Gamma = \Gamma_0 + \mathcal{A}_1$ also generates (see, e.g., [39, p. 77]) a C_0 semigroup, $\{S^t\}_{t\geq 0}$, which is the unique solution of the integral equation

$$S^t f = e^{t\Gamma_0} f + \int_0^t e^{(t-\tau)\Gamma_0} \mathcal{A}_1 S^\tau f \, d\tau, \quad f \in C_0(\Theta, X),\ t \geq 0. \tag{2.9}$$

We show that $S^t = T^t$. Indeed, for $f \in C_0(\Theta, X)$ define $g = \int_0^t e^{(t-\tau)\Gamma_0}\mathcal{A}_1 T^\tau f\, d\tau$. Then (2.8) implies

$$\begin{aligned} g(\theta) &= \int_0^t e^{(t-\tau)A_0} A_1(\varphi^\tau(\varphi^{-t}\theta))\Phi(\varphi^{-t}\theta,\tau) f(\varphi^{-t}\theta)\, d\tau \\ &= \Phi(\varphi^{-t}\theta, t) f(\varphi^{-t}\theta) - e^{tA_0} f(\varphi^{-t}\theta) \\ &= (T^t f)(\theta) - \left(e^{t\Gamma_0} f\right)(\theta). \end{aligned}$$

Therefore, $\{T^t\}_{t\geq 0}$ satisfies (2.9), and so $T^t = S^t$. The generator, $\Gamma = \Gamma_0 + \mathcal{A}_1$, of this semigroup is given by (2.7). □

Example 2.7. *Norm continuous line setting.* Let $\Theta = \mathbb{R}$, and assume $A\colon \mathbb{R} \to \mathcal{L}(X)$ is a bounded continuous function. Let $U(\tau, s)$, $\tau, s \in \mathbb{R}$, denote the propagator for the equation

$$\frac{dx}{dt} = A(t)x(t), \quad t \in \mathbb{R}. \tag{2.10}$$

This means that the solution $x(\cdot)$ of (2.10) satisfies $x(\tau) = U(\tau, s)x(s)$. Denote $\varphi^t(\tau) = \tau + t$ and $\Phi(\tau, t) = U(\tau + t, \tau)$, for $\tau, t \in \mathbb{R}$ (cf. [10]). Then Φ is a cocycle. In this case, the group (2.2) is given by $(T^t f)(\tau) = U(\tau, \tau - t) f(\tau - t)$, $f \in C_0(\mathbb{R}, X)$, and the generator Γ is given by the formula:

$$(\Gamma f)(\tau) = -\frac{df}{d\tau} + A(\tau) f(\tau), \quad \mathcal{D}(\Gamma) = \{f \in C_0(\mathbb{R}, X) : f' \in C_0(\mathbb{R}, X)\}.$$

Example 2.8. *Strongly continuous line setting.* In the case of $\Theta = \mathbb{R}$, consider a well-posed differential equation (2.10) with, generally, unbounded operators $A(\tau)$. Let $\{U(t, s)\}_{t\geq s}$ be the associated strongly continuous evolutionary family of operators in $\mathcal{L}(X)$. This means that $(\tau, s) \mapsto U(\tau, s)x$ is continuous for each $x \in X$, $U(\tau, s) = U(\tau, r)U(r, s)$ for $\tau \geq r \geq s$, $U(s, s) = I$, and $\|U(\tau, s)\| \leq Ce^{\omega(\tau-s)}$ for some $C, \omega > 0$. The flow φ^t, the cocycle Φ, and the evolutionary semigroup can be defined exactly as in the previous example. If, in particular, $A(\tau) \equiv A_0$ is a generator of a C_0 semigroup $\{e^{tA_0}\}_{t\geq 0}$ on X, then $U(\tau, s) = e^{(\tau-s)A_0}$, and Γ is the closure of

$$\Gamma' = -\frac{d}{dt} + A_0, \quad \mathcal{D}(\Gamma') = \{f \in C_0(\mathbb{R}, X) : f' + A_0 f \in C_0(\mathbb{R}, X), f : \mathbb{R} \to \mathcal{D}(A_0)\}.$$

For $\Theta = \mathbb{R}$ consider now the case where each of the operators $A(t)$, $t \in \mathbb{R}$ in (2.10) is an (unbounded) operator on X with dense domain.

For each $t \in \mathbb{R}$, denote the domain of $A(t)$ by $\mathcal{D}_t := \mathcal{D}(A(t))$. Consider the following nonautonomous abstract Cauchy problem (cf. [34]):

$$\frac{dx}{dt} = A(t)x(t) \text{ for } t \geq s \in \mathbb{R}, \quad \text{and } x(s) = x_s, \tag{2.11}$$

where $x_s \in \mathcal{D}_s$, $s \in \mathbb{R}$. We say (see [34]) that the evolutionary family $\{U(\tau, s)\}_{\tau \geq s}$ solves (2.11) if $x(\cdot) = U(\cdot\,, s)x_s$ is a unique solution of (2.11) for every $x_s \in \mathcal{D}_s$, that is $x(\cdot)$ is differentiable, $x(t) \in \mathcal{D}_t$ for $t \geq s$, and (2.11) holds.

Proposition 2.9. *Assume that a strongly continuous evolutionary family $\{U(\tau, s)\}_{\tau \geq s}$ solves (2.11). Then the generator Γ of the evolutionary semigroup $(T^t f)(\tau) = U(\tau, \tau - t)f(\tau - t)$ on $C_0(\mathbb{R}, X)$ is the closure of the operator Γ', with domain consisting of differentiable functions f such that $f(t) \in \mathcal{D}_t$ and $\Gamma' f \in C_0(\mathbb{R}, X)$, defined as follows:*

$$(\Gamma' f)(\tau) = -\frac{df}{d\tau} + A(\tau)f(\tau), \quad \tau \in \mathbb{R}. \tag{2.12}$$

Proof. Fix $s \in \mathbb{R}$ and $x_s \in \mathcal{D}_s$. For any smooth $\alpha : \mathbb{R} \to \mathbb{R}$ with compact support $\operatorname{supp} \alpha \subset [s, \infty)$, consider a function $f \in C_0(\mathbb{R}, X)$, defined by

$$f(\tau) = \alpha(\tau)U(\tau, s)x_s \text{ for } \tau > s \text{ and } f(\tau) = 0 \text{ for } \tau \leq s. \tag{2.13}$$

Then $\Gamma f = \Gamma' f$. Indeed

$$(T^t f)(\tau) = \alpha(\tau - t)U(\tau, \tau - t)U(\tau - t, s)x_s = \alpha(\tau - t)U(\tau, s)x_s$$

for $\tau - t > s$, and zero otherwise. Hence,

$$\frac{d}{dt}(T^t f)(\tau)\bigg|_{t=0} = -\alpha'(\tau)U(\tau, s)x_s, \quad \tau \in \mathbb{R}.$$

On the other hand, since $\tau \mapsto U(\tau, s)x_s$ satisfies (2.11), one has:

$$\frac{df}{d\tau} = \alpha'(\tau)U(\tau, s)x_s + \alpha(\tau)\frac{d}{d\tau}U(\tau, s)x_s = \alpha'(\tau)U(\tau, s)x_s + \alpha(\tau)A(\tau)U(\tau, s)x_s.$$

To finish the proof, we need to show that linear combinations of functions f, as in (2.13), are dense in $C_0(\mathbb{R}, X)$. To see this, first observe that the set of finite sums $\sum \beta_j v_j$ with arbitrary $v_j \in X$ and smooth $\beta_j : \mathbb{R} \to \mathbb{R}$ with compact support, is dense in $C_0(\mathbb{R}, X)$.

Now, consider any function $g = \beta v$ with fixed $v \in X$ and smooth $\beta : \mathbb{R} \to \mathbb{R}$ with compact support. We show that g can be approximated by a sum of functions f as in (2.13).

To see that, fix $\epsilon > 0$. For every $\tau_0 \in \operatorname{supp}\beta$, the map $(\tau, s) \mapsto U(\tau, s)v$, $\tau \geq s$ is continuous at the point (τ_0, τ_0) by the definition of $\{U(\tau, s)\}$. Hence, there exist $s_0 \leq s_0'$ such that for the interval $I_0 = I(\tau_0) := (s_0, s_0')$, containing τ_0, one has $\|U(\tau, s_0)v - v\| \leq \epsilon/2$ for $\tau \in I_0$. Thus, we obtain an open covering of $\operatorname{supp}\beta$, formed by $I(\tau_0)$, $\tau_0 \in \operatorname{supp}\beta$. Take a finite subcovering $\{I_j\}_{j=1}^n$. For $I_j := (s_j, s_j')$, one has $\|U(\tau, s_j)v - v\| \leq \epsilon/2$ for $\tau \in I_j$, $j = 1, \ldots, n$.

Consider a smooth partition of unity $\{\gamma_j\}_{j=1}^n$ for $\{I_j\}_{j=1}^n$, that is, smooth functions $\gamma_j : \mathbb{R} \to [0, 1]$ such that

$$\sum_{j=1}^{n} \gamma_j(\tau) = 1 \text{ for } \tau \in \operatorname{supp}\beta, \text{ and } \operatorname{supp}\gamma_j \subset I_j, j = 1, \ldots, n.$$

Since the $\mathcal{D}_{s_j}$ are dense in X, one can choose $v_j \in \mathcal{D}_{s_j}$ such that $\|v - v_j\| \leq \epsilon/2$.

Define the function

$$h(\tau) := \sum_{j=1}^{n} \beta(\tau)\gamma_j(\tau)U(\tau, s_j)v_j, \quad \tau \in \mathbb{R}.$$

Since $\operatorname{supp}\gamma_j \subset (s_j, s_j')$, the function h is a sum of functions f as in (2.13). Also,

$$\begin{aligned} \|g(\tau) - h(\tau)\| &= \left\| \beta(\tau) \sum_{j=1}^{n} \gamma_j(\tau)v - \sum_{j=1}^{n} \beta(\tau)\gamma_j(\tau)U(\tau, s_j)v_j \right\| \\ &\leq |\beta(\tau)| \left(\sum_{j=1}^{n} \gamma_j(\tau)\|v - v_j\| + \sum_{j=1}^{n} \gamma_j(\tau)\|v_j - U(\tau, s_j)v_j\| \right) \\ &\leq \epsilon \max_{\tau} |\beta(\tau)|, \end{aligned}$$

and the proof is completed. □

2.3. **An algebra of weighted translation operators.** In Section 4 we study the spectrum of $T = T^1$, that is, the invertibility of $b = \lambda I - T$. This operator belongs to an algebra, $\mathfrak{B}$, of weighted translation operators which we now define.

Let $C_b(\Theta; \mathcal{L}_s(X))$ denote the set of strongly continuous and bounded functions $a \colon \Theta \to \mathcal{L}_s(X)$. For $a \in C_b(\Theta, \mathcal{L}_s(X))$, set $\|a\|_u := \sup_{\theta\in\Theta} \|a(\theta)\|_{\mathcal{L}(X)}$.

Such a function induces a multiplication operator on $C_0(\Theta, X)$ defined by $(M_a f)(\theta) = a(\theta)f(\theta)$. The mapping $a \mapsto M_a$ is an isometry from the Banach space $C_b(\Theta, \mathcal{L}_s(X))$ to $B(C_0(\Theta, X))$ and so the operator M_a will be denoted simply by a, its norm given by $\|a\| = \|a\|_{B(C_0(\Theta,X))} = \|a\|_u$. Let $\mathfrak{A}$ denote the set of all such multiplication operators $a = M_a \in B(C_0(\Theta, X))$.

Now let $\varphi = \varphi^1$ be a homeomorphism on Θ. Denote by V the translation operator on $C_0(\Theta, X)$ given by $(Vf)(\theta) = f(\varphi^{-1}\theta)$. Define $\mathfrak{B}$ to be the set of all operators $b \in B(C_0(\Theta, X))$ such that

$$b = \sum_{k=-\infty}^{\infty} a_k V^k, \quad \text{where } a_k \in \mathfrak{A}, \text{ and } \|b\|_1 := \sum_{k=-\infty}^{\infty} \|a_k\| < \infty. \tag{2.14}$$

Proposition 2.10. *If the set of aperiodic points of φ is dense in Θ, then the representation* (2.14) *of an element $b \in \mathfrak{B}$ is unique.*

Proof. This follows from the observation that for any polynomial $b_N = \sum_{k=-N}^{N} a_k V^k$ in $B(C_0(\Theta, X))$, where $a_k \in \mathfrak{A}$ and $N \in \mathbb{N}$,

$$\|b_N\|_{B(C_0(\Theta,X))} \geq \|a_k\|_{B(C_0(\Theta,X))}, \quad |k| \leq N. \tag{2.15}$$

To prove (2.15), first note that by replacing b by bV^{-k} it suffices to consider only the case $k = 0$. Fix $\epsilon > 0$. For $a \in \mathfrak{A}$, $\|a\|_{B(C_0(\Theta,X))} = \sup_{\theta\in\Theta} \sup_{\|x\|_X=1} \|a(\theta)x\|_X$, so there exists a nonperiodic point θ_0 of $\{\varphi^t\}_{t\in\mathbb{R}}$, and a $x \in X$, $\|x\| = 1$, such that

$$\|a_0(\theta_0)x\|_X \geq \|a_0\|_{B(C_0(\Theta,X))} - \epsilon.$$

Choose $\delta > 0$ such that for $B = B(\theta_0, \delta)$, $\varphi^k(B) \cap \varphi^j(B) = \emptyset$ for $k \neq j$, $|k|$, $|j| \leq N$. Choose a continuous function $\alpha : \Theta \to [0, 1]$ such that $\alpha(\theta_0) = 1$ and $\alpha(\theta) = 0$ for $\theta \notin B$. Define $f \in C_0(\Theta, X)$ by

$$f(\theta) = \begin{cases} \alpha(\varphi^{-k}\theta)x, & \text{if } \theta \in \varphi^k(B),\ |k| \leq N \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|f\|_{C_0(\Theta,X)} = 1$ and

$$\begin{aligned}\|b\|_{B(C_0(\Theta,X))} \geq \|bf\|_{C_0(\Theta,X)} &= \max_{\theta\in\Theta} \left\| \sum_{k=-N}^{N} a_k(\theta)\alpha(\varphi^{-k}\theta)x \right\|_X \\ &= \max_{\theta\in\Theta} \max_{|k|\leq N} \|a_k(\theta)\alpha(\varphi^{-k}\theta)x\|_X \geq \|a_0(\theta_0)x\|_X \\ &\geq \|a_0\|_{B(C_0(\Theta,X))} - \epsilon.\end{aligned}$$

□

Proposition 2.11. *$(\mathfrak{B}, \|\cdot\|_1)$ is a Banach algebra.*

Proof. The norm $\|\cdot\|_1$ is an algebra norm. To see that it is complete, consider a $\|\cdot\|_1$-Cauchy sequence $\{b^{(n)}\}_{n=1}^{\infty}$ in $\mathfrak{B}$, $b^{(n)} = \sum_{k=-\infty}^{\infty} a_k^{(n)} V^k$. Set $a_k = \|\cdot\|_1\text{-}\lim_{m\to\infty} a_k^{(m)}$, $k \in \mathbb{Z}$. For every m, n in $\mathbb{Z}$,

$$\left| \sum_{k=-\infty}^{\infty} \|a_k^{(m)}\| - \sum_{k=-\infty}^{\infty} \|a_k^{(n)}\| \right| \leq \|b^{(m)} - b^{(n)}\|_1.$$

So a sequence $(\|a_k^{(m)}\|)_{k\in\mathbb{Z}} \in \ell_1(\mathbb{Z}, \mathbb{R})$ converges to $(\|a_k\|)_{k\in\mathbb{Z}}$ in $\ell_1(\mathbb{Z}, \mathbb{R})$. Hence $b = \sum_k a_k V^k$ is an element of $\mathfrak{B}$. For every $\epsilon > 0$ and for sufficiently large m, n, the inequality

$$\epsilon \geq \|b^{(m)} - b^{(n)}\|_1 = \sum_{k=-\infty}^{\infty} \|a_k^{(m)} - a_k^{(n)}\| \geq \sum_{k=-N}^{N} \|a_k^{(m)} - a_k^{(n)}\|$$

holds for every $N \in \mathbb{N}$. Taking the limit as $n \to \infty$, and then the limit as $N \to \infty$ shows $\|b^{(m)} - b\|_1 \to 0$. □

In Section 4, we consider the relationship between the invertibility of a weighted translation operator b in $\mathfrak{B}$ and a family of representations (weighted shift operators), $\pi_\theta(b)$, in $B(\ell_\infty(\mathbb{Z}, X))$. To define the representation π_θ of $\mathfrak{B}$, denote by S the shift operator on $\ell_\infty(\mathbb{Z}, X)$:

$$S: \ (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n-1})_{n\in\mathbb{Z}}, \quad (x_n)_{n\in\mathbb{Z}} \in \ell_\infty(\mathbb{Z}, X).$$

For $a \in \mathfrak{A}$ and $\theta \in \Theta$, let $\pi_\theta(a) = \operatorname{diag}\{a(\varphi^n\theta)\}_{n\in\mathbb{Z}}$. Defining $\pi_\theta(aV) = \pi_\theta(a)S$ gives a continuous homomorphism from $\mathfrak{B}$ into $B(\ell_\infty(\mathbb{Z}, X))$: for $b = \sum_k a_k S^k$ in $\mathfrak{B}$,

$$\pi_\theta(b) = \sum_{k=-\infty}^{\infty} \pi_\theta(a_k) S^k. \tag{2.16}$$

With this in mind, let $\mathfrak{C}$ denote the set of operators in $B(\ell_\infty(\mathbb{Z}, X))$ of the form $d = \operatorname{diag}\{d^{(n)}\}_{n\in\mathbb{Z}}$, where $d^{(n)} \in \mathcal{L}(X)$, and consider the Banach algebra $(\mathfrak{D}, \|\cdot\|_1)$ of operators on $\ell_\infty(\mathbb{Z}, X)$ of the form

$$d = \sum_{k=-\infty}^{\infty} d_k S^k \ \text{ where } \ d_k \in \mathfrak{C}, \ \text{ and } \ \|d\|_1 := \sum_{k=-\infty}^{\infty} \|d_k\|_{B(\ell_\infty(\mathbb{Z},X))} < \infty.$$

The following lemma points out, in particular, that $\mathfrak{D}$ is an inverse-closed subalgebra of $B(\ell_\infty(\mathbb{Z}, X))$. The proof follows exactly as in Lemma 1.6 and Remark 1.7 of [24].

Proposition 2.12. *Let* $d(\theta) = \sum_{k=-\infty}^{\infty} d_k(\theta)S^k$ *be an operator in* $\mathfrak{D}$*, and assume that*

$$\sum_{k=-\infty}^{\infty} \sup_{\theta\in\Theta} \|d_k(\theta)\|_{B(\ell_\infty(\mathbb{Z},X))} < \infty.$$

If $d(\theta)^{-1} \in B(\ell_\infty(\mathbb{Z}, X)))$ *for each* $\theta \in \Theta$ *and if for some* $C > 0$*,*

$$\sup_{\theta\in\Theta} \|d(\theta)^{-1}\|_{B(\ell_\infty(\mathbb{Z},X))} < C,$$

then $d(\theta)^{-1} \in \mathfrak{D}$ *for each* $\theta \in \Theta$*. Moreover, if* $d(\theta)^{-1} = \sum_{k=-\infty}^{\infty} c_k(\theta)S^k$*, then*

$$\sum_{k=-\infty}^{\infty} \sup_{\theta\in\Theta} \|c_k(\theta)\|_{B(\ell_\infty(\mathbb{Z},X))} < \infty.$$

3. Spectral Mapping Theorem

In this section we prove that the spectral mapping property

$$e^{t\sigma(\Gamma)} = \sigma(T^t)\backslash\{0\}, \quad t > 0, \tag{3.1}$$

holds for any evolutionary semigroup (2.2) with generator Γ. As usual, the nonperiodic points of $\{\varphi^t\}_{t\in\mathbb{R}}$ are assumed to be dense in Θ. For a general C_0 semigroup $\{T^t\}_{t\geq 0}$ generated by an operator Γ, the spectral inclusion $e^{t\sigma(\Gamma)} \subseteq \sigma(T^t)\backslash\{0\}$ is well known, as are examples showing that the inclusion is, in general, proper. Moreover, equality holds for the point spectrum and residual spectrum [33], so to prove (3.1) we focus on the approximate point spectrum, $\sigma_{ap}(\cdot)$. By a standard rescaling technique [33], it suffices to consider only $T = T^1$.

We build on the ideas of [30] and observe, in particular, that the ϵ-eigenfunctions of T obey the following "localization" principle: if $1 \in \sigma_{ap}(T)$, then for every $N \in \mathbb{N}$, there exists a point $\theta_0 \in \Theta$, a function $f \in C_0(\Theta, X)$ with $\|f\| = 1$, and an open neighborhood D of θ_0 such that $\operatorname{supp} f \subseteq \cup_{j=0}^{2N}\varphi^j(D)$ and $\|Tf - f\| = O(\frac{1}{N})$. Using f, we construct a function $g \in C_0(\Theta, X)$ such that $\|\Gamma g\| = O(\frac{1}{N})\|g\|$, hence showing that $0 \in \sigma_{ap}(\Gamma)$.

Before beginning the proof of (3.1), we prove two technical lemmas which verify the existence of an appropriate $f \in C_0(\Theta, X)$ and aid in the subsequent construction of g, as described above.

Lemma 3.1. *Let $A \in \mathcal{L}(X)$ and assume $1 \in \sigma_{ap}(A)$. For each $N \geq 2$, there exists $x \in X$, $\|x\| = 1$, such that*

a) $\|A^N x - x\|_X \leq \frac{1}{8}$;
b) $\|A^k x\|_X \leq 2 \quad$ *for* $k = 0, 1, \ldots, 2N$.

Proof. Set $c = \sum_{k=0}^{2N} \|A\|^k$. From the identity $A^k - I = (A^{k-1} + \ldots + A + I)(A - I)$, for $k = 2, \ldots, 2N$, it follows that for any $x \in X$,

$$\|(A^k - I)x\| \leq c\|(A - I)x\|, \quad k = 0, 1, \ldots, 2N. \tag{3.2}$$

Since $1 \in \sigma_{ap}(A)$, there exists a $x \in X$, $\|x\| = 1$, such that $\|(A-I)x\| \leq \frac{1}{8c}$. Set $k = N$ in (3.2) to obtain a). Moreover, (3.2) shows that for $k = 0, 1, \ldots, 2N$,

$$\|A^k x\| \leq \|(A^k - I)x\| + \|x\| \leq c\|(A - I)x\| + 1 \leq c\frac{1}{8c} + 1 \leq 2.$$

□

Now let $\theta_0 \in \Theta$ be a nonperiodic point of φ^t, and fix $s \in (0, 1)$ and $N \geq 2$. If B is a neighborhood of θ_0, then for $\theta \in \Theta$ we use the notation $R(\theta) = \{t \in \mathbb{R} : |t| \leq N,\ \varphi^t x \in B\}$. Lebesgue measure on $\mathbb{R}$ will be denoted by m.

Lemma 3.2. *There exists a sufficiently small open neighborhood, B, of θ_0 and a continuous "bump" function $\alpha \colon \Theta \to [0, 1]$ such that:*

a) $\alpha(\varphi^t \theta_0) = 1, \quad$ *for* $|t| \leq \frac{s}{4}$;
b) $\alpha(\varphi^t \theta_0) = 0, \quad$ *for* $s \leq |t| \leq 2N$;
c) $m(R(\theta)) \leq 2s, \quad$ *for all* $\theta \in \Theta$.

For the important case $\Theta = \mathbb{R}$ with $\varphi^t \theta = \theta + t$, this is trivial: for any $\theta_0 \in \mathbb{R}$ take $B = (\theta_0 - s, \theta_0 + s)$, and $\alpha : \mathbb{R} \to [0, 1]$ with $\operatorname{supp} \alpha \subset B$ and $\alpha(\theta) = 1$ for $x \in (\theta_0 - s/4, \theta_0 + s/4)$.

Proof. We begin with:

Claim 1 . For sufficiently small $\epsilon_* > 0$, and all $\epsilon \leq \epsilon_*$, if $\delta < \epsilon$, then $\theta \in B' := B(\theta_0, \delta)$ implies $\varphi^t \theta \notin B'$ provided $\frac{s}{2} \leq |t| \leq 5N$.

Proof of Claim 1. Suppose, to the contrary, that there exists a sequence $\delta_n \downarrow 0$ such that for some $\theta_n \in B_n := B(\theta_0, \delta_n)$ and some t_n with $\frac{s}{2} \leq |t_n| \leq 5N$, one has $\varphi^{t_n} \theta_n \in B_n$. Passing to a subsequence, one can assume $t_n \to t_*$ for some t_* with $\frac{s}{2} \leq |t_*| \leq 5N$. Since the map $(\theta, t) \mapsto \varphi^t \theta$ from $\Theta \times \mathbb{R}$ to Θ is continuous at the point (θ_0, t_*), and since $\theta_n \in B_n$, it follows that $\theta_n \to \theta_0$. Thus $t_n \to t_*$ implies

$\varphi^{t_n}\theta_n \to \varphi^{t_*}\theta_0$. On the other hand, $\varphi^{t_n}\theta_n \in B_n$ implies $\varphi^{t_n}\theta_n \to \theta_0$, as $t_n \to t_*$. But then $\varphi^{t_*}\theta_0 = \theta_0$, which contradicts the assumption that θ_0 is a nonperiodic point of φ^t. This proves Claim 1. □

Now let $\delta < \epsilon_*$, and set $B' = B(\theta_0, \delta)$ as in Claim 1. Choose an open neighborhood B'' of θ_0 such that $\overline{B''} \subset B'$. Denote:

$$B := \bigcup_{|t|\le \frac{s}{4}} \varphi^t(B'), \quad C := \bigcup_{|t|\le \frac{s}{4}} \varphi^t(B''). \tag{3.3}$$

Since $\overline{C} \subset B$, there exists a continuous $\alpha\colon \Theta \to [0,1]$ such that $\alpha(\theta) = 1$ for $\theta \in C$, and $\alpha(\theta) = 0$ for $\theta \notin B$.

Note that since $\theta_0 \in B''$, it follows that $\{\varphi^t\theta_0 : |t| \le \frac{s}{4}\} \subset C$ and so $\alpha(\varphi^t\theta_0) = 1$ whenever $|t| \le \frac{s}{4}$. This proves *a*).

To prove *b*) we first prove

Claim 2 . If $\theta \in B$ and if $s \le |t| \le 2N$, then $\varphi^t\theta \notin B$.

Proof of Claim 2. Suppose, to the contrary, that there exists a $\theta \in B$ and a $t_* \in \mathbb{R}$ satisfying $s \le |t_*| \le 2N$ and $\varphi^{t_*}\theta \in B$. By the definition (3.3) of B, there exist θ_1 and θ_2 in B' such that

$$\theta = \varphi^{t_1}\theta_1 \quad \text{and} \quad \varphi^{t_*}\theta = \varphi^{t_2}\theta_2,$$

for some numbers $t_1,\ t_2$ with $|t_1|,\ |t_2| \le \dfrac{s}{4}$. This implies

$$\varphi^{t_*+t_1-t_2}\theta_1 \in B'. \tag{3.4}$$

Note that

$$|t_* + t_1 - t_2| \le 2N + \frac{s}{4} + \frac{s}{4} \le 5N$$

and

$$|t_* + t_1 - t_2| \ge |t_*| - |t_1| - |t_2| \ge s - \frac{s}{4} - \frac{s}{4} = \frac{s}{2},$$

and so (3.4) contradicts Claim 1. This proves Claim 2. □

Since, in particular, $\theta_0 \in B$ and $\alpha(\theta) = 0$ for $\theta \notin B$, Claim 2 proves part *b*) of the lemma.

To prove *c*), fix $\theta_1 \in \Theta$ and denote the orbit by $O(\theta_1) = \{\varphi^t\theta_1 : |t| \le N\}$. Clearly, $m(R(\theta_1)) = 0$ provided $O(\theta_1) \cap B = \emptyset$, so consider the case $O(\theta_1) \cap B \ne \emptyset$. Fix any $\theta \in O(\theta_1) \cap B$. Then $\theta = \varphi^{t_1}\theta_1$ for some t_1 with $|t_1| \le N$. Further, for any $t_* \in R(\theta_1)$, one has $|t_* - t_1| \le 2N$ and

$$\varphi^{t_*}\theta_1 = \varphi^{t_*-t_1}\theta \in B. \tag{3.5}$$

Since $\theta \in B$, Claim 2 shows that (3.5) can only hold provided $|t_* - t_1| < s$, that is, $t_* \in (t_1 - s, t_1 + s)$. Since t_* was chosen arbitrarily in $R(\theta_1)$, this shows $R(\theta_1) \subseteq (t_1 - s, t_1 + s)$. Thus, $m(R(\theta_1)) \leq 2s$. This completes the proof of the lemma. □

We now proceed with the proof of the Spectral Mapping Theorem.

Theorem 3.3. *Let* Γ *be the generator of the evolutionary semigroup* (2.2). *Then*

$$e^{t\sigma(\Gamma)} = \sigma(T^t) \setminus \{0\}, \quad t > 0. \tag{3.1}$$

Moreover, the spectrum $\sigma(\Gamma)$ *is invariant with respect to translations along the imaginary axis, and the spectrum* $\sigma(T^t)$, $t > 0$, *is invariant with respect to rotations centered at origin.*

Proof. As noted in the beginning of this section, to prove (3.1) it suffices to show: $1 \in \sigma_{ap}(T)$ implies $0 \in \sigma_{ap}(\Gamma)$. In fact, we show that $\sigma_{ap}(\Gamma)$ contains the entire imaginary axis whenever $1 \in \sigma_{ap}(T)$. Since $\sigma_{ap}(\Gamma)$ contains the boundary of $\sigma(\Gamma)$, all assertions of the theorem follow from this.

Let $1 \in \sigma_{ap}(T)$, and let $\xi \in \mathbb{R}$. We begin by using Lemma 3.1 with $A = T$ to obtain, for any $N \geq 2$, a function $f \in C_0(\Theta, X)$ satisfying:

(3.6a) $\|f\|_{C_0(\Theta,X)} = 1$;
(3.6b) $\|T^N f - f\|_{C_0(\Theta,X)} \leq \frac{1}{8}$;
(3.6c) $\|T^k f\|_{C_0(\Theta,X)} \leq 2$, for $k = 0, 1, \ldots, 2N$.

For this f, fix $s \in (0,1)$ such that

$$\|T^{t+N} f - T^N f\|_{C_0(\Theta,X)} \leq \frac{1}{16}, \quad \text{for } |t| \leq s, \tag{3.7}$$

and at the same time

$$|e^{-i\xi t} - 1| \leq \frac{1}{32}, \quad \text{for } |t| \leq s. \tag{3.8}$$

With the goal of constructing an approximate eigenfunction g, for Γ, corresponding to $i\xi$, choose a smooth function $\gamma\colon \mathbb{R} \to [0,1]$ such that:

(3.9a) $\gamma(t) = 0$, for $t \notin (0, 2N)$;
(3.9b) $|\gamma'(t)| \leq \frac{2}{N}$, for all $t \in \mathbb{R}$;
(3.9c) $\gamma(t) = 1$, for $|t - N| \leq s$.

Now fix $\theta_0 \in \Theta$, a nonperiodic point of φ^t which satisfies

$$\|f(\theta_0)\| \geq \frac{7}{8}\|f\|_{C_0(\Theta,X)} = \frac{7}{8}. \tag{3.10}$$

Set $\theta_N := \varphi^{-N}\theta_0$, and use Lemma 3.2 to obtain an open neighborhood, B, of θ_N and a "bump"function α with the properties $a) - c)$ listed there. Define $g \in C_0(\Theta, X)$ by

$$g(\theta) = \int_{-\infty}^{\infty} e^{-i\xi t}\gamma(t)(T^t\alpha f)(\theta)\, dt, \quad \theta \in \Theta. \tag{3.11}$$

Note that due to (3.9a), the integration goes from 0 to $2N$. Also, note that

$$\Gamma g = i\xi g - \int_{-\infty}^{\infty} e^{-i\xi t}\gamma'(t)(T^t\alpha f)\, dt.$$

Indeed,

$$\begin{aligned}
\Gamma g &= \frac{d}{d\tau}\Big|_{\tau=0} T^\tau g = \frac{d}{d\tau}\Big|_{\tau=0} \int_{-\infty}^{\infty} e^{-i\xi t}\gamma(t)(T^{t+\tau}\alpha f)\, dt \\
&= \frac{d}{d\tau}\Big|_{\tau=0} \int_{-\infty}^{\infty} e^{-i\xi (t-\tau)}\gamma(t-\tau)(T^t\alpha f)\, dt \\
&= \int_{-\infty}^{\infty} \left[i\xi e^{-i\xi t}\gamma(t)(T^t\alpha f) - e^{-i\xi t}\gamma'(t)(T^t\alpha f)\right] dt.
\end{aligned}$$

We now proceed to show that $\|\Gamma g - i\xi\| = O(\frac{1}{N})\|g\|$ with the following two claims.

Claim 1 . Set $C := \max_{0\le t\le 1} \|T^t\|_{\mathcal{L}(X)}$. Then $\|\Gamma g - i\xi g\|_{C_0(\Theta,X)} \le \dfrac{8Cs}{N}$.

Proof of Claim 1. First use (3.9a) and (3.9b) to obtain

$$\begin{aligned}
\|\Gamma g - i\xi g\| &= \left\| - \int_0^{2N} e^{-i\xi t}\gamma'(t)\alpha(\varphi^{-t}\cdot)T^t f(\cdot)\, dt \right\| \\
&\le \frac{2}{N} \cdot \max_{0\le t\le 2N} \|T^t f\| \cdot \max_{x\in\Theta} \int_0^{2N} \alpha(\varphi^{-t}\theta)\, dt.
\end{aligned} \tag{3.12}$$

Using (3.6c), note that

$$\max_{0\le t\le 2N} \|T^t f\| \le \max_{0\le k\le 2N} \max_{0\le \tau\le 1} \|T^\tau T^k f\| \le 2C. \tag{3.13}$$

Also, the change of variable $t \mapsto -t + N$ gives

$$\max_{x\in\Theta} \int_0^{2N} \alpha(\varphi^{-t}\theta)\, dt = \max_{x\in\Theta} \int_{-N}^{N} \alpha(\varphi^t(\varphi^{-N}\theta))\, dt = \max_{x\in\Theta} \int_{-N}^{N} \alpha(\varphi^t\theta)\, dt.$$

Now, for fixed $\theta \in \Theta$, recall from Lemma 3.2 that the set $R(\theta) = \{t \in \mathbb{R} : |t| \le N,\ \varphi^t\theta \in B\}$ has measure $m(R(\theta)) \le 2s$. Consider $|t| \le N$; then $\alpha(\varphi^t\theta) \le 1$ for $t \in R(\theta)$, and $\alpha(\varphi^t\theta) = 0$ for $t \notin R(\theta)$, and so

$$\max_{\theta\in\Theta} \int_{-N}^{N} \alpha(\varphi^t\theta)\, dt \le \max_{\theta\in\Theta} \int_{R(\theta)} dt \le 2s. \tag{3.14}$$

Using (3.13) and (3.14) in the inequality (3.12) gives

$$\|\Gamma g - i\xi g\| \leq \frac{2}{N} \cdot 2C \cdot 2s = \frac{8Cs}{N}.$$

This proves Claim 1. □

Claim 2 . $\|g\|_{C_0(\Theta,X)} \geq \frac{s}{128}$.

Proof of Claim 2. Recall $\theta_N = \varphi^{-N}\theta_0$. Using (3.9a) and a change of variable $t \mapsto t+N$ gives:

$$\begin{aligned} g(\theta_0) &= \int_0^{2N} e^{-i\xi t}\gamma(t)\alpha(\varphi^{-t}\theta_0)(T^t f)(\theta_0)\,dt \\ &= \int_{-N}^{N} e^{-i\xi(t+N)}\gamma(t+N)\alpha(\varphi^{-t}\theta_N)(T^{t+N} f)(\theta_0)\,dt. \end{aligned}$$

And so, Lemma 3.2 *b*) and (3.9c) show that

$$\begin{aligned} g(\theta_0) &= \int_{-s}^{s} e^{-i\xi(t+N)}\gamma(t+N)\alpha(\varphi^{-t}\theta_N)(T^{t+N} f)(\theta_0)\,dt \\ &= e^{-i\xi N}\int_{-s}^{s} e^{-i\xi t}\alpha(\varphi^{-t}\theta_N)(T^{t+N} f)(\theta_0)\,dt \\ &= e^{-i\xi N}(I_1 + I_2 + I_3), \end{aligned}$$

where we have denoted for brevity:

$$\begin{aligned} I_1 &= f(\theta_0)\int_{-s}^{s} e^{-i\xi t}\alpha(\varphi^{-t}\theta_N)\,dt, \\ I_2 &= (T^N - I)f(\theta_0)\int_{-s}^{s} e^{-i\xi t}\alpha(\varphi^{-t}\theta_N)\,dt, \\ I_3 &= \int_{-s}^{s} e^{-i\xi t}\alpha(\varphi^{-t}\theta_N)[(T^{t+N} - T^N)f](\theta_0)\,dt. \end{aligned}$$

It follows from Lemma 3.2 *a*) (and recall $0 \leq \alpha(\theta) \leq 1$), and (3.8) and (3.10) that

$$\begin{aligned} \|I_1\| &= \|f(\theta_0)\| \left|\int_{-s}^{s} e^{-i\xi t}\alpha(\varphi^{-t}\theta_N)\,dt\right| \\ &\geq \|f(\theta_0)\| \left(\left|\int_{-s}^{s} \alpha(\varphi^{-t}\theta_N)\,dt\right| - \left|\int_{-s}^{s} (e^{-i\xi t} - 1)\alpha(\varphi^{-t}\theta_N)\,dt\right|\right) \\ &\geq \|f(\theta_0)\| \left(\left|\int_{\frac{s}{4}}^{-\frac{s}{4}} \alpha(\varphi^{-t}\theta_N)\,dt\right| - \int_{-s}^{s} \left|(e^{-i\xi t} - 1)\right|\,dt\right) \\ &\geq \|f(\theta_0)\| \left(\frac{s}{2} - \frac{1}{32}2s\right) \geq \frac{7}{8}\left(\frac{7s}{16}\right) = \frac{49s}{128}. \end{aligned}$$

On the other hand, (3.6b) and (3.7) show that

$$\|I_2\| \le \left\|(T^N - I)f\right\| \left|\int_{-s}^{s} e^{-i\xi t}\alpha(\varphi^{-t}\theta_N)\,dt\right| \le \frac{1}{8}2s = \frac{s}{4},$$

$$\|I_3\| \le \int_{-s}^{s} \left|e^{-i\xi t}\alpha(\varphi^{-t}\theta_0)\right| \left\|(T^{t+N} - T^N)f\right\|\,dt \le \frac{1}{16}2s = \frac{s}{8}.$$

As a result,

$$\|g\| \ge \|g(\theta_0)\| = \|I_1+I_2+I_3\| \ge \|I_1\| - \|I_2\| - \|I_3\| \ge \frac{49s}{128} - \frac{s}{4} - \frac{s}{8} = \frac{s}{128}.$$

This proves Claim 2. □

Returning to the proof of the theorem, we combine Claim 1 with Claim 2 to obtain

$$\|\Gamma g - i\xi g\| \le \frac{8Cs}{N} \le \frac{8C}{N}\cdot 128\|g\|.$$

Since $N \ge 2$ was arbitrary, this shows $i\xi \in \sigma_{ap}(\Gamma)$. □

4. Exponential Dichotomy

Let $\{T^t\}_{t\ge 0}$ be an evolutionary semigroup (2.2). In this section, we relate the hyperbolicity of the weighted translation operator $T = T^1 = aV$, where $a(\theta) = \Phi(\varphi^{-1}\theta, 1)$, to the exponential dichotomy of the LSPF $\hat{\varphi}^t$. One can assume in this section, that $t \in \mathbb{Z}$.

We also relate the hyperbolicity of T to the hyperbolicity of weighted shift operators $\pi_\theta(T)$, $\theta \in \Theta$. The representation π_θ of $\mathfrak{B}$ was introduced in Section 2 (see (2.16)). For each $\theta \in \Theta$, the weighted shift operator $\pi_\theta(T)$ acts on $\ell_\infty(\mathbb{Z}, X)$ by the rule

$$\pi_\theta(T) = \operatorname{diag}\{\Phi(\varphi^{n-1}\theta, 1)\}_{n\in\mathbb{Z}} S, \quad \text{where } S\colon (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n-1})_{n\in\mathbb{Z}}.$$

It is useful to make the observation that for $k \in \mathbb{Z}$,

$$\pi_\theta(T^k) = [\pi_\theta(T)]^k = \operatorname{diag}\{\Phi(\varphi^{n-k}\theta, k)\}_{n\in\mathbb{Z}} S^k = S^k \operatorname{diag}\{\Phi(\varphi^n\theta, k)\}_{n\in\mathbb{Z}}. \tag{4.1}$$

For $\bar{x} = (x_n)_{n\in\mathbb{Z}}$, and $\bar{y} = (y_n)_{n\in\mathbb{Z}}$ in $\ell_\infty(\mathbb{Z}, X)$, note that the equation $[I - \pi_\theta(T)]\bar{x} = \bar{y}$ can be expressed componentwise as

$$x_{n+1} - \Phi(\varphi^n\theta, 1)x_n = y_{n+1}, \quad n \in \mathbb{Z}.$$

As seen in Section 5.4 (see also [25, Prop. 3.22]), the hyperbolicity of $\pi_\theta(T)$ is equivalent to the existence of exponential dichotomy for $\hat{\varphi}^t$ along the orbit through θ, i.e., "pointwise" dichotomy (see [16, Theorem 7.6.5] and [8]).

We begin with the definition of exponential dichotomy on Θ (or "global" dichotomy [8, 9, 16, 15, 28, 48]).

Definition 4.1. A linear skew product flow $\hat{\varphi}^t$ has *exponential dichotomy* on Θ if there exists a continuous projection $P\colon \Theta \to \mathcal{L}_s(X)$ such that for $\theta \in \Theta$ and $t \geq 0$,

a) $P(\varphi^t\theta)\Phi(\theta,t) = \Phi(\theta,t)P(\theta)$;
b) $\Phi_Q(\theta,t)$ is invertible from $\operatorname{Im} Q(\theta)$ to $\operatorname{Im} Q(\varphi^t\theta)$;
c) there exist constants M, $\beta > 0$ such that for $t > 0$

$$\|\Phi_P(\theta,t)\| \leq Me^{-\beta t}, \quad \|[\Phi_Q(\theta,t)]^{-1}\| \leq Me^{-\beta t}.$$

If a projection $P\colon \Theta \to \mathcal{L}(X)$ and a cocycle Φ satisfy a), above, then $\Phi_P(\theta,t)$ and $\Phi_Q(\theta,t)$ will be used to denote the restrictions $\Phi(\theta,t)P(\theta)\colon \operatorname{Im} P(\theta) \to \operatorname{Im} P(\varphi^t\theta)$ and $\Phi(\theta,t)Q(\theta)\colon \operatorname{Im} Q(\theta) \to \operatorname{Im} Q(\varphi^t\theta)$, respectively.

The main result of this section follows. As usual, the set of nonperiodic points of φ^t is assumed to be dense in Θ.

Theorem 4.1. *The following are equivalent:*

i) $\sigma(T) \cap \mathbb{T} = \emptyset$ *in* $C_0(\Theta, X)$*;*
ii) $\sigma(\pi_\theta(T)) \cap \mathbb{T} = \emptyset$*, on* $\ell_\infty(\mathbb{Z}, X)$*, for all* $\theta \in \Theta$ *and there exists a constant* $C > 0$*, such that*

$$\|[\pi_\theta(T) - \lambda I]^{-1}\|_{B(\ell_\infty(\mathbb{Z},X))} \leq C \text{ for all } \theta \in \Theta,\ \lambda \in \mathbb{T}; \tag{4.2}$$

iii) The LSPF $\hat{\varphi}^t$ *has exponential dichotomy.*

We note (see Remark 3 in Section 5.6) that condition (4.2) often follows automatically from $\sigma(\pi_\theta(T)) \cap \mathbb{T} = \emptyset$, $\theta \in \Theta$. For results related to *i)* $\Leftrightarrow$ *iii)*, see [42, 43, 44, 45] where a quite different method was used.

We begin the proof of the theorem with an observation about the operators $\pi_\theta(T)$.

Proposition 4.2. $\sigma(\pi_\theta(T))$ *is invariant with respect to rotations centered at the origin. Moreover, for* $\sigma(\pi_\theta(T)) \cap \mathbb{T} = \emptyset$*,*

$$\|[\lambda - \pi_\theta(T)]^{-1}\|_{B(\ell_\infty(\mathbb{Z},X))} = \|[I - \pi_\theta(T)]^{-1}\|_{B(\ell_\infty(\mathbb{Z},X))}, \quad \lambda \in \mathbb{T}.$$

Proof. For $\omega \in \mathbb{T}$, define $\Lambda = \operatorname{diag}\{\omega^n\}_{n\in\mathbb{Z}}$. Then Λ is an invertible operator on $\ell_\infty(\mathbb{Z}, X)$, and

$$\Lambda\pi_\theta(\lambda - T)\Lambda^{-1} = \lambda - \omega \operatorname{diag}\{a(\varphi^n\theta)\}_{n\in\mathbb{Z}}S = \omega(\omega^{-1}\lambda - \pi_\theta(T)).$$

Hence, $\|[\lambda - \pi_\theta(T)]^{-1}\| = \|[\omega^{-1}\lambda - \pi_\theta(T)]^{-1}\|$ and the proposition follows. □

As a result, statement *ii)* in Theorem 4.1 can be replaced by the statement

ii′) $I - \pi_\theta(T)$ is invertible for all $\theta \in \Theta$, and there exists $C > 0$ such that $\|[I - \pi_\theta(T)]^{-1}\|_{B(\ell_\infty(\mathbb{Z},X))} \le C$ for all $\theta \in \Theta$.

The proof of Theorem 4.1 will follow from a series of lemmas.

The *Proof of i)* $\Rightarrow$ *ii′)* is given by the following lemma.

Lemma 4.3. *If $b = I - T$ is an invertible operator on $C_0(\Theta, X)$, then $\pi_\theta(b)$ is invertible on $\ell_\infty(\mathbb{Z}, X)$ for all $\theta \in \Theta$. Moreover,*

$$\|[\pi_\theta(b)]^{-1}\|_{B(\ell_\infty(\mathbb{Z},X))} \le \|b^{-1}\|_{B(C_0(\Theta,X))}$$

for all $\theta \in \Theta$.

Proof. We first show that $\pi_\theta(b)$ is injective, uniformly for $\theta \in \Theta$, by showing that for any $\bar{x} \in \ell_\infty(\mathbb{Z}, X)$,

$$\|\pi_\theta(b)\bar{x}\|_{\ell_\infty(\mathbb{Z},X)} \ge (\|b^{-1}\|_{B(C_0(\Theta,X))})^{-1}\|\bar{x}\|_{\ell_\infty(\mathbb{Z},X)}. \tag{4.3}$$

It suffices to prove (4.3) for finitely supported $\bar{x} = (x_n)_{n=-N}^{N}$.

Let $\theta_0 \in \Theta$ be a nonperiodic point of φ^t. Let $\epsilon > 0$. Since $\theta \mapsto a(\theta)x$ is continuous for all $x \in X$, there exists $\delta > 0$ such that for $B := B(\theta_0, \delta)$,

$$\theta \in B \ \text{ implies } \ \|[a(\theta_0) - a(\theta)]x_n\|_X \ < \epsilon, \ \text{ for all } |n| \le N. \tag{4.4}$$

Moreover, δ can be chosen so that, in addition,

$$\varphi^n(B) \cap \varphi^k(B) = \emptyset \ \text{ for all } k \ne n, \ |n|, |k| \le N.$$

Choose a continuous function $\alpha\colon \Theta \to [0, 1]$ such that $\alpha(\theta_0) = 1$, and $\alpha(\theta) = 0$ for $\theta \notin B$. Define $f \in C_0(\Theta, X)$ by

$$f(\theta) = \begin{cases} \alpha(\varphi^{-n}\theta)x_n, & \text{if } \theta \in \varphi^n(B), \ |n| \le N \\ 0, & \text{otherwise.} \end{cases}$$

Since $b^{-1} \in B(C_0(\Theta, X))$, we have

$$\|b^{-1}\| \cdot \|bf\| \ge \|f\|_{C_0(\Theta,X)} = \sup_{\theta\in\Theta} \|f(\theta)\|_X = \|\bar{x}\|_{\ell_\infty(\mathbb{Z},X)}. \tag{4.5}$$

Also, since $f(\varphi^{-1}\theta) = \alpha(\varphi^{-n}\theta)x_{n-1}$ for $\theta \in \varphi^n(B)$, $|n| \leq N$, and $f(\varphi^{-1}\theta) = 0$ otherwise, we have

$$\begin{aligned}
\|bf\|_{C_0(\Theta,X)} &= \sup_{\theta\in\Theta} \|f(\theta) - a(\theta)f(\varphi^{-1}\theta)\|_X \\
&= \sup_{|n|\leq N}\sup_{\theta\in\Theta} \|\alpha(\varphi^{-n}\theta)x_n - a(\theta)\alpha(\varphi^{-n}\theta)x_{n-1}\|_X \\
&\leq \sup_{|n|\leq N}\sup_{\theta\in\Theta} \|x_n - a(\varphi^n\theta)x_{n-1}\|_X \\
&\leq \sup_{|n|\leq N}\sup_{\theta\in\Theta}\{\|x_n - a(\varphi^n\theta_0)x_{n-1}\|_X + \|a(\varphi^n\theta_0)x_{n-1} - a(\varphi^n\theta)x_{n-1}\|_X\} \\
&< \|\pi_{\theta_0}(b)\bar{x}\|_{\ell_\infty(\mathbb{Z},X)} + \epsilon,
\end{aligned}$$

using (4.4) in the last inequality. So,

$$\|bf\|_{C_0(\Theta,X)} \leq \|\pi_{\theta_0}(b)\bar{x}\|_{\ell_\infty(\mathbb{Z},X)}. \tag{4.6}$$

Combining (4.5) and (4.6) shows

$$\|\pi_{\theta_0}(b)\bar{x}\|_{\ell_\infty(\mathbb{Z},X)} \geq \|bf\|_{C_0(\Theta,X)} \geq \frac{1}{\|b^{-1}\|_{B(C_0(\Theta,X))}}\|\bar{x}\|_{\ell_\infty(\mathbb{Z},X)}.$$

We have shown that (4.3) holds for all nonperiodic points of φ^t. If θ_0 is a periodic point of φ^t, then choose a sequence of nonperiodic points $\{\theta_k\}_{k=1}^\infty$ in Θ converging to θ_0. Since the map $\theta \mapsto \pi_\theta(b)\bar{x}$ is continuous, (4.3) holds, and hence $\pi_\theta(b)$ is uniformly injective for all $\theta \in \Theta$.

We now check that $\pi_\theta(b)$ is surjective for all $\theta \in \Theta$. Let $\bar{x} \in \ell_\infty(\mathbb{Z}, X)$. As before, consider $\bar{x} = (x_n)_{n=-N}^N$. First suppose θ_0 is a nonperiodic point of φ^t. Define $f \in C_0(\Theta, X)$ as above and note that since b is surjective, there exists a $g \in C_0(\Theta, X)$ such that $bg = f$; that is,

$$g(\theta) - a(\theta)g(\varphi^{-1}\theta) = f(\theta) \quad \text{for all } \theta \in \Theta.$$

Substituting $\theta = \varphi^n\theta_0$ into this equation and defining $\bar{y} \in \ell_\infty(\mathbb{Z}, X)$ by $y_n = g(\varphi^n\theta_0)$ for $|n| \leq N$, gives $\pi_{\theta_0}(b)\bar{y} = \bar{x}$.

To address the case for which θ_0 is a periodic point of φ^t, so that $\varphi^k\theta_0 = \theta_0$ for some $k \in \mathbb{Z}$, we begin with

Case $k = 1$. Assume $\varphi\theta_0 = \theta_0$. If $T = aV$ is hyperbolic on $C_0(\Theta, X)$, then $\pi_{\theta_0}(T) = \mathrm{diag}\{a(\theta_0)\}_{n\in\mathbb{Z}}S$ is hyperbolic on $\ell_\infty(\mathbb{Z}, X)$.

Proof. By hypothesis, $I - aV$ is surjective on $C_0(\Theta, X)$, and so $I - a(\theta_0)$ is surjective on X. Indeed, given $x \in X$, choose $f \in C_0(\Theta, X)$ with $f(\theta_0) = x$ and then choose $g \in C_0(\Theta, X)$ so that $(I - aV)g = f$; setting $y = g(\theta_0)$ gives $(I - a(\theta_0))y = x$. Therefore, $a(\theta_0)$ is hyperbolic on X. As in Lemma 2.4 of [22], this implies that $\operatorname{diag}\{a(\theta_0)\}_{n\in\mathbb{Z}}S$ is hyperbolic on $\ell_\infty(\mathbb{Z}, X)$.

Case $k > 1$. Assume $\varphi^k\theta_0 = \theta_0$. If T is hyperbolic on $C_0(\Theta, X)$, then $\pi_{\theta_0}(T)$ is hyperbolic on $\ell_\infty(\mathbb{Z}, X)$.

Proof. By standard spectral properties, it suffices to consider hyperbolicity for T^k. First note (see (4.1)) that

$$\pi_{\theta_0}(T^k) = S^k \operatorname{diag}\{\Phi(\varphi^n\theta_0, k)\}_{n\in\mathbb{Z}}.$$

This can be expressed as the operator

$$\hat{S} \operatorname{diag}\{d\}_{n\in\mathbb{Z}}, \quad \text{on} \quad \ell_\infty(\mathbb{Z}, X^k) \approx \ell_\infty(\mathbb{Z}, X), \tag{4.7}$$

where $\hat{S}$, on $\ell_\infty(\mathbb{Z}, X^k)$, and d, on X^k, are defined as

$$\begin{pmatrix} x_{n+1} \\ \vdots \\ x_{n+k} \end{pmatrix}_{n\in\mathbb{Z}} \mapsto \begin{pmatrix} x_{n+k+1} \\ \vdots \\ x_{n+2k} \end{pmatrix}_{n\in\mathbb{Z}} \quad \text{and } d = \begin{bmatrix} \Phi(\theta_0, k) & & \\ & \ddots & \\ & & \Phi(\varphi^{k-1}\theta_0, k) \end{bmatrix}.$$

As in Lemma 2.4 of [22], (4.7) is hyperbolic on $\ell_\infty(\mathbb{Z}, X^k)$ provided d is hyperbolic on X^k. Therefore, it suffices to show that $\Phi(\theta_0, k), \dots, \Phi(\varphi^{k-1}\theta_0, k)$ are hyperbolic on X. Applying *Case* $k = 1$ with $\phi := \varphi^k$ and the representation $\pi_{\theta,\phi}\colon a \mapsto \operatorname{diag}\{a(\phi^{-n}\theta)\}_{n\in\mathbb{Z}}$ shows that

$$\pi_{\theta_0,\phi}(T^k) = S^k \operatorname{diag}\{\Phi(\varphi^{kn}\theta_0, k)\}_{n\in\mathbb{Z}} = S^k \operatorname{diag}\{\Phi(\theta_0, k)\}_{n\in\mathbb{Z}}$$

is hyperbolic. As before, expressing this operator as $\hat{S} \operatorname{diag}\{\Phi(\theta_0, k)\}_{n\in\mathbb{Z}}$ on $\ell_\infty(\mathbb{Z}, X^k)$ and applying Lemma 2.4 of [22] shows that $\Phi(\theta_0, k)$ is hyperbolic. This proves the case $k > 1$ □

Thus, the lemma is proved. □

The *Proof of ii)* $\Rightarrow$ *i)* requires showing that the invertibility of b in $\mathfrak{B}$ can be derived from the invertibility of all its images $\pi_\theta(b)$ in $B(\ell_\infty(\mathbb{Z}, X))$. To do this, we consider the algebra $\mathfrak{D}$ of operators in $B(\ell_\infty(\mathbb{Z}, X))$ as defined in Section 2. The next lemma proves a slightly stronger statement than $ii) \Rightarrow i)$.

Lemma 4.4. *Let* $b = \lambda I - T$. *Assume* $\pi_\theta(b)$ *is invertible in* $B(\ell_\infty(\mathbb{Z}, X))$ *and that there exists a* $C > 0$ *such that*

$$\|[\pi_\theta(b)]^{-1}\|_{B(\ell_\infty(\mathbb{Z},X))} \le C \quad \text{for all } \theta \in \Theta.$$

Then b^{-1} *exists and is an element of* $\mathfrak{B}$.

Proof. Let $b = I - T$; the proof remains the same for $b = \lambda I - T$, $\lambda \in \mathbb{T}$. Using the notation of Proposition 2.12, set $d(\theta) := \pi_\theta(b) = I - \pi_\theta(a)S$. Since $\sup_{\theta\in\Theta} \|a(\theta)\|_{\mathcal{L}(X)} < \infty$, it follows that

$$\sum_k \sup_{\theta\in\Theta} \|d_k(\theta)\|_{B(\ell_\infty(\mathbb{Z},X))} = 1 + \sup_{\theta\in\Theta} \|\pi_\theta(a)\|_{B(\ell_\infty(\mathbb{Z},X))} < \infty,$$

and hence $d(\theta)$ satisfies the conditions of Proposition 2.12. Consequently, for each θ, $\pi_\theta(b) = d(\theta)$ has an inverse that is in $\mathfrak{D}$:

$$[\pi_\theta(b)]^{-1} = \sum_{k=-\infty}^{\infty} c_k(\theta)S^k, \quad \text{for some } c_k(\theta) = \mathrm{diag}\{c_k^{(n)}(\theta)\}_{n\in\mathbb{Z}} \text{ in } \mathfrak{C},\ k \in \mathbb{Z}.$$

Moreover, since $\sup_{\theta\in\Theta} \|\pi_\theta(b)\|_{B(\ell_\infty(\mathbb{Z},X))} \le C$, Proposition 2.12, shows that

$$\sum_k \sup_{\theta\in\Theta} \|c_k(\theta)\|_{B(\ell_\infty(\mathbb{Z},X))} < \infty.$$

The lemma is proved by showing that for each $k \in \mathbb{Z}$,

$$\theta \mapsto c_k^{(0)}(\theta) \text{ is a continuous, bounded function from } \Theta \to \mathcal{L}_s(X), \tag{4.8}$$

(in the notation of Section 2, $c_k^{(0)} \in \mathfrak{A}$) and then showing that the operator $r := \sum_{k=-\infty}^{\infty} c_k^{(0)} V^k$ is in $\mathfrak{B}$ and satisfies $r = b^{-1}$.

Let $k \in \mathbb{Z}$, and fix $x \in X$ and $\theta_0 \in \Theta$. Define $\bar{x} = (x_n)_{n\in\mathbb{Z}}$ in $\ell_\infty(\mathbb{Z}, X)$ by $x_n = x$ if $n = -k$, and $x_n = 0$ if $n \neq -k$. Then for any $\theta \in \Theta$,

$$\begin{aligned}
\|[c_k^{(0)}(\theta) - c_k^{(0)}(\theta_0)]x\|_X &= \left\| \sum_{j=-\infty}^{\infty} [c_j^{(0)}(\theta) - c_j^{(0)}(\theta_0)]x_{-j} \right\|_X \\
&\le \sup_{n\in\mathbb{Z}} \left\| \sum_{j=-\infty}^{\infty} [c_j^{(n)}(\theta) - c_j^{(n)}(\theta_0)]x_{n-j} \right\|_X \\
&= \left\| \left(\sum_{j=-\infty}^{\infty} \mathrm{diag}\{c_j^{(n)}(\theta) - c_j^{(n)}(\theta_0)\}_{n\in\mathbb{Z}} S^j \right) \bar{x} \right\|_{\ell_\infty(\mathbb{Z},X)} \\
&= \left\| \left(\sum_{j=-\infty}^{\infty} [c_j(\theta) - c_j(\theta_0)] S^j \right) \bar{x} \right\|_{\ell_\infty(\mathbb{Z},X)} \\
&= \left\| [\pi_\theta(b)]^{-1}[\pi_{\theta_0}(b) - \pi_\theta(b)][\pi_{\theta_0}(b)]^{-1}\bar{x} \right\|_{\ell_\infty(\mathbb{Z},X)} \\
&\le C \left\| [\pi_{\theta_0}(b) - \pi_\theta(b)]\bar{y} \right\|_{\ell_\infty(\mathbb{Z},X)},
\end{aligned}$$

where the last inequality comes from letting $\bar{y} = [\pi_{\theta_0}(b)]^{-1}\bar{x}$ and using the assumption $\|[\pi_\theta(b)]^{-1}\| \le C$. Since the map $\theta \mapsto \pi_\theta(b)\bar{y}$ is continuous for any $\bar{y} \in \ell_\infty(\mathbb{Z}, X)$, $c_k^{(0)}$ is continuous at θ_0.

Moreover, note that $c_k(\theta) = \operatorname{diag}\{c_k^{(n)}(\theta)\}_{n\in\mathbb{Z}}$ satisfies, for each $n \in \mathbb{Z}$, the inequality

$$\|c_k(\theta)\|_{B(\ell_\infty(\mathbb{Z},X))} \ge \|c_k^{(n)}(\theta)\|_{\mathcal{L}(X)}.$$

Therefore, $\sum_k \sup_{\theta\in\Theta} \|c_k(\theta)\|_{B(\ell_\infty(\mathbb{Z},X))} < \infty$ implies, in particular, that $\sup_{\theta\in\Theta} \|c_k^{(0)}(\theta)\|_{\mathcal{L}(X)} < \infty$, and so each $c_k^{(0)}$ is bounded. Thus, (4.8) holds.

Also, since $\|c_k^{(0)}\|_{B(C_0(\Theta,X))} = \sup_{\theta\in\Theta} \|c_k^{(0)}(\theta)\|_{\mathcal{L}(X)}$, it follows that

$$\sum_{k=-\infty}^{\infty} \|c_k^{(0)}\|_{B(C_0(\Theta,X))} = \sum_{k=-\infty}^{\infty} \sup_{\theta\in\Theta} \|c_k^{(0)}(\theta)\|_{\mathcal{L}(X)} \le \sum_{k=-\infty}^{\infty} \sup_{\theta\in\Theta} \|c_k(\theta)\|_{B(\ell_\infty(\mathbb{Z},X))} < \infty.$$

Therefore, the operator $r := \sum_{k=-\infty}^{\infty} c_k^{(0)} V^k$ is in $\mathfrak{B}$.

Now observe that

$$c_k^{(n)}(\theta) = c_k^{(0)}(\varphi^n\theta), \quad n \in \mathbb{Z}. \tag{4.9}$$

Indeed, for any $n \in \mathbb{Z}$, $\pi_\theta(a)S^n = S^n\pi_{\varphi^n\theta}(a)$, and so

$$\pi_{\varphi^n\theta}(b) = I - \operatorname{diag}\{a(\varphi^{n+k}\theta)\}_{k\in\mathbb{Z}}S = S^{-n}\pi_\theta(b)S^n.$$

Therefore,

$$\begin{aligned}[\pi_{\varphi^n\theta}(b)]^{-1} = S^{-n}[\pi_\theta(b)]^{-1}S^n &= S^{-n}\left(\sum_{k=-\infty}^{\infty} \operatorname{diag}\{c_k^{(i)}(\theta)\}_{i\in\mathbb{Z}}S^k\right)S^n \\ &= \sum_{k=-\infty}^{\infty} \operatorname{diag}\{c_k^{(i+n)}(\theta)\}_{i\in\mathbb{Z}}S^k.\end{aligned}$$

On the other hand,

$$[\pi_{\varphi^n\theta}(b)]^{-1} = \sum_{k=-\infty}^{\infty} c_k(\varphi^n\theta)S^k = \sum_{k=-\infty}^{\infty} \operatorname{diag}\{c_k^{(i)}(\varphi^n\theta)\}_{i\in\mathbb{Z}}S^k,$$

so (4.9) holds.

As a consequence, $rb = br = I$. Indeed, using (4.9),

$$\begin{aligned}\pi_\theta(r) &= \sum_{k=-\infty}^{\infty} \pi_\theta(c_k^{(0)})S^k = \sum_{k=-\infty}^{\infty} \operatorname{diag}\{c_k^{(0)}(\varphi^n\theta)\}_{n\in\mathbb{Z}}S^k \\ &= \sum_{k=-\infty}^{\infty} \operatorname{diag}\{c_k^{(n)}(\theta)\}_{n\in\mathbb{Z}}S^k = \sum_{k=-\infty}^{\infty} c_k(\theta)S^k = [\pi_\theta(b)]^{-1}.\end{aligned}$$

Therefore, $I = \pi_\theta(r)\pi_\theta(b) = \pi_\theta(rb)$, and so $\pi_\theta(rb - I) = 0$ for all θ. One checks directly that for $\pi_\theta\colon \mathfrak{B} \to \mathfrak{D}$, $\bigcap_{\theta\in\Theta} \operatorname{Ker}\pi_\theta = \{0\}$ and so $rb = I$. □

The proof $i) \Rightarrow iii)$ uses the following corollary to Lemma 4.4.

Corollary 4.5. *If $\sigma(T) \cap \mathbb{T} = \emptyset$, then the Riesz projection, $\mathcal{P}$, corresponding to $\sigma(T) \cap \mathbb{D}$ has the form $\mathcal{P}f(\theta) = P(\theta)f(\theta)$, $f \in C_0(\Theta, X)$, for some bounded, continuous projection-valued function $P : \Theta \to \mathcal{L}_s(X)$.*

Proof. If $\sigma(T) \cap \mathbb{T} = \emptyset$, then statement $ii)$ of Theorem 4.1 holds, and so Lemma 4.4 applies and shows $(\lambda I - T)^{-1}$ is in $\mathfrak{B}$ for all $\lambda \in \mathbb{T}$. Consequently, $\mathcal{P} = \frac{1}{2\pi i}\int_{\mathbb{T}}(\lambda I - T)^{-1}\, d\lambda$ is an element of $\mathfrak{B}$. Proceeding exactly as in [25] or [21, Lemma 3], one can show that $\mathcal{P} \in \mathfrak{A}$. □

Before proceeding, we make an additional observation.

Proposition 4.6. *Assume $\sigma(T) \cap \mathbb{T} = \emptyset$, and let $\mathcal{P}$ be the Riesz projection corresponding to $\sigma(T) \cap \mathbb{D}$, where $\mathcal{P}f(\theta) = P(\theta)f(\theta)$, as above. Then the Riesz projection $\mathcal{P}_\theta$ in $B(\ell_\infty(\mathbb{Z}, X))$ corresponding to $\sigma(\pi_\theta(T)) \cap \mathbb{D}$ is given by*

$$\pi_\theta(\mathcal{P}) = \operatorname{diag}\{P(\varphi^n\theta)\}_{n\in\mathbb{Z}}.$$

Proof. As in the previous corollary, $\mathcal{P} \in \mathfrak{B}$. Moreover, $\pi_\theta((\lambda I - T)^{-1}) = [\lambda I - \pi_\theta(T)]^{-1}$ is in $\mathfrak{D}$, for $\lambda \in \mathbb{T}$, and the Riesz projection corresponding to $\sigma(\pi_\theta(T)) \cap \mathbb{D}$, given by $\mathcal{P}_\theta = \frac{1}{2\pi i}\int_{\mathbb{T}}[\lambda I - \pi_\theta(T)]^{-1}\, d\lambda$, is in $\mathfrak{D}$. Since $\pi_\theta\colon \mathfrak{B} \to \mathfrak{D}$ is a continuous homomorphism, $\mathcal{P}_\theta = \pi_\theta(\mathcal{P})$. Since $\mathcal{P}$ is given by $P(\cdot)$, if follows that $\mathcal{P}_\theta = \pi_\theta(\mathcal{P}) = \operatorname{diag}\{P(\varphi^n\theta)\}_{n\in\mathbb{Z}}$. □

The *Proof of $i) \Rightarrow iii)$* is a consequence of the next lemma.

Lemma 4.7. *If $\sigma(T) \cap \mathbb{T} = \emptyset$, then the LSPF $\widehat{\varphi}^t$ has exponential dichotomy.*

Proof. Let $\mathcal{P}$ be the Riesz projection corresponding to $\sigma(T) \cap \mathbb{D}$. By Corollary 4.5, $\mathcal{P}$ is given by $(\mathcal{P}f)(\theta) = P(\theta)f(\theta)$ for some projection-valued function $P\colon \Theta \to X$. We show that P satisfies the properties of Definition 4.1. As already shown, the statement $\sigma(T) \cap \mathbb{T} = \emptyset$ implies statement $ii)$ of Theorem 4.1. We use the latter to show that $\Phi(\theta, t)$ is invertible as an operator from $\operatorname{Im} Q(\theta)$ to $\operatorname{Im} Q(\varphi^t\theta)$.

Fix $\theta \in \Theta$. Statement *ii*) implies that $\pi_\theta(T) \cap \mathbb{T} = \emptyset$, and Proposition 4.6 shows that the corresponding Riesz projection is given by $\mathcal{P}_\theta = \pi_\theta(\mathcal{P}) = \mathrm{diag}\{P(\varphi^n\theta)\}_{n\in\mathbb{Z}}$. Set $\mathcal{Q}_\theta = I - \mathcal{P}_\theta$, and let $k \in \mathbb{Z}$. Since $\pi_\theta(T)\mathcal{Q}_\theta$ is invertible on $\mathrm{Im}\,\mathcal{Q}_\theta$, so is $[\pi_\theta(T)\mathcal{Q}_\theta]^k = \pi_\theta(T^k)\mathcal{Q}_\theta$. Hence, for any $\bar{y} = (y_n)_{n\in\mathbb{Z}} \in \mathrm{Im}\,\mathcal{Q}_\theta$, there exists a unique $\bar{x} = (x_n)_{n\in\mathbb{Z}} \in \mathrm{Im}\,\mathcal{Q}_\theta$ such that (see (4.1))

$$\{\Phi(\varphi^{n-k}\theta, k)x_{n-k}\}_{n\in\mathbb{Z}} = \pi_\theta(T^k)\bar{x} = \bar{y}.$$

The fact $\bar{y} \in \mathrm{Im}\,\mathcal{Q}_\theta$ means precisely that $y_n \in \mathrm{Im}\,Q(\varphi^n\theta)$ for every $n \in \mathbb{Z}$. So fix $y \in \mathrm{Im}\,Q(\varphi^k\theta)$ and define $\bar{y}$ by $y_n = y$ for $n = k$, and $y_n = 0$, for $n \neq k$. There exists a unique $\bar{x} \in \mathrm{Im}\,\mathcal{Q}_\theta$ such that

$$\Phi(\varphi^{n-k}\theta, k)x_{n-k} = y_n, \quad n \in \mathbb{Z}.$$

In particular, for $n = k$, $\Phi(\theta, n)x_0 = y$. This shows that $\Phi(\theta, n)$ is invertible from $\mathrm{Im}\,Q(\theta)$ to $\mathrm{Im}\,Q(\varphi^n\theta)$ for all $\theta \in \Theta$, $n \in \mathbb{Z}$.

To see that $\Phi(\theta, t)$ is invertible for all $t \in \mathbb{R}$, fix t and choose $n \in \mathbb{Z}$ such that $t \in [n, n+1)$. Since $\Phi(\theta, n)$ is invertible, the identity

$$\Phi(\theta, t) = \Phi(\varphi^n\theta, t-n)\Phi(\theta, n)$$

shows that it suffices to prove $\Phi(\varphi^n\theta, t-n)$ is invertible. But the identities

$$\Phi(\varphi^n\theta, 1) = \Phi(\varphi^t\theta, n+1-t)\Phi(\varphi^n\theta, t-n)$$
$$\Phi(\varphi^{t-1}\theta, 1) = \Phi(\varphi^n\theta, t-n)\Phi(\varphi^{t-1}\theta, n-(t-1))$$

show, respectively, that $\Phi(\varphi^n\theta, t-n)$ has a left and a right inverse. Hence $\Phi(\theta, t)$ is invertible from $\mathrm{Im}\,Q(\theta)$ to $\mathrm{Im}\,Q(\varphi^t\theta)$.

Part *b*) for $t \in \mathbb{Z}_+$ follows from (4.1) and the fact that $\pi_\theta(T)\mathcal{P} = \pi_\theta(T)\mathcal{P}$. As in [25, Prop. 3.10], the statement is seen to hold for $t \in \mathbb{R}_+$.

Finally, since $\sigma(T^t) \cap \mathbb{T} = \emptyset$, there exist $M, \beta > 0$ such that

$$\sup_{t\in\mathbb{R}} \|\Phi(\theta, t)P(\theta)\|_{\mathcal{L}(X)} = \|\mathcal{P}T^t\mathcal{P}\|_{B(C_0(\Theta, X))} \leq Me^{-\beta t}.$$

The first inequality in Definition 4.1 *c*) follows. The second inequality is shown similarly. □

The *Proof of iii*) $\Rightarrow$ *i*) is trivial. This completes the proof of Theorem 4.1.

Let us make the following observation concerning the dynamical spectrum, Σ (see the Introduction for the definition). As a result of Theorem 4.1 and Corollary 4.5, the dynamical spectrum Σ coincides with $\ln|\sigma(T)\setminus\{0\}|$. An application of Theorem 3.3 gives the formula (1.3) in the introduction. Also, the spectral subbundles for $\hat{\varphi}^t$ are determined by the spectral projections for T.

5. Consequences

In this section we formulate a variety of consequences which follow, almost immediately, from the results of the previous sections.

5.1. **Invertibility of Γ.** An immediate consequence of the Theorem 3.3 and Theorem 4.1 (see *i*) $\Leftrightarrow$ *iii*)) is the fact that a LSPF has exponential dichotomy if and only if the spectrum of the generator Γ of the corresponding evolutionary semigroup (2.2) satisfies $\sigma(\Gamma)\cap i\mathbb{R}=\emptyset$. Since $\sigma(\Gamma)$ is invariant with respect to translations along $i\mathbb{R}$, we conclude:

Corollary 5.1. *The LSPF $\hat{\varphi}^t$ has exponential dichotomy if and only if Γ is invertible.*

Now recall that in the norm continuous compact setting, the generator of the evolutionary semigroup is given by (2.5). Consequently, the LSPF generated by the cocycle in Example 2.2 is hyperbolic if and only if the equation

$$\frac{d}{dt} f\circ\varphi^t(\theta)\bigg|_{t=0} - A(\theta)f(\theta) = g(\theta)$$

has a unique solution f for every $g\in C_0(\Theta,X)$.

Applying Corollary 5.1 to Example 2.8, we note that an evolutionary family $\{U(\tau,s)\}$, $\tau\geq s$ has exponential dichotomy if and only if Γ is invertible on $C_0(\mathbb{R},X)$ (cf. [22, 23, 24]). In the special case where $A(\tau)\equiv A_0$ generates a C_0 semigroup $\{e^{tA_0}\}_{t\geq0}$ on X (here, $U(\tau,s)=e^{(\tau-s)A_0}$), this gives a result from [41]: a C_0 semigroup $\{e^{tA_0}\}_{t\geq0}$ is hyperbolic if and only if the equation $f'-A_0f=g$ has a unique solution for every $g\in C_0(\mathbb{R},X)$.

On the other hand, for the norm continuous line setting (Example 2.7), we conclude that (2.10) has exponential dichotomy if and only if $\Gamma=-d/dt+A(\cdot)$ is invertible [3, 37], or, equivalently, the equation $f'(t)-A(t)f(t)=g(t)$, $t\in\mathbb{R}$, has a unique solution for every $g\in C_0(\mathbb{R},X)$ (see [11, 31]).

5.2. **Roughness of the dichotomy.** In this subsection we give a very short proof of the facts that the exponential dichotomy persists under small perturbations.

Theorem 5.2. *Assume that the LSPF $\hat{\varphi}_1^t$ over the flow φ^t generated by a cocycle $\Phi_1(\theta, t)$ has exponential dichotomy. Then there exists $\epsilon > 0$ such that for every cocycle $\Phi_2(\theta, t)$ satisfying*

$$\sup_{\theta\in\Theta} \|\Phi_1(\theta, 1) - \Phi_2(\theta, 1)\|_{\mathcal{L}(X)} < \epsilon, \tag{5.1}$$

the LSPF $\hat{\varphi}_2^t$ over φ^t generated by $\Phi_2(\theta, t)$ also has the exponential dichotomy.

Proof. By Theorem 4.1, $i) \Leftrightarrow iii)$, the operator T_1, $(T_1 f)(\theta) = \Phi_1(\varphi^{-1}\theta, 1)f(\varphi^{-1}\theta)$, is hyperbolic in $C_0(\Theta, X)$. Note that for $(T_2 f)(\theta) = \Phi_2(\varphi^{-1}\theta, 1)f(\varphi^{-1}\theta)$,

$$\begin{aligned}\|T_1 - T_2\|_{B(C_0(\Theta,X))} &= \sup_{\|f\|_{C_0(\Theta,X)}=1} \left\| \left[\Phi_1(\varphi^{-1}\cdot\,, 1) - \Phi_2(\varphi^{-1}\cdot\,, 1)\right] f(\varphi^{-1}\cdot)\right\|_{C_0(\Theta,X)} \\ &\le \sup_{\theta\in\Theta} \|\Phi_1(\theta, 1) - \Phi_2(\theta, 1)\|_{\mathcal{L}(X)}.\end{aligned}$$

For sufficiently small $\epsilon > 0$, (5.1) implies $\sigma(T_2) \cap \mathbb{T} = \emptyset$. □

¿From Corollary 5.1 we derive the following.

Corollary 5.3. *Let $\hat{\varphi}_1^t$ and $\hat{\varphi}_2^t$ be two LSPFs over φ^t, and let Γ_1 and Γ_2 denote the respective generators of the corresponding evolutionary semigroups* (2.2). *If $\hat{\varphi}_1^t$ has exponential dichotomy, then there exists $\epsilon > 0$, $\epsilon = \epsilon(\hat{\varphi}_1^t)$ such that $\|\Gamma_1 - \Gamma_2\| < \epsilon$ implies $\hat{\varphi}_2^t$ also has exponential dichotomy.*

Since the "typical" generator Γ for a variational equation (2.4) has the form (2.5), this can be used when the unbounded operators $A(\theta)$ are perturbed by bounded operators such that $\Gamma_1 - \Gamma_2$ is bounded.

For the strongly continuous setting, we have the following roughness result (cf. [38]) for the variational equation in Example 2.5. Recall that in that example the generator of the evolutionary semigroup is identified in (2.7).

Corollary 5.4. *Let A_0 be a generator of a C_0 semigroup $\{e^{tA_0}\}_{t\ge 0}$ on X, and assume A_1 and $A_2 \colon \Theta \to \mathcal{L}_s(X)$ be bounded and continuous. Consider the equations:*

$$\frac{dx}{dt} = A_0 x(t) + A_1(\varphi^t\theta)x(t), \quad \frac{dx}{dt} = A_0 x(t) + A_2(\varphi^t\theta)x(t), \quad \theta \in \Theta,\ t \in \mathbb{R}$$

(see (2.4)–(2.6)). Assume that the LSPF $\hat{\varphi}_1^t$ over φ^t generated by the first equation has exponential dichotomy. Then there exists $\epsilon > 0$, $\epsilon = \epsilon(\hat{\varphi}_1^t)$ such that $\|A_1(\theta) - A_2(\theta)\| < \epsilon$ implies the LSPF $\hat{\varphi}_2^t$ over φ^t generated by the second equation has exponential dichotomy, provided the both cocycles are strongly continuous.

This gives an alternate way of addressing the situation of Theorem 5.2 in [8].

5.3. **Green's function.** We describe the existence of exponential dichotomy for $\hat{\varphi}^t$ in terms of the existence and uniqueness of a Green's function. Let $P\colon \Theta \to \mathcal{L}_s(X)$ be a bounded continuous projection-valued function that satisfies the following properties for $\theta \in \Theta$ and $t > 0$:

a) $\Phi(\theta,t)P(\theta) = P(\varphi^t\theta)\Phi(\theta,t)$;
b) $\Phi(\theta,t)$ is invertible as an operator from $\operatorname{Im} Q(\theta)$ to $\operatorname{Im} Q(\varphi^t\theta)$.

Define

$$G(\theta,t) = \begin{cases} \Phi_P(\theta,t), & t > 0 \\ \Phi_Q(\theta,t), & t < 0. \end{cases} \tag{5.2}$$

Here we have denoted:

$$\Phi_P(\theta,t) = P(\varphi^t\theta)\Phi(\theta,t)P(\theta) \quad \text{for } t > 0$$
$$\Phi_Q(\theta,t) = \left[Q(\varphi^{-t}\theta)\Phi(\theta,-t)Q(\theta)\right]^{-1} \quad \text{for } t < 0,$$

and used a) and b).

Definition 5.1. We say that the LSPF $\hat{\varphi}^t$ has a Green's function if there exists a bounded continuous projection $P\colon \Theta \to \mathcal{L}_s(X)$, such that for G from (5.2) the operator

$$(\widehat{G}f)(\theta) = \int_{-\infty}^{\infty} G(\varphi^{-t}\theta,t)f(\varphi^{-t}\theta)\,dt, \quad \theta \in \Theta, \tag{5.3}$$

is bounded on $C_0(\Theta,X)$.

The following result generalizes [49] (see also [21]).

Theorem 5.5. *The LSPF $\hat{\varphi}^t$ has exponential dichotomy if and only if the Green's function exists and is unique.*

Proof. For every $P(\cdot)$ satisfying a)-b), define $\mathcal{P} \in B(C_0(\Theta,X))$ by $\mathcal{P}f(\theta) = P(\theta)f(\theta)$, and set $\mathcal{Q} = I - \mathcal{P}$. Then $\mathcal{P}$ commutes with

T^t, and $T^t_Q \equiv \mathcal{Q}T^t\mathcal{Q}$ is invertible in $\operatorname{Im}\mathcal{Q}$. We note that (5.3) can be rewritten as $\widehat{G} = -\tilde{G}$, where

$$\tilde{G}f = \int_0^\infty T_Q^{-t} f\, dt - \int_0^\infty T_Q^t f\, dt. \tag{5.4}$$

If Γ denotes the generator of T^t, then the Spectral Mapping Theorem, 3.3, shows that $\Gamma^{-1} \in B(C_0(\Theta, X))$ if and only if T^t is hyperbolic.

Assume the Green's function, G, exists and is unique. Then $\tilde{G}$ is bounded. By [24, Lemma 4.2] then Γ is invertible and $\Gamma^{-1} = \tilde{G}$. Since T^t is hyperbolic, $i) \Rightarrow iii)$ of Theorem 4.1 shows that $\hat{\varphi}^t$ has exponential dichotomy.

Conversely, assume $\hat{\varphi}^t$ has exponential dichotomy. Then T^t is hyperbolic with Riesz projection $\mathcal{P} = P(\cdot)$. By [2] this projection is unique. The standard norm-estimates in (5.4) shows that $\tilde{G}$ and $\widehat{G}$ are bounded. $\square$

5.4. **Pointwise dichotomy.** In this subsection we consider the interrelation between "global" and "pointwise" dichotomies. Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{Z}$. Fix a point $\theta_0 \in \Theta$. We define an exponential dichotomy of $\hat{\varphi}^t$ over the orbit through θ_0 as follows.

Definition 5.2. The LSPF $\hat{\varphi}^t$ has *exponential dichotomy* over $\mathbb{K}$ at θ_0 if for all $\tau \in \mathbb{K}$, there exists a projection $P(\varphi^\tau\theta_0) \in \mathcal{L}(X)$ such that

a) $P(\varphi^\tau\theta_0)\Phi(\theta_0, \tau) = \Phi(\theta_0, \tau)P(\theta_0)$;
b) $\Phi_Q(\varphi^\tau\theta_0, t)$ is invertible from $\operatorname{Im} Q(\varphi^\tau\theta_0)$ to $\operatorname{Im} Q(\varphi^{\tau+t}\theta_0)$ for all $t > 0$ and $\tau \in \mathbb{K}$;
c) there exist positive constants $M = M(\theta_0)$, $\beta = \beta(\theta_0)$ such that for all $\tau \in \mathbb{R}$ and $t > 0$:

$$\|\Phi_P(\varphi^\tau\theta_0, t)\| \leq Me^{-\beta t}, \quad \|\left[\Phi_Q(\varphi^\tau\theta_0, t)\right]^{-1}\| \leq Me^{-\beta t}.$$

For $\mathbb{K} = \mathbb{R}$ we require, in addition, that $\tau \mapsto P(\varphi^\tau\theta_0)$ is a (strongly) continuous function from $\mathbb{R}$ to $\mathcal{L}_s(X)$.

For invertible-valued cocycles $\Phi\colon \Theta \times \mathbb{K} \to \mathcal{L}(X)$, Definition 5.2 is equivalent to the classical definition of exponential dichotomy of a LSPF at a point: There exists a projection P and constants M, β, such that $\|\Phi(\theta_0, s')P\Phi^{-1}(\theta_0, s)\| \leq Me^{-\beta(s'-s)}$ for $s' \geq s$, and $\|\Phi(\theta_0, s')Q\Phi^{-1}(\theta_0, s)\| \leq Me^{-\beta(s-s')}$ for $s \geq s'$ (see, e.g., [46]).

The following assertion has appeared previously. For the sake of completeness, we include a brief proof that is consistent with the approach of the present paper (see also [8], [16, Thm. 7.6.5], and [25, Thm. 3.22]).

Lemma 5.6. *The LSPF $\hat{\varphi}^t$ has exponential dichotomy over $\mathbb{Z}$ at $\theta_0 \in \Theta$ if and only if $\pi_{\theta_0}(T)$ is hyperbolic in $l_\infty(\mathbb{Z}; X)$.*

Proof. Indeed, the spectral radius r of $\pi_{\theta_0}(T)\mathcal{P}_{\theta_0}$ is given by the formula

$$r = \lim_{k\to\infty} \left(\sup_{n\in\mathbb{Z}} \|\Phi(\varphi^n\theta_0, k)P(\varphi^n\theta_0)\| \right)^{1/k},$$

and the exponential dichotomy of $\hat{\varphi}^t$ at θ_0 implies the hyperbolicity of $\pi_{\theta_0}(T)$ with Riesz projection

$$\mathcal{P}_{\theta_0} = \mathrm{diag}\{P_n\}_{n\in\mathbb{Z}} \tag{5.5}$$

for $P_n := P(\varphi^n\theta_0)$.

Conversely, assume $\pi_{\theta_0}(T) \cap \mathbb{T} = \emptyset$. Then for all $\lambda \in \mathbb{T}$, $[\lambda I - \pi_{\theta_0}(T)]^{-1} \in \mathfrak{D}$. Therefore, as in Corollary 4.5, the Riesz projection $\mathcal{P}_{\theta_0}$ corresponding to $\sigma(\pi_{\theta_0}(T) \cap \mathbb{D}$ is an element of $\mathfrak{D}$. Proceeding as in [21, Lemma 3], one can show that $\mathcal{P}_{\theta_0} \in \mathfrak{C}$. I.e., there exist operators P_n in $\mathcal{L}(X)$ such that $\mathcal{P}_{\theta_0} = \mathrm{diag}\{P_n\}_{n\in\mathbb{Z}}$. Now define $P(\varphi^n\theta_0) := P_n$ for each $n \in \mathbb{Z}$. This defines projections in $\mathcal{L}(X)$, which, by the fact that $\pi_{\theta_0}(T)\mathcal{P}_{\theta_0} = \mathcal{P}_{\theta_0}\pi_{\theta_0}(T)$, are seen to satisfy part a) of Definition 5.2:

$$P(\varphi^\tau\theta_0)\Phi(\theta_0, \tau) = \Phi(\theta_0, \tau)P(\theta_0), \quad \tau \in \mathbb{Z}$$

(use (4.1)). Also, as in the proof of Lemma 4.7, $\Phi_Q(\theta_0, k)\colon \mathrm{Im}\, Q(\theta_0) \to \mathrm{Im}\, Q(\varphi^k\theta_0)$ is invertible for all $\theta \in \Theta$, $k \in \mathbb{Z}$. In particular, $\Phi_Q(\varphi^n\theta_0, k)\colon \mathrm{Im}\, Q(\varphi^n\theta_0) \to \mathrm{Im}\, Q(\varphi^{n+k}\theta_0)$ is invertible for all $n, k \in \mathbb{Z}$.

The estimates in Definition 5.2 c) are also verified as in Lemma 4.7. □

Combining this with Theorem 4.1 gives the following fact.

Corollary 5.7. *The LSPF $\hat{\varphi}^t$ has exponential dichotomy on Θ if and only if it has exponential dichotomy at every $\theta \in \Theta$ and $\|[\pi_\theta(T) - \lambda I]^{-1}\| \le C$, $\theta \in \Theta$, $\lambda \in \mathbb{T}$.*

5.5. **Evolutionary semigroups along trajectories.** Next we relate the exponential dichotomy of $\hat{\varphi}^t$ to an associated evolutionary semigroup defined on $C_0(\mathbb{R}, X)$ along each trajectory of the flow (cf. [21, 25]). Fix $\theta \in \Theta$, and consider the semigroup $\{\Pi_\theta^t\}_{t\geq 0}$ defined by

$$(\Pi_\theta^t f)(s) = \Phi(\varphi^{s-t}\theta, t)f(s-t), \quad s \in \mathbb{R},\ t \geq 0,\ f \in C_0(\mathbb{R}, X).$$

Let L_θ denote the generator. The semigroup $\{\Pi_\theta^t\}_{t\geq 0}$ is, in fact, an evolutionary semigroup in the sense of the strongly continuous line setting. Indeed, for $s \geq \tau$, set $U(s,\tau) = \Phi(\varphi^\tau\theta, s-\tau)$. Then $U(s,s) = I$, and for $s \geq r \geq \tau$,

$$\begin{aligned} U(s,\tau) = \Phi(\varphi^\tau\theta, s-t) &= \Phi(\varphi^{r-\tau}(\varphi^\tau\theta), s-r)\Phi(\varphi^\tau\theta, r-t) \\ &= \Phi(\varphi^r\theta, s-r)\Phi(\varphi^\tau\theta, r-\tau) \\ &= U(s,r)U(r,\tau). \end{aligned}$$

Hence, $\{U(s,\tau)\}_{s\geq\tau}$ is an evolutionary family on X, and $(\Pi_\theta^t f)(s) = U(s, s-t)f(s-t)$. As shown in [22, 23] (see also [44, 45]), any such evolutionary semigroup has the following properties:

Lemma 5.8. *$\sigma(\Pi_\theta^t)$ is invariant under rotations centered at origin, $\sigma(L_\theta)$ is invariant under translation along $i\mathbb{R}$, and the spectral mapping theorem holds:*

$$\sigma(\Pi_\theta^t)\backslash\{0\} = e^{t\sigma(L_\theta)}, \quad t > 0.$$

Further, when $\sigma(\Pi_\theta^1) \cap \mathbb{T} = \emptyset$, the corresponding Riesz projection $\mathcal{P}$ is an operator of multiplication: $(\mathcal{P}f)(s) = P(s)f(s)$, for some bounded, continuous projection-valued $P : \mathbb{R} \to \mathcal{L}_s(X)$.

Theorem 4.1 and Lemma 5.6 give the following result.

Theorem 5.9. *The following are equivalent*

i) The LSPF $\widehat{\varphi}^t$ has exponential dichotomy;

ii) $\sigma(\Pi_\theta^1) \cap \mathbb{T} = \emptyset$ for all $\theta \in \Theta$, and there exists $C > 0$ such that

$$\| \left[\Pi_\theta^1 - \lambda I\right]^{-1} \|_{B(C_0(\mathbb{R};X))} \leq C, \quad \theta \in \Theta,\ \lambda \in \mathbb{T};$$

iii) $\sigma(L_\theta) \cap i\mathbb{R} = \emptyset$ for all $\theta \in \Theta$, and there exists $C > 0$ such that

$$\| \left[i\xi - L_\theta\right]^{-1} \|_{B(C_0(\mathbb{R};X))} \leq C, \quad \theta \in \Theta,\ \xi \in \mathbb{R}.$$

As in Theorem 4.1 (see also [LMS1]), one can replace the inequalities in *ii*) and *iii*) by the inequalities

$$\|[\Pi_\theta^1 - I]^{-1}\|_{B(C_0(\mathbb{R};X))} \leq C, \ \text{ and } \ \|L_\theta^{-1}\|_{B(C_0(\mathbb{R};X))} \leq C, \quad \theta \in \Theta,$$

respectively. We note that the operators L_θ are differential operators of the first order, and $i) \Leftrightarrow iii)$ reduces the problem of the existence of exponential dichotomy on Θ to the problem of invertibility of these operators.

Proof. Lemma 5.8 proves the equivalence of $iii)$ and $ii)$. To prove $i) \Leftrightarrow ii)$, we introduce operators on the space

$$C_0(\mathbb{R}\times\Theta, X) = C_0(\Theta, C_0(\mathbb{R}, X)) = C_0(\mathbb{R}, C_0(\Theta, X))$$

defined as

$$\begin{aligned}
(\Pi h)(s,\theta) &= \Phi(\varphi^{s-1}\theta, 1)h(s-1,\theta),\\
(\hat{T}h)(s,\theta) &= \Phi(\varphi^{-1}\theta, 1)h(s-1,\varphi^{-1}\theta),\\
(Jh)(s,\theta) &= h(s,\varphi^{s}\theta), (J^{-1}h)(s,\theta) = h(s,\varphi^{-s}\theta),
\end{aligned}$$

for $s\in\mathbb{R}$, $\theta\in\Theta$, $h\in C_0(\mathbb{R}\times\Theta, X)$. These operators satisfy

$$J^{-1}\Pi J = \hat{T}. \tag{5.6}$$

Indeed, for $r_1(s,\theta) = h(s,\varphi^s\theta)$ one has $(\Pi r_1)(s,\theta) = \Phi(\varphi^{s-1}\theta,1)h(s-1,\varphi^{s-1}\theta)$, and $(J^{-1}\Pi J h)(s,\theta) = \Phi(\varphi^{s-1}(\varphi^{-s}\theta),1)h(s-1,\varphi^{s-1}(\varphi^{-s}\theta) = (\hat{T}h)(s,\theta)$.

Next, note that for a function $F\colon \Theta\to C_0(\mathbb{R},X)$, the operator Π acts as the multiplication by Π^1_θ: $(\Pi F)(\theta) = \Pi^1_\theta F(\theta)$. Hence, for $\lambda\in\mathbb{T}$, the operator $\lambda-\Pi$ of multiplication by $\lambda-\Pi^1_\theta$ is invertible on $C_0(\Theta, C_0(\mathbb{R},X))$ if and only if $\lambda-\Pi^1_\theta$ is invertible on $C_0(\mathbb{R},X)$ for each $\theta\in\Theta$, and $\|(\lambda-\Pi^1_\theta)^{-1}\|\le C$ for some $C>0$. This means, that the statement $\sigma(\Pi)\cap\mathbb{T}=\emptyset$ is equivalent to $ii)$.

By Theorem 4.1, $i)$ is equivalent to the statement $\sigma(T)\cap\mathbb{T}=\emptyset$ on $C_0(\Theta,X)$. From [22] (Theorem 2.5, (1) $\Leftrightarrow$ (2)), we conclude that $\sigma(T)\cap\mathbb{T}=\emptyset$ on $C_0(\Theta,X)$ is equivalent to $\sigma(\hat{T})\cap\mathbb{T}=\emptyset$ on $C_0(\mathbb{R},C_0(\Theta,X))$. Indeed, for $f\colon\mathbb{R}\to C_0(\Theta,X)$, $\hat{T}$ acts as $(\hat{T}f)(s) = Tf(s-1)$, and this is exactly the case considered in Theorem 2.5 of [22]. Hence, $i)$ is equivalent to $\sigma(\hat{T})\cap\mathbb{T}=\emptyset$. Equation (5.6) shows that $\sigma(\Pi)=\sigma(\hat{T})$, and hence $i)\Leftrightarrow ii)$. □

5.6. **Open Problems and Concluding Remarks.**

Remark 1. An open problem is to prove analogues of Theorems 3.3 and 4.1 in the L_p-setting: Let μ be a Borel measure on Θ, positive on open sets, and quasi-invariant with respect to the flow φ^t. The semigroup (2.2) could be replaced by

$$(T^t f)(\theta) = \left(\frac{d\mu \circ \varphi^t}{d\mu}(\theta)\right)^{1/p} \Phi(\varphi^{-t}\theta, t) f(\varphi^{-t}\theta), \quad f \in L_p(\Theta, \mu, X).$$

The proof of Theorem 3.3 would require minor changes. However, the proof of Lemma 4.3 should be modified essentially. Theorem 4.1 has been proven for the strongly continuous line setting for L_p [22, 24]; in this setting, the statement *i)* $\Leftrightarrow$ *iii)* is proved in [42] using an interesting alternative C_0-semigroup approach.

Remark 2. It would be interesting to determine under which conditions on the cocycle Φ and in which sense (in strongly continuous locally compact setting) the generator Γ is given by formula (2.5) (see [34, 35] for the line setting).

Remark 3. We conjecture that the condition (4.2) in Theorem 4.1 and the corresponding inequalities in Corollary 5.7 and Theorem 5.9 are redundant, at least for the norm continuous compact setting. This means that the LSPF is exponentially dichotomic on Θ if and only if it is exponentially dichotomic at every $\theta \in \Theta$. This conjecture is true for finite dimensional X [47, Lemma 2A] and for the norm continuous compact setting where X is a Hilbert space [25]. The corresponding algebraic question here is whether the set of representations $\{\pi_\theta\}_{\theta\in\Theta}$ of the algebra $\mathfrak{B}$ is sufficient, that is, whether $b \in \mathfrak{B}$ is invertible if and only if $\pi_\theta(b)$ is invertible for all $\theta \in \Theta$.

Remark 4. A problem related to Remark 3 is to consider, instead of $\pi_\theta(T)$, weighted shift operators $\pi_{\bar\theta}(T)$ along ϵ-trajectories $\bar\theta$. Recall, that a sequence $\bar\theta = \{\theta_n\}_{n\in\mathbb{Z}}$ is called an ϵ-trajectory for $\varphi = \varphi^1$ if $\operatorname{dist}(\varphi\theta_n, \theta_{n+1}) \le \epsilon$ for $n \in \mathbb{Z}$. The operator $\pi_{\bar\theta}(T)$ can be defined on $l_\infty(\mathbb{Z}, X)$ as $\pi_{\bar\theta}(T) = \operatorname{diag}\{\Phi(\theta_{n-1}, 1)\}_{n\in\mathbb{Z}} S$. We suspect that the LSPF has exponential dichotomy over Θ if and only if $\pi_{\bar\theta}(T)$ is hyperbolic for all ϵ-trajectories with sufficiently small ϵ.

Remark 5. An area open for investigation is the case when Φ is a (semi)cocycle over a *semiflow* $\{\varphi^t\}_{t\in\mathbb{R}_+}$. For the line setting, this corresponds to the dichotomy on the semiaxis.

Remark 6. We showed in this paper that exponential dichotomy persists under small perturbations of the cocycle. A theorem by R. Sacker and G. Sell [47, Thm. 6] says that dichotomy persists under "small" perturbations of a φ^t-invariant compact subset $\Theta_0 \subset \Theta$. The proof in [47] is essentially finite-dimensional. It would be of interest to know whether the theorem holds in the strongly continuous setting. We note that Remark 4 helps to prove the theorem for the uniformly continuous setting when X is a Hilbert space (in preparation).

Remark 7. We note that Lyapunov numbers (see [8, 20, 47]) for the cocycle Φ belong to the dynamical spectrum $\Sigma = \sigma(\Gamma) \cap \mathbb{R}$. As proven in [20] for the finite dimensional setting and in [25] for the uniformly continuous setting on a Hilbert space X, the boundaries of Σ can be computed via the *exact* Lyapunov-Oseledets exponents given by the multiplicative ergodic theorem. This theorem is now available for Banach spaces [29]. A natural question then is to characterize the boundaries of Σ for the compact-valued cocycles on a Banach space.

Remark 8. Another natural question is to relax our main assumption that the aperiodic trajectories of φ^t are dense in Θ. Without this assumption the Spectral Mapping Theorem does not hold (see [4] and [25]). Let $p(\theta) = \inf\{t : \varphi^t\theta = \theta\}$ denote the prime period of $\theta \in \Theta$, and set

$$p_0(\theta) := \inf_U \sup_{y \in U} p(y),$$

where U denotes an open set containing θ. Let $\mathcal{H}(S)$ denote the union of the circles, centered at origin, intersecting a set $S \subset \mathbb{C}$.

Assume $p_0(\theta) \geq c > 0$ for all $\theta \in \Theta$ and some c. We conjecture that the following Annular Hull Theorem (see [4]) is valid:

$$\exp t\sigma(\Gamma) \subset \sigma(T^t) \setminus \{0\} \subset \mathcal{H}(\exp t\sigma(\Gamma)).$$

There are infinite-dimensional counterexamples (similar to one in [32]) showing the theorem fails without the assumption $p_0(\theta) \geq c > 0$.

To prove this conjecture it might be helpful to consider the following function:

$$y(\theta) = \int_0^{p_0(\theta)} \rho(t/p_0(\theta))(T^t w)(\theta) + (1 - \rho(t/p_0(\theta)))(T^{t+p_0(\theta)} w)(\theta) dt,$$

instead of (3.11) in the proof of the Spectral Mapping Theorem. The function $\rho : [0,1] \to [0,1]$ here (see [23, p. 46]) is a smooth function such that that $\rho(\tau) = 0$ for $\tau \in [0, 1/3]$ and $\rho(\tau) = 1$ for $\tau \in [2/3, 1]$.

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References

[1] A. B. Antonevich, Two methods for investigating the invertibility of operators from C^*-algebras generated by dynamical systems, *Math. USSR-Sb* **52** (1985), 1–20.

[2] H. Bart, I. Gohberg, and M. A. Kaashoek, Wiener-Hopf factorization, inverse Fourier transform and exponentially dichotomous operators, *J. Funct. Anal.* **68** (1986),1-42.

[3] A. Ben-Artzi and I. Gohberg, Dichotomy of systems and invertibility of linear ordinary differential operators, *Oper. Theory Adv. Appl.* **56** (1992), 90–119.

[4] C. Chicone and R. Swanson, Spectral theory for linearization of dynamical systems, *J. Differential Equations* **40** (1981), 155–167.

[5] C. Chicone, Y. Latushkin and S. Montgomery-Smith, The spectrum of the kinematic dynamo operator for an ideally conducting fluid, submitted.

[6] S.-N. Chow, X.-B. Lin, and K. Lu, Smooth invariant foliations in infinite dimensional spaces, *J. Differential Equations* **94** (1991), 266–291.

[7] S.-N. Chow, and K. Lu, Invariant manifolds for flows in Banach spaces, *J. Differential Equations* **74** (1988), 285–317.

[8] S.-N. Chow and H. Leiva, Existence and roughness of the exponential dichotomy for linear skew-product semiflows in Banach spaces, *J. Differential Equations*, to appear.

[9] S.-N. Chow and H. Leiva, Two definitions of the exponential dichotomy for skew-product semiflow in Banach spaces, preprint.

[10] S.-N. Chow and H. Leiva, Dynamical spectrum for time dependent linear systems in Banach spaces, *Japan J. Industr. Appl. Math.*, **11**, no. 3 (1994).

[11] J. Daleckij and M. Krein, "Stability of Differential Equations in Banach Space," Amer. Math. Soc., Providence, RI, 1974.

[12] D. E. Evans, Time dependent perturbations and scattering of strongly continuous groups on Banach spaces, *Math. Ann.* **221** (1976), 275–290.

[13] J. Goldstein, Asymptotics for bounded semigroups on Hilbert space, *in* "Aspects of Positivity in Functional Analysis," Mathematics Studies, Vol. 122 (1986).

[14] J. Hale, "Asymptotic Behavior of Dissipative Systems," Math. Surveys Mongoraphs Vol. 25, Amer. Math. Soc., Providence, RI, 1988.

[15] J. Hale and S. M. Verduyn Lunel, "Introduction to Functional Differential Equations," Appl. Math. Sci. **99**, Springer-Verlag, New York, 1993.

[16] D. Henry, "Geometric Theory of Nonlinear Parabolic Equations," Lecture Notes in Math., no. 840, Springer-Verlag, New York, 1981.

[17] D. Henry, "Topics in analysis," *Publ. Mat.* **31** (1) (1987), 29–84.

[18] J. S. Howland, Stationary scattering theory for time-dependent hamiltonians, *Math. Ann.* **207** (1974), 315–335.

[19] R. Johnson, Analyticity of spectral subbundles, *J. Diff. Eqns.* **35** (1980), 366–387.

[20] R. Johnson, K. Palmer, and G. Sell, Ergodic properties of linear dynamical systems, *SIAM J. Math. Anal.* **1**, no. 1 (1987), 1–33.

[21] Y. Latushkin, Green's function, continual weighted composition operators along trajectories, and hyperbolicity of linear extensions for dynamical systems, *J. Dynamics Differential Equations* **6** (1) (1994), 1–21.

[22] Y. Latushkin and S. Montgomery-Smith, Evolutionary semigroups and Lyapunov theorems in Banach spaces, *J. Funct. Anal.*, to appear.

[23] Y. Latushkin and S. Montgomery-Smith, Lyapunov theorems for Banach spaces, *Bull. Amer. Math. Soc. (N.S.)*, **31**, no. 1 (1994), 44–49.

[24] Y. Latushkin and T. Randolph, Dichotomy of differential equations on Banach spaces and an algebra of weighted composition operators, submitted.

[25] Y. Latushkin and A. M. Stepin, Weighted composition operators and linear extensions of dynamical systems, *Russian Math. Surveys*, **46**, no. 2 (1992), 95–165.

[26] X.-B. Lin, Exponential dichotomies in intermediate spaces with applications to a diffusively perturbed predator-prey model, *J. Differential Equations* **108** (1994), 36–63.

[27] G. Lumer, Equations de diffusion dans le domaines (x, t) non-cylindriques et semigroupes "espace-temps", Lecture Notes in Math., no. 1393, Springer-Verlag, New York, 1989, 161–179.

[28] L. T. Magalhães, Persistence and smoothness of hyperbolic invariant manifold for functional differential equations, *SIAM J. Math. Anal.*, **18** (3) (1987), 670–693.

[29] R. Mãné, Lyapunov exponents and stable manifolds for compact transformations, *in* "Geometric Dynamics," J. Palis (ed.), Lecture Notes in Math., no. 1007, Springer-Verlag, New York, 1983, 522–577.

[30] J. Mather, Characterization of Anosov diffeomorphisms, *Indag. Math.*, **30** (1968), 479–483.

[31] J. Massera and J. Schaffer, "Linear Differential Equations and Function Spaces," Academic Press, NY, 1966.

[32] S. Montgomery-Smith, Stability and dichotomy of positive semigroups on L_p, preprint.

[33] R. Nagel (ed.) "One Parameters Semigroups of Positive Operators", Lecture Notes in Math., no. 1184, Springer-Verlag, Berlin, 1984.

[34] R. Nagel, Semigroup methods for non-autonomous Cauchy problems, *Tubingen Berichte zur Funktionalanalysis*, Heft 2, Jahrang 1992/93, (1993) 103–118.

[35] R. Nagel and A. Rhandi, A characterization of Lipschitz continuous evolution families on Banach spaces, *Integral Equations Operator Theory*, to appear.

[36] H. Neidhardt, On abstract linear evolution equations, I, *Math. Nachr.* **103** (1981), 283–298.

[37] K. Palmer, Exponential dichotomy and Fredholm operators, *Proc. Amer. Math. Soc.*, **104** (1988), 149-156.

[38] K. Palmer, Exponential dichotomies and transversal homoclinic points, *J. Differential Equations* **55** (1984), 225–256.

[39] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, N.Y./Berlin, 1983.

[40] Ya. B. Pesin, "Hyperbolic theory," *in* Encyclop. Math. Sci. Dynamical Systems **2** (1988).
[41] J. Prüss, On the spectrum of C_0-semigroups, *Trans. Amer. Math. Soc.* **284**, no. 2 (1984), 847–857.
[42] F. Räbiger and R. Schnaubelt, A spectral characterization of exponentially dichotomic and hyperbolic evolution families, preprint.
[43] R. Rau, Hyperbolic linear skew-product semiflows, preprint.
[44] R. Rau, Hyperbolic evolutionary semigroups on vector-valued function spaces, *Semigroup Forum*, **48** (1994), 107–118.
[45] R. Rau, Hyperbolic evolution groups and exponential dichotomy of evolution families, *J. Funct. Anal.*, to appear.
[46] R. Sacker and G. Sell, Existence of dichotomies and invariant splitting for linear differential systems, I,II,III *J. Differential Equations* **15, 22** (1974,1976) 429–458, 478–522.
[47] R. Sacker and G. Sell, A spectral theory for linear differential systems, *J. Differential Equations* **27** (1978), 320–358.
[48] R. Sacker and G. Sell, "Dichotomies for Linear Evolutionary Equations in Banach Spaces," IMA Preprint No. 838 (1991).
[49] Y. A. Mitropolskij, A. M. Samojlenko, and V. L.Kulik, Issledovanija Dichotomii Lineinyh Sistem Differencial'nyh Uravnenij, Russian (Dichotomy of Systems of Linear Differential Equations), Naukova Dumka, Kiev, 1990.
[50] J. Selgrade, Isolated invariant sets for flows on vector bundles, *Trans. Amer. Math. Soc.* **203** (1975) 359–390.
[51] H. Tanabe, "Equations of Evolution," Pitman, London, 1979.
[52] L. Weis, The stability of positive semigroups on L_p spaces, *Proc. Amer. Math. Soc.*, to appear.

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