Time decay for the bounded mean oscillation of solutions of the Schrödinger and wave equations

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Abstract: Let $u(x,t)$ be the solution of the Schrödinger or wave equation with L_2 initial data. We provide counterexamples to plausible conjectures involving the decay in t of the BMO norm of $u(t,\cdot)$. The proofs make use of random methods, in particular, Brownian motion.

1. Introduction

Consider the wave equation:

$$\begin{aligned} \partial_t^2 u(t,x) &= \Delta u(t,x) \\ u(0,x) &= 0 \\ \partial_t u(0,x) &= f, \end{aligned}$$

where $t \in \mathbf{R}$, $x \in \mathbf{R}^d$, and $f \in L_2(\mathbf{R}^d)$. Let us write once and for all $B_t f = u(t,\cdot)$, so that $\widehat{B_t f}(\zeta) = \dfrac{\sin(t\,|\zeta|)}{|\zeta|}\hat f(\zeta)$. In this paper we will be discussing the endpoints of the following assertion.

($B_{d,p,q}$): *if $f \in L_2(\mathbf{R}^d)$, then $t \mapsto B_t f$ is in $L_q(\mathbf{R}, L_p(\mathbf{R}^d))$, and there exists a constant c independent of f such that*

$$\left(\int_{-\infty}^{\infty} \|B_t f\|_p^q \, dt\right)^{1/q} \le c\,\|f\|_2 .$$

If $p,q \ge 1$, then by standard arguments, it is easy to show that this is equivalent to its dual assertion. Here, as in the rest of the paper, $1/p + 1/p' = 1/q + 1/q' = 1$.

($B^*_{d,p',q'}$): *if $t \mapsto f_t$ is in $L_{q'}(\mathbf{R}, L_{p'}(\mathbf{R}^d))$, then $\int_{-\infty}^{\infty} B_t f_t \, dt$ (exists almost everywhere and) is in $L_2(\mathbf{R}^d)$, and there exists a constant c independent of f_t such that*

$$\left\| \int_{-\infty}^{\infty} B_t f_t \, dt \right\|_2 \le c \left(\int_{-\infty}^{\infty} \|f_t\|_{p'}^{q'} \, dt\right)^{1/q'} .$$

In stating these assertions, we will always suppose that d, p and q satisfy the following conditions:

$$\left.\begin{aligned} \frac{d}{p} + \frac{1}{q} &= \frac{d}{2} - 1 \\ p,q &\ge 2 \\ d &\ge 3. \end{aligned}\right\} \qquad (*_B)$$

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Indeed the first condition can easily be shown to be necessary if ($B_{d,p,q}$) is to hold. This follows from considering the substitution $f(x) \to f(rx)$ for any $0 < r < \infty$. (Physicists might call this a dimensional argument.)

It is known that if conditions ($*_B$) hold, then assertions ($B_{d,p,q}$) and ($B^*_{d,p',q'}$) are true whenever $p < \infty$. Results of this form are commonly known as Strichartz inequalities, and were proven in [St1] in the case $p = q$. The other cases follow by the same argument. The main application of these results is to show existence and uniqueness results for non-linear wave equations. (See [Br], [Ka], [Ke] and [Ru] for applications of this and similar results, and references to other such results.)

The only case left is $p = \infty$, $q = 2$ and $d = 3$. The assertion ($B_{3,\infty,2}$) and its dual ($B^*_{3,1,2}$) were shown to be false by Klainerman and Machedon [Kl].

It was then conjectured that a weaker assertion may hold, that is, ($B_{3,\mathrm{BMO},2}$). The space $\mathrm{BMO}(\mathbf{R}^d)$ has long been studied by many authors. Many results that are not true for L_∞ turn out to be true for BMO. The space BMO is slightly larger than L_∞, with a smaller norm. There are several equivalent definitions for this space (see for example [Se]), and for our purposes it will be convenient to define BMO in terms of its dual space, H_1, that is, $\mathrm{BMO}(\mathbf{R}^d)$ is the space of measurable functions f from $\mathbf{R}^d$ such that if $g \in H_1(\mathbf{R}^d)$, then $fg \in L_1(\mathbf{R}^d)$. Furthermore, BMO is equipped with the dual norm:

$$\|f\|_{\mathrm{BMO}} = \sup \left\{ \int_{\mathbf{R}^d} fg : \|g\|_{H_1} \le 1 \right\}.$$

Thus it only remains to define $H_1(\mathbf{R}^d)$. Again, the literature gives several definitions (see for example [Se]). We will pick a definition in terms of maximal functions based upon the heat kernel:

$$Mg(x) = \sup_{t>0} \int_{\mathbf{R}^d} \frac{1}{(4\pi t)^{d/2}} \exp(-|y|^2/t) g(x-y)\, dy.$$

Then we say that $g \in H_1(\mathbf{R}^d)$ if $Mg \in L_1(\mathbf{R}^d)$, and set

$$\|g\|_{H_1} = \|Mg\|_1 .$$

It is clear that $|g| \le |Mg|$ almost everywhere, and hence H_1 is a subspace of L_1 with larger norm.

For definiteness, let us explicitly state the assertions involving these norms.

($B_{3,\mathrm{BMO},2}$): *if $f \in L_2(\mathbf{R}^3)$, then $t \mapsto B_t f$ is in $L_2(\mathbf{R}, \mathrm{BMO}(\mathbf{R}^3))$, and there exists a constant c independent of f such that*

$$\left(\int_{-\infty}^{\infty} \|B_t f\|^2_{\mathrm{BMO}}\, dt \right)^{1/2} \le c\, \|f\|_2 .$$

This is equivalent to its dual assertion:

($B^*_{3,H_1,2}$): *if $t \mapsto f_t$ is in $L_2(\mathbf{R}, H_1(\mathbf{R}^3))$, then $\int_{-\infty}^{\infty} B_t f_t\, dt$ is in $L_2(\mathbf{R}^3)$, and there exists a constant c independent of f_t such that*

$$\left\| \int_{-\infty}^{\infty} B_t f_t\, dt \right\|_2 \le c \left(\int_{-\infty}^{\infty} \|f_t\|^2_{H_1}\, dt \right)^{1/2} .$$

The purpose of this paper is to provide a counterexample to assertion $(B_{3,\mathrm{BMO},2})$. We will proceed by considering the dual assertion $(B^*_{3,H_1,2})$. We will be using random methods, and so we do not provide an explicit counterexample. The tool from probability theory we shall use is Brownian motion. We refer the reader to [Pe] for details. However the only property of Brownian motion we shall use is that it is a randomly chosen continuous function $t \mapsto b_t$ from $\mathbf{R}$ to $\mathbf{R}$ such that $b_t - b_s$ is a gaussian random variable with mean 0 and standard deviation $\sqrt{|s-t|}$. Throughout the paper, we will use the notation EX to denote the expected value of a random variable X.

We shall also consider the Schrödinger equation with zero potential:

$$\begin{aligned} \partial_t u(t,x) &= -i\Delta u(t,x) \\ u(0,x) &= f(x), \end{aligned}$$

where, once again, $t \in \mathbf{R}$, $x \in \mathbf{R}^d$, and $f \in L_2(\mathbf{R}^d)$. Let us write once and for all $A_t f = u(t,\cdot)$, so that $\widehat{A_t f}(\zeta) = \exp(i\,|\zeta|^2)\hat{f}(\zeta)$. Then we get a similar assertion, with its dual.

$(A_{d,p,q})$: *if* $f \in L_2(\mathbf{R}^d)$, *then* $t \mapsto A_t f$ *is in* $L_q(\mathbf{R}, L_p(\mathbf{R}^d))$, *and there exists a constant* c *independent of* f *such that*

$$\left(\int_{-\infty}^{\infty} \|A_t f\|_p^q \, dt \right)^{1/q} \le c \, \|f\|_2 .$$

$(A^*_{d,p',q'})$: *if* $t \mapsto f_t$ *is in* $L_{q'}(\mathbf{R}, L_{p'}(\mathbf{R}^d))$, *then* $\int_{-\infty}^{\infty} A_t f_t \, dt$ *is in* $L_2(\mathbf{R}^d)$, *and there exists a constant* c *independent of* f_t *such that*

$$\left\| \int_{-\infty}^{\infty} A_t f_t \, dt \right\|_2 \le c \left(\int_{-\infty}^{\infty} \|f_t\|_{p'}^{q'} \, dt \right)^{1/q'} .$$

In stating these assertions, we will always suppose that p, q and d satisfy the following conditions:

$$\left.\begin{aligned} \frac{d}{p} + \frac{2}{q} &= \frac{d}{2} \\ p, q &\ge 2 \\ d &\ge 2. \end{aligned}\right\} \qquad (*_A)$$

Once again, the first condition can be shown to be necessary.

Rather less is known about these assertions than the corresponding ones for the wave equation. It is known [Gi] that if conditions $(*_A)$ hold, and $q > 2$, then $(A_{d,p,q})$ holds.

However the case $q = 2$ seems to be open. This paper tackles one of these problems, that is, the case when $q = 2$, $d = 2$ and $p = \infty$. We will demonstrate that the assertion $(A_{2,\infty,2})$ does not hold. (We will also include an alternative proof of this fact due to Tony Carbery and Steve Hofmann.) The problem when $q = 2$ and $d \ge 3$ seems to be very difficult, and at the time of writing is apparently unknown.

We will also deal with the assertions involving the space BMO, again showing that these are false.

$(A_{2,\mathrm{BMO},2})$: *if $f \in L_2(\mathbf{R}^2)$, then $t \mapsto A_t f$ is in $L_2(\mathbf{R}, \mathrm{BMO}(\mathbf{R}^2))$, and there exists a constant c independent of f such that*

$$\left(\int_{-\infty}^{\infty} \|A_t f\|_{\mathrm{BMO}}^2 \, dt \right)^{1/2} \le c \, \|f\|_2 .$$

This is equivalent to its dual assertion:

$(A^*_{2,H_1,2})$: *if $t \mapsto f_t$ is in $L_2(\mathbf{R}, H_1(\mathbf{R}^2))$, then $\int_{-\infty}^{\infty} A_t f_t \, dt$ is in $L_2(\mathbf{R}^2)$, and there exists a constant c independent of f_t such that*

$$\left\| \int_{-\infty}^{\infty} A_t f_t \, dt \right\|_2 \le c \left(\int_{-\infty}^{\infty} \|f_t\|_{H_1}^2 \, dt \right)^{1/2} .$$

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2. Solutions of the Schrödinger equation

We will start by considering the Schrödinger equation, because the techniques are simpler. Our first result will be eclipsed by Theorem 2, given later. However, we will prove the following result because the method of the proof is simpler, and illustrates the main ideas that will be used. Because this will not be a definitive result, we will not be completely rigorous.

Throughout this section, we will make great use of the fact that if $\hat f(\zeta) = \exp(-\alpha \, |\zeta|^2)$ ($\zeta \in \mathbf{R}^2$), where α is a complex number with $\mathrm{Re}(\alpha) > 0$, then $f(x) = (4\pi\alpha)^{-1} \exp(-\,|x|^2 / \alpha)$. If one is considering tempered distributions, the result remains true if $\mathrm{Re}(\alpha) = 0$.

Theorem 1. *Assertion $(A_{2,\infty,2})$ is not true.*

In fact, what we will do is to show that assertion $(A^*_{2,1,2})$ is not true. Our counterexample is $f_t(x) = \alpha_t \delta(x - p_t)$, where $\int_{-\infty}^{\infty} |\alpha_t|^2 \, dt = 1$, $p_t \in \mathbf{R}^2$ will be chosen later, and δ is the Dirac delta function on $\mathbf{R}^2$. Of course, δ is not a function, and thus is not in $L_1(\mathbf{R}^2)$. However, we will sacrifice rigor for the sake of clarity. A more rigorous argument may be formed by setting $\hat\delta(\zeta) = \exp(-\,|\zeta|^2)$, and following the argument used in the proof of Theorem 2.

Note that if $t \in \mathbf{R}$, and $p, q \in \mathbf{R}^2$, then

$$\begin{aligned} \int_{\mathbf{R}^2} (A_t \delta)(x - p) \overline{\delta(x - q)} \, dx &= \int_{\mathbf{R}^2} \exp(it \, |\zeta|^2) \exp(ip \cdot \zeta) \exp(-iq \cdot \zeta) \, d\zeta \\ &= -\frac{1}{4\pi i t} \exp\left(\frac{|p - q|^2}{it} \right) . \end{aligned}$$

Then

$$\left\| \int_{-\infty}^{\infty} A_t f_t \, dt \right\|_2^2 = \int_{\mathbf{R}^2} \int_{-\infty}^{\infty} A_t f_t(x) \, dt \int_{-\infty}^{\infty} \overline{A_s f_s(x)} \, ds \, dx$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbf{R}^2} (A_{t-s} f_t)(x) \overline{f_s(x)} \, dx \, ds \, dt$$
$$= - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\alpha_t \bar{\alpha}_s}{4\pi i (t-s)} \exp\left(\frac{|p_t - p_s|^2}{i(t-s)} \right) ds \, dt.$$

Now we let $p_t = (\sqrt{\theta} b_t, 0)$, where θ is chosen so that if γ is a gaussian random variable with mean 0 and standard deviation 1, then $a = E(\sin(\theta\gamma^2) \neq 0$. Thus

$$E \sin\left(\frac{|p_t - p_s|^2}{(t-s)} \right) = E \sin(\operatorname{sgn}(t-s)\theta\gamma^2) = a \operatorname{sgn}(t-s).$$

Then, if α_t is real,

$$- \operatorname{Re} E \left\| \int_{-\infty}^{\infty} A_t f_t \, dt \right\|_2^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\alpha_t \alpha_s}{4\pi i (t-s)} E \sin\left(\frac{|p_t - p_s|^2}{(t-s)} \right) ds \, dt$$
$$= a \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\alpha_t \alpha_s}{4\pi i \, |t-s|} \, ds \, dt.$$

Taking $\alpha_t = 1$ if $|t| \leq 1$, and 0 otherwise, we obtain an unbounded integral, and hence the desired counterexample.

We will also give a different proof of this last result due to Tony Carbery and Steve Hofmann. This proof was found shortly after the one just given, and is reproduced here with their permission. Again, we are sacrificing some rigor to obtain clarity. The starting point is the same, except that we shall take $f_t(x) = \delta(x - \sqrt{r} p_t)$ for $t \in I$, and 0 otherwise, where I is any interval. We shall suppose that p_t is any path in $\mathrm{Lip}_{1/2}$, with $\|p\|_{\mathrm{Lip}_{1/2}} = 1$, and that $r \in \mathbf{R}$. Then it is sufficient to find a counterexample to the following assertion. For every $r \in \mathbf{R}$ and interval I we have

$$\left| \int_I \int_I \frac{1}{4\pi i (t-s)} \exp\left(\frac{r \, |p_t - p_s|^2}{i(t-s)} \right) ds \, dt \right| \leq |I| \, .$$

Now we apply an idea from [Co]. Pick $F \in C_0^\infty(\mathbf{R})$ such that $F(\rho) = \rho$ for $|\rho| \leq 1$. Then

$$\frac{|p_t - p_s|^2}{t-s} = \int_{-\infty}^{\infty} \hat{F}(-r) \exp\left(\frac{r \, |p_t - p_s|^2}{i(t-s)} \right) dr,$$

and so combining the last two displayed equations, and rearranging the integrals, we see that

$$\left| \int_I \int_I \frac{|p_t - p_s|^2}{4\pi(t-s)^2} \, ds \, dt \right| \le |I| \int_{-\infty}^{\infty} |\hat{F}(r)| \, dr \le C \, |I| \, ,$$

where C is some universal constant. But by a result of Strichartz [St2], we have that if $D^{1/2} p \in \mathrm{BMO}$, then

$$\sup_I \frac{1}{|I|} \int_I \int_I \frac{|p_t - p_s|^2}{(t-s)^2} \, ds \, dt \approx \|D^{1/2} p\|_{\mathrm{BMO}}.$$

Here $\widehat{D^{1/2} p}(\tau) = |\tau|^{1/2} \hat{p}(\tau)$. Hence we have our counterexample by picking p with $\|p\|_{\mathrm{Lip}_{1/2}} = 1$, and $\|D^{1/2} p\|_{\mathrm{BMO}}$ arbitrarily large.

Now we will improve this result. In the arguments that follow, the reader may feel uncomfortable with the cavalier and implicit use of Fubini's Theorem. The use of Fubini's Theorem requires all the integrals to be absolutely convergent, and this is not the case, as it is exactly the opposite that we are trying to show. For this reason, the proofs should really be seen as proofs by contradiction, that is, the reader should suppose initially that the assertions stated in the theorems are true.

Theorem 2. *Assertion* $(A_{2,\mathrm{BMO},2})$ *is not true.*

Once again, we provide a counterexample to the dual assertion $(A^*_{2,H_1,2})$. In this case, our counterexample will be $f_t(x) = \alpha_t g(x - p_t)$, where $\int_{-\infty}^{\infty} |\alpha_t|^2 \, dt = 1$, $p_t \in \mathbf{R}^2$ will be chosen later, and $\hat{g}(\zeta) = |\zeta|^2 \exp(-|\zeta|^2)$.

Let us first show that $g \in H_1(\mathbf{R}^2)$. Using Fourier transforms, we see that

$$\begin{aligned} Mg(x) &= \sup_{t>0} \frac{1}{4\pi t} \int_{\mathbf{R}^2} \exp(-|x-y|^2/t) g(y) \, dy \\ &= \sup_{t>0} \frac{4(1+t) - 2|x|^2}{4\pi(1+t)^3} \exp\left(-\frac{|x|^2}{1+t}\right) \\ &= O(\frac{1}{1+|x|^2}), \end{aligned}$$

which is in $L_1(\mathbf{R}^2)$.

Note also that for $t \in \mathbf{R}$ and $p, q \in \mathbf{R}^2$

$$\begin{aligned} &\int_{\mathbf{R}^2} (A_t g)(x-p) \overline{g(x-q)} \, dx \\ &\qquad = \int_{\mathbf{R}^2} |\zeta|^4 \exp((-2+it)|\zeta|^2) \exp(\zeta \cdot (q-p)) \, d\zeta \\ &\qquad = \frac{1}{4\pi(2-it)} \Delta^2 \exp\left(\frac{|x|^2}{-2+it}\right) \Big|_{x=q-p} \end{aligned}$$

$$= \frac{1}{4\pi(2-it)^3} P(|q-p|^2/(-2+it)) \exp\left(\frac{|q-p|^2}{-2+it}\right)$$
$$= \frac{1}{4\pi(2-it)^3} P(|q-p|^2/(-2+it)) \exp\left(-\frac{2|q-p|^2}{4+t^2} - \frac{it|q-p|^2}{4+t^2}\right),$$

where $P(t) = 16t^2 - 64t + 32$.

Hence

$$\left\| \int_{-\infty}^{\infty} A_t f_t \, dt \right\|_2^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha_t \bar{\alpha}_s K(t,s) \, ds \, dt,$$

where

$$K(t,s) = \frac{1}{4\pi(2-i(t-s))^3} P(|p_t - p_s|^2/(-2+i(t-s))) \times$$
$$\exp\left(-\frac{2|p_t-p_s|^2}{4+(t-s)^2} - \frac{i(t-s)|p_t-p_s|^2}{4+(t-s)^2}\right).$$

Now we will choose $p_t = (t, \sqrt{\theta} b_t)$, where b_t and θ are defined as in the previous section. In order to demonstrate that we have a counterexample, we need to show that $E(K(t,s))$ is not the kernel of bounded operator from $L_2(\mathbf{R}) \to L_2(\mathbf{R})$.

We shall be interested in the behavior of $E(K(t,s))$ as $t - s \to \pm\infty$. Let $b_t - b_s = \sqrt{|t-s|}\gamma$, where γ is a gaussian random variable with mean 0 and standard deviation 1. Denote $\langle t \rangle = 1 + |t|$.

First see that

$$P(|p_t - p_s|^2/(-2+i(t-s))) = -16(t-s)^2 + O(\gamma^4 \langle t-s \rangle).$$

Also, there exists a constant c_1 such that

$$\exp\left(-\frac{2|p_t-p_s|^2}{4+(t-s)^2}\right) = \exp(-(2+O(\gamma^2\langle t-s\rangle^{-1})) = \exp(-2) + O(\exp(c_1\gamma^2\langle t-s\rangle^{-1}) - 1).$$

Furthermore,

$$\frac{(t-s)|p_t-p_s|^2}{4+(t-s)^2} = (t-s) + \operatorname{sgn}(t-s)\theta\gamma^2 + O(\gamma^2\langle t-s\rangle^{-1}),$$

and hence there is a constant c_2 such that

$$\exp\left(-i\frac{(t-s)|p_t-p_s|^2}{4+(t-s)^2}\right) = \exp(i(s-t+\operatorname{sgn}(s-t)\theta\gamma^2)) + O(\exp(c_2\gamma^2\langle t-s\rangle^{-1}) - 1).$$

Therefore, for some constant c_3,

$$K(t,s) = \frac{-16\exp(-2)(t-s)^2}{4\pi(-2+i(t-s))^3}\exp(i(s-t))\exp(i\,\mathrm{sgn}(s-t)\theta\gamma^2) + O\big(\gamma^4\langle t-s\rangle^{-3} + \langle t-s\rangle^{-2}(\exp(c_3\gamma^2\langle t-s\rangle^{-1})-1)\big).$$

Hence

$$E(K(t,s)) = \frac{-16\exp(-2)(t-s)^2}{4\pi(2-i(t-s))^3}\exp(i(s-t))(a_1 + ia_2\,\mathrm{sgn}(s-t)) + O(\langle t-s\rangle^{-3}),$$

where $a_1 + ia_2 = E(\exp(i\theta\gamma^2))$. By considering the examples $\alpha_t = \exp(it)/\sqrt{N}$ if $|t| \le N$, and 0 otherwise, and letting $N \to \infty$, it may be readily seen that $E(K(t,s))$ is not the kernel of a bounded map from $L_2(\mathbf{R}) \to L_2(\mathbf{R})$, and the desired counterexample has been obtained.

We will now present a second proof of the same result. This proof goes via the Fourier transform. Although this proof is less intuitive, it also has less technical difficulties. To recap, it is sufficient to find a counterexample to the following statement: there is a constant c such that if $\int_{-\infty}^{\infty} |\alpha_t|^2\,dt \le 1$, and if $g \in H_1(\mathbf{R}^2)$, then for any path $p_t \in \mathbf{R}^2$, we have that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(s,t)\alpha_t\bar\alpha_s\,ds\,dt \le c,$$

where

$$\begin{aligned} K(s,t) &= \int_{\mathbf{R}^2} (A_{t-s}g)(x-p_t)\overline{g(x-p_s)}\,dx\,ds\,dt \\ &= \int_{\mathbf{R}^2} \exp(i(t-s)\,|\zeta|^2 + i(p_s-p_t)\cdot\zeta)\,|\hat g(\zeta)|^2\,d\zeta. \end{aligned}$$

That is, we are asking whether $K(s,t)$ is the kernel of a bounded operator from $L_2(\mathbf{R})$ to $L_2(\mathbf{R})$.

We take $p_t = (t,b_t)$, where b_t is Brownian motion. It is sufficient to show that $E(K(s,t))$ is not the kernel of a bounded operator on $L_2(\mathbf{R}^2)$. Note that if γ is a gaussian random variable with mean 1 and standard deviation 0, then $E(\exp(i\gamma)) = \exp(-t^2/2)$. Hence $E(K(s,t)) = L(t-s)$, where

$$L(t) = \int_{\mathbf{R}^2} \exp(it\,|\zeta|^2 - it\zeta_1 - |t|\,\zeta_2^2/2)\,|\hat g(\zeta)|^2\,d\zeta.$$

But $L(t-s)$ fails to be the kernel of a bounded operator on $L_2(\mathbf{R})$ if and only if $\hat L(\omega)$ fails to be bounded almost everywhere. Furthermore

$$\hat L(\omega) = \int_{\mathbf{R}^2} k(\omega,\zeta)\,|g(\zeta)|^2\,d\zeta,$$

where $k(\cdot,\zeta) = k_1(\cdot,\zeta) * k_2(\cdot,\zeta)$ with

$$k_1(\omega,\zeta) = \delta(\omega + |\zeta|^2 - \zeta_1)$$

and

$$k_2(\omega,\zeta) = \frac{4\zeta_2^2}{4\omega^2 + \zeta_2^4},$$

that is,

$$k(\omega,\zeta) = \frac{4\zeta_2^2}{4(\omega + |\zeta|^2 - \zeta_1)^2 + \zeta_2^4}.$$

Clearly k enjoys enough continuity properties so that it is sufficient to show that $\hat{L}(\omega)$ is unbounded for $\omega = 1/4$. So

$$k(1/4,\zeta) = \frac{4\zeta_2^2}{4((\zeta_1 - 1/2)^2 + \zeta_2^2)^2 + \zeta_2^4} \geq \frac{\zeta_2^2}{2\,|\zeta - (1/2,0)|^4}.$$

It is clear that $k(1/4,\zeta)$ has an $L_1(\mathbf{R}^2)$ singularity at $(1/2,0)$, and hence taking $\hat{g}(\zeta) = |\zeta|^2 \exp(-|\zeta|^2)$, we see that $\hat{L}(1/4)$ is unbounded.

3. Solutions of the wave equation

This section is devoted to the following result.

Theorem 3. *Assertion $(B_{3,\mathrm{BMO},2})$ is not true.*

We will show that assertion $(B^*_{3,H_1,2})$ is not true. The methods will be essentially the same as in the previous proofs, but the details will be more difficult, and so we will break it into steps.

To start, let us define an operator on a dense subspace of $L_2(\mathbf{R}^3)$ for each $t \in \mathbf{R}$ given by

$$\widehat{C_t f}(\zeta) = \frac{\cos(t\,|\zeta|)}{|\zeta|^2}\hat{f}(\zeta).$$

Let us also set

$$K_t(x) = \frac{1}{4\pi\,|x|} I_{|x|\geq|t|} \quad (x \in \mathbf{R}^3).$$

Lemma 4. *If $f \in L_1(\mathbf{R}^3) \cap L_2(\mathbf{R}^3)$, and $t \neq 0$, then*

$$C_t f(x) = \int_{\mathbf{R}^3} K_t(x-y) f(y)\,dy,$$

and the operator norm of C_t from $L_1(\mathbf{R}^3)$ to $L_1(\mathbf{R}^3)$ is $(4\pi\,|t|)^{-1}$.

To show this when f is C^∞ with compact support, it is sufficient to show that $\hat{K}_t = \cos(t\,|\zeta|)/\,|\zeta|^2$ (as tempered distributions). Then by Young's convolution formula, as an operator from $L_1(\mathbf{R}^3)$ to $L_1(\mathbf{R}^3)$, the operator norm of C_t is given by

$$\|C_t\| = \|K_t\|_\infty = \frac{1}{4\pi\,|t|}.$$

It is clear that $\hat{K}_t(\zeta)$ depends only upon t and $|\zeta|$. So without loss of generality, we may suppose that $\zeta = (\zeta_1, 0, 0)$, where $\zeta_1 = |\zeta|$. In performing the following integral, we will use the following change of variables: $x_1 = u$, $x_2 = \sqrt{v^2 - u^2}\cos(\theta)$, and $x_3 = \sqrt{v^2 - u^2}\sin(\theta)$. Thus the Jacobian $\partial(x_1, x_2, x_3)/\partial(u, v, \theta) = v = |x|$. Then

$$\begin{aligned}
\hat{K}_t(\zeta) &= \int_{|x|\ge|t|} \frac{\exp(-ix\cdot\zeta)}{4\pi\,|x|}\,dx \\
&= \frac{1}{4\pi}\int_{v=|t|}^{\infty}\int_{u=-v}^{v}\int_0^{2\pi} \cos(u\,|\zeta|)\,d\theta\,dv\,du \\
&= \frac{\cos(t\,|\zeta|)}{|\zeta|^2}.
\end{aligned}$$

In the last line we have used the assertion that $\lim_{R\to\infty}\cos(R\,|\zeta|) = 0$, which is true in the space of tempered distributions.

Corollary 5. *If $f \in L_2(\mathbf{R}^3)$ is bounded with compact support, and $s, t \in \mathbf{R}$, then*

$$B_s B_t f(x) = \frac{C_{s+t}f(x) - C_{s-t}f(x)}{2} = \int_{\mathbf{R}^3} \frac{K_{s+t}(x-y) - K_{s-t}(x-y)}{2} f(y)\,dy.$$

This follows because $\sin(t\,|\zeta|)\sin(s\,|\zeta|) = \frac{1}{2}(\cos((s+t)\,|\zeta|) - \cos((s-t)\,|\zeta|))$.

Lemma 6. *There is a universal constant c such that if $t \mapsto f_t$ is in $L_2(\mathbf{R}, L_1(\mathbf{R}^3))$, then*

$$\int_0^\infty\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} K_{s+t}(x-y) f_t(x) f_s(y)\,dy\,dx\,dt\,ds \le c\int_0^\infty \|f_t\|_1^2\,dt.$$

The proof of this result depends upon the boundedness of the Hardy operators defined on $L_2([0,\infty))$:

$$\begin{aligned}
H\alpha(t) &= \frac{1}{t}\int_0^t \alpha(s)\,ds \\
H^*\alpha(t) &= \int_t^\infty \frac{\alpha(s)}{s}\,ds.
\end{aligned}$$

If $\alpha \in L_2([0,\infty))$, then both $H\alpha$ and $H^*\alpha$ are in $L_2([0,\infty))$, with $\|H\alpha\|_2, \|H^*\alpha\|_2 \le 2\|\alpha\|_2$. The result for H may be found in [Ha], and the result for H^* follows because H^* is the adjoint operator to H.

Next, by Lemma 4, if $s, t > 0$

$$\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} K_{s+t}(x-y) f_t(x) f_s(y)\,dy\,dx \le \frac{1}{4\pi(s+t)}\|f_t\|_1\|f_s\|_1.$$

Thus, setting $\alpha(t) = \|f_t\|_1$, we see that

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} K_{s+t}(x-y) f_t(x) f_s(y)\,dy\,dx\,dt\,ds \\
&\le \frac{1}{4\pi}\int_0^\infty (H\alpha(t) + H^*\alpha(t))\alpha(t)\,dt \\
&\le \frac{1}{\pi}\|\alpha\|_2^2.
\end{aligned}$$

Now we will consider the following assertion.

(C): *There is a universal constant c such that if $t \mapsto f_t$ is in $L_2(\mathbf{R}, H_1(\mathbf{R}^3))$, then*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} K_{s-t}(x-y) f_t(x) f_s(y)\, dy\, dx\, dt\, ds \le c \int_{-\infty}^{\infty} \|f_t\|_{H_1}^2 \, dt.$$

Let us first demonstrate that the failure of assertion (C) implies the failure of assertion $(B_{3,H_1,2})$. Let us suppose that we have a sequence $t \mapsto f_t$ in $L_2(\mathbf{R}, H_1(\mathbf{R}^3))$ such that

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} K_{s-t}(x-y) f_t(x) f_s(y)\, dy\, dx\, dt\, ds$$

is unbounded. We may split the integral up into four pieces:

$$\begin{aligned} I &= I_1 + I_2 + I_3 + I_4 \\ &= \int_0^{\infty} \int_0^{\infty} \dots \, dt\, ds + \int_{-\infty}^{0} \int_0^{\infty} \dots \, dt\, ds + \int_0^{\infty} \int_{-\infty}^{0} \dots \, dt\, ds + \int_{-\infty}^{0} \int_{-\infty}^{0} \dots \, dt\, ds. \end{aligned}$$

By Lemma 6, we know that I_2 and I_3 are bounded. Therefore one of I_1 or I_4 is unbounded, and without loss of generality we may suppose that I_1 is unbounded. So without loss of generality, we may suppose that $f_t = 0$ if $t < 0$.

Now, multiplying out, and applying Corollary 5, we see that

$$\begin{aligned} \left\| \int_{-\infty}^{\infty} B_t f_t \, dt \right\|_2^2 &= \left\| \int_0^{\infty} B_t f_t \, dt \right\|_2^2 \\ &= \int_0^{\infty} \int_0^{\infty} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} \frac{K_{s+t}(x-y) - K_{s-t}(x-y)}{2} f_t(x) f_s(y)\, dy\, dx\, dt\, ds. \end{aligned}$$

Again, by Lemma 6, the last integral differs from $I_1/2$ by a bounded amount, and we have produced the desired counterexample.

All that remains to be shown is the following.

Lemma 7. *Assertion (C) is false.*

Our counterexample will be $f_t(x) = \alpha_t g(x - p_t)$, where $\int_0^\infty |\alpha_t|^2 \, dt = 1$, and $p_t = (t, b_t, b'_t)$. The function $g \in H_1(\mathbf{R}^3)$ will be selected later. Here b_t and b'_t are two independent Brownian motions. We will show that

$$J = E\left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} K_{s-t}(x-y) \alpha_t \bar{\alpha}_s g(x-p_t) \overline{g(x-p_s)}\, dy\, dx\, dt\, ds \right)$$

cannot be universally bounded. This quantity is more easily computed using the Fourier transform:

$$J = E\left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \alpha_t \bar{\alpha}_s \int_{\mathbf{R}^3} \frac{\cos((t-s)\,|\zeta|)}{|\zeta|^2} \exp(i(p_s - p_t)\cdot \zeta)\, |\hat{g}(\zeta)|^2 \, d\zeta\, dt\, ds \right).$$

Notice that

$$\begin{aligned} E(\exp(i(p_s - p_t)) &= \exp(i(s-t)\zeta_1) E(\exp(i(b_s - b_t)\zeta_2) + i(b'_s - b'_t)\zeta_3) \\ &= \exp(i(s-t)\zeta_1)\exp(-|s-t|\,(\zeta_2^2 + \zeta_3^2)/2). \end{aligned}$$

Hence

$$J = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L(s-t)\alpha_t \bar{\alpha}_s \, dt\, ds,$$

where

$$L(t) = \int_{\mathbf{R}^3} \frac{\cos(t\,|\zeta|)}{|\zeta|^2} \exp(it\zeta_1)\exp(-|t|\,(\zeta_2^2+\zeta_3^2)/2)\,|\hat{g}(\zeta)|^2 \, d\zeta.$$

Saying that J is bounded for all α_t in $L_2(\mathbf{R})$ is equivalent to saying that convolving with L gives a bounded operator on $L_2(\mathbf{R})$. But this is equivalent to $\hat{L}$ being in $L_\infty(\mathbf{R})$. Now

$$\hat{L}(\omega) = \int_{\mathbf{R}^3} k(\omega,\zeta)\,|\hat{g}(\zeta)|^2 \,/\, |\zeta|^2 \, d\zeta,$$

where $k(\cdot,\zeta) = h_1(\cdot,\zeta) * h_2(\cdot,\zeta) * h_3(\cdot,\zeta)$ with

$$\begin{aligned} h_1(\omega,\zeta) &= \frac{\delta(\omega + |\zeta|) + \delta(\omega - |\zeta|)}{2} \\ h_2(\omega,\zeta) &= \delta(\omega - \zeta_1) \\ h_3(\omega,\zeta) &= \frac{4(\zeta_2^2+\zeta_3^2)}{4\omega^2 + (\zeta_2^2+\zeta_3^2)^2}. \end{aligned}$$

Thus

$$k(\omega,\zeta) = \frac{k_1(\omega,\zeta) + k_2(\omega,\zeta)}{2},$$

where

$$k_1(\omega,\zeta) = \frac{4(\zeta_2^2+\zeta_3^2)}{4(\omega - \zeta_1 + |\zeta|)^2 + (\zeta_2^2+\zeta_3^2)^2},$$

and

$$k_2(\omega,\zeta) = \frac{4(\zeta_2^2+\zeta_3^2)}{4(\omega - \zeta_1 - |\zeta|)^2 + (\zeta_2^2+\zeta_3^2)^2}.$$

Since all the expressions involved are positive, we will have found a counterexample if we can show that

$$\int_U k_1(0,\zeta)\,|\hat{g}(\zeta)|^2 \,/\, \left|\zeta^2\right| \, d\zeta = \infty,$$

where U is any subset of $\mathbf{R}^3$. We will take

$$U = \{\zeta : \sqrt{\zeta_2^2+\zeta_3^2} \le \zeta_1 \le 1\},$$

and

$$g(x) = \begin{cases} 1 & \text{if } 0 < x_1 \le 1 \text{ and } -1 \le x_2, x_3 \le 1 \\ -1 & \text{if } -1 \le x_1 < 0 \text{ and } -1 \le x_2, x_3 \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\hat{g}(\zeta) = \frac{8(1 - \cos(\zeta_1)) \sin(\zeta_2) \sin(\zeta_3)}{\zeta_1 \zeta_2 \zeta_3}.$$

In this case, we see that $|\hat{g}(\zeta)|^2 / |\zeta|^2$ is bounded from below by a positive number on U. Therefore, we need to show that

$$\int_U k_1(0, \zeta)\, d\zeta = \infty.$$

To do this, let us compute the integral using cylindrical coordinates, that is, we will write $z = \zeta_1$, and $r = \sqrt{\zeta_2^2 + \zeta_3^2}$. Then we see that the last integral is

$$\int_{z=0}^{1} \int_{r=0}^{z} \frac{4r^2}{4(\sqrt{z^2 + r^2} - z)^2 + r^4} 2\pi r \, dr \, dz.$$

But in U,

$$\sqrt{z^2 + r^2} - z \le r^2 / z,$$

and hence

$$\frac{4r^2}{4(\sqrt{z^2 + r^2} - z)^2 + r^4} \ge \frac{4z^2}{r^2(4 + z^2)}.$$

Hence the integral in question becomes bounded below by

$$\int_{z=0}^{1} \int_{r=0}^{z} \frac{8\pi z^2}{r(4 + z^2)} \, dr \, dz,$$

and this is easily seen to be infinite.

4. References

There is an extensive literature on the positive results, and we do not desire to give a complete list. Instead, we refer the reader to the papers chosen principally because they are recent, and provide many references on past work.

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