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Random Rearrangements and Operators

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Dedicated to Professor Selim Krein on his 80th birthday

0. Introduction

Let n be a positive integer, $x = (x_{ij})_{1\le i,j\le n}$, and S_n the group of permutations of $\{1, 2, \dots, n\}$. Denote by $x_1^*, x_2^*, \dots, x_{n^2}^*$ the rearrangement of $|x_{ij}|$ in decreasing order. S. Kwapien and C. Schütt proved the following results.

Theorem A [**KS1**]. *We have that*

$$\frac{1}{2n}\sum_{k=1}^{n} x_k^* \le \frac{1}{n!}\sum_{\pi\in S_n}\max_{1\le i\le n}\left|x_{i,\pi(i)}\right| \le \frac{1}{n}\sum_{k=1}^{n} x_k^*.$$

Theorem B [**Sc**]. *If $1 \le p \le q < \infty$, then*

$$\begin{aligned}
&\frac{1}{10}\left(\left(\frac{1}{n}\sum_{k=1}^{n}(x_k^*)^p\right)^{1/p} + \left(\frac{1}{n}\sum_{k=n+1}^{n^2}(x_k^*)^q\right)^{1/q}\right)\\
&\le \left(\frac{1}{n!}\sum_{\pi\in S_n}\left(\sum_{i=1}^{n}\left|x_{i,\pi(i)}\right|^q\right)^{p/q}\right)^{1/p}\\
&\le \left(\frac{1}{n}\sum_{k=1}^{n}(x_k^*)^p\right)^{1/p} + \left(\frac{1}{n}\sum_{k=n+1}^{n^2}(x_k^*)^q\right)^{1/q}.
\end{aligned}$$

There are two ways to generalize these results. They were presented in [**S1**], [**S2**] and [**M2**]. This article is devoted to the development of these methods.

Let $1 \le q \le \infty$, and ℓ be a one-to-one correspondence of S_n to $\{1, 2, \dots, n!\}$. We define the quasi-linear operator T_q as follows:

$$T_q x(t) = \left(\sum_{i=1}^{n}\left|x_{i,\pi(i)}\right|^q\right)^{1/q}, \qquad t \in \left[\frac{\ell(\pi)-1}{n!}, \frac{\ell(\pi)}{n!}\right),$$

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with the usual modification for $q=\infty$. The operator T_q acts from the set of $n\times n$ matrices into the step functions. Clearly it depends upon the choice of ℓ. However, if E is a rearrangement invariant space (see Section 1 for the definitions), then $\|T_q x\|_E$ does not depend upon ℓ. We define the operator

$$Ux(t)=\sum_{k=1}^{n} x_k^* \chi_{((k-1)/n,k/n)}(t)$$

on the set of $n\times n$ matrices $x=(x_{ij})$, where χ_e is the characteristic function of $e\subset[0,1]$.

(We would like to mention that there is another modification of this construction. If E is an r.i. space, we can construct a space $\tilde{E}$ on the group S_n equipped with the Haar measure. In this case it is not necessary to introduce the function ℓ. However, some additional difficulties appear.)

The inequality

$$\frac{1}{12}\left(\|U_x\|_E+\left(\frac{1}{n}\sum_{k=n+1}^{n^2} x_k^{*q}\right)^{1/q}\right)\le \|T_q x\|_E \tag{1}$$

was proved [**S1**] for any matrix x, rearrangement invariant space E, and $1\le q<\infty$. A more exact estimate is valid for $q=\infty$:

$$\frac{1}{2}\|Ux\|_E\le\|T_\infty x\|_E\le\|Ux\|_E\,. \tag{2}$$

The inverse inequality

$$\|T_q x\|_E\le C\left(\|Ux\|_E+\left(\frac{1}{n}\sum_{k=n+1}^{n^2} x_k^{*q}\right)^{1/q}\right) \tag{3}$$

was established under the additional assumptions $\alpha_E>0$, where α_E is the lower Boyd index of an r.i. space E, and C does not depend upon x or n. It is evident that (3) fails for $E=L_\infty$ and $1\le q<\infty$.

In this article we generalize these results. In Chapter 4, we give a complete criterion for Lorentz spaces for which (3) holds. In Chapter 5, we consider other r.i. spaces for which (3) holds, and give some interpolation results. All this is based on Chapter 3, where it is shown that one can reduce (3) to the special case of diagonal matrices.

In Chapter 7, we consider a completely different generalization, where in effect E is L_1, but T_q is replaced by something analogous to T_X, where X is a symmetric sequence space. For the analogous result to (3), one needs the concept of a D^*-convex space. For this reason, in Chapter 6, we develop the theory of D and D^*-convex spaces, building on earlier work of Kalton [**K**]. In Chapter 8 we develop this idea further, and classify which Lorentz spaces are D or D^*-convex.

Some results from this article were announced in [**S2**] and [**M2**].

1. Preliminaries

If $x(t)$ is a measurable function on $[0,1]$, we denote by $x^*(t)$ the decreasing rearrangement of $|x(t)|$. A Banach space E on $[0,1]$ is said to be rearrangement invariant (r.i.) if $y \in E$ and $x^* \le y^*$ implies that $x \in E$ and $\|x\|_E \le \|y\|_E$.

The embeddings $L_\infty \subset E \subset L_1$ are true for every r.i. space E. In fact, $1 \in E$. Without loss of generality (except in Section 7 and parts of Section 6), we may assume that

$$\|1\|_E = 1. \tag{4}$$

We write $x \prec y$ if

$$\int_0^\tau x^*(t)\,dt \le \int_0^\tau y^*(t)\,dt$$

for each $\tau \in [0,1]$. If E is separable or isometric to the conjugate of some separable r.i. space, that $x \prec y$ implies $\|x\|_E \le \|y\|_E$. Denote

$$E' = \left\{ x : x \in L_1,\ \int_0^1 xy\,dt < \infty \quad \forall y \in E \right\},$$

and equip it with the norm

$$\|x\|_{E'} = \sup_{\|y\|_E \le 1} \int_0^1 xy\,dt.$$

Then E' is an r.i. space. The embedding $E \subset E''$ is isometric. In fact $\|x\|_E = \|x\|_{E''}$ for all $x \in L_\infty$. The function $\left\|\chi_{(0,s)}\right\|_E$ is called the fundamental function of E.

Throughout this paper we will assume that all r.i. spaces are either maximal or minimal in the sense of Lindenstrauss and Tzafriri [**LT**] (or one may restrict to the case that the r.i. space is separable orisometric to the dual of a separable r.i. space). Thus for $x \in E$ with $x \ge 0$, we have that $\lim_{t\to\infty} \|\min\{x,t\}\|_E = \|x\|_E$.

Let $\varphi(t)$ be an increasing function from $[0,1]$ to $[0,1]$, with $\varphi(0) = 0$ and $\varphi(1) = 1$, and continuous on $(0,1]$. Let $1 \le r < \infty$. The Lorentz space $\Lambda_r(\varphi)$ consists of those functions on $[0,1]$ for which the functional

$$\|x\|_{\Lambda_r(\varphi)} = \left(\int_0^1 (x^*(t))^r \, d(\varphi(t))^r \right)^{1/r}$$

is finite.

Let us set Φ_r to be the collection of those φ satisfying the above conditions, and also that $\varphi(t)^r$ is concave. Then we see that if φ is in Φ_r, then $\Lambda_r(\varphi)$ satisfies the triangle inequality, and hence is an r.i. space.

Also, for $r > 1$, it is also known that this is equivalent to a norm satisfying the triangle inequality if and only if there exist $c > 0$ and $\epsilon > 0$ such that $\varphi(ts) \ge c^{-1} t^{1-\epsilon} \varphi(s)$ for $0 \le s,t \le 1$ (see [**Sa**]).

In the earlier sections, we shall primarily be interested in the case when $r = 1$, and so we will write $\Lambda(\varphi)$ for $\Lambda_1(\varphi)$, and Φ for Φ_1.

If $\varphi \in \Phi$ is continuous at 0, then $(\Lambda(\varphi))^*$ is equal to the Marcinkiewicz space, $M(\varphi)$, where $M(\varphi)$ is the space of functions on $[0,1]$ for which the functional

$$\|x\|_{M(\varphi)} = \sup_{0<s\le 1} \frac{\int_0^s x^*(t)\,dt}{\varphi(s)}$$

is finite. This space is also an r.i. space. The spaces $(\Lambda(\varphi))'$ and $M(\varphi)$ coincide for every $\varphi \in \Phi$.

If $\varphi \in \Phi$, then $\varphi(t)$ and $t/\varphi(t)$ increase on $[0,1]$. A function having these properties is said to be quasi-concave. If ψ is quasi-concave, then there exists $\varphi \in \Phi$ such that $\frac{1}{2}\varphi \leq \psi \leq \varphi$. Indeed, φ may be chosen as the concave majorant of ψ.

Let $M(t)$ be a concave even function on $(-\infty, \infty)$, $M(0) = 0$. Then we define another r.i. space, the Orlicz space L_M, to consist of all functions on $[0,1]$ for which the functional

$$\|x\|_{L_M} = \inf\left\{\lambda : \ \lambda > 0, \int_0^1 M\left(\frac{|x(t)|}{\lambda}\right) dt \leq 1\right\}$$

is finite.

We shall use Peetre's K-method in this article. Therefore we present the main definitions. Let (E_0, E_1) be a compatible pair of Banach spaces. The K-functional is defined for each $x \in E_0 + E_1$ and $t > 0$ by

$$K(t,x) = K(t,x,E_0,E_1) = \inf(\|x_0\|_{E_0} + t\,\|x_1\|_{E_1}),$$

where the infimum is taken over all representations $x = x_0 + x_1$ with $x_0 \in E_0$ and $x_1 \in E_1$. If $0 < \theta < 1$ and $1 \leq p \leq \infty$, then the space $(E_0, E_1)_{\theta,p}$ consists of all $x \in E_0 + E_1$ for which the functional

$$\|x\|_{\theta,p} = \left(\int_0^\infty (t^{-\theta} K(t,x))^p \, \frac{dt}{t}\right)^{1/p}$$

is finite. The space $(E_0, E_1)_{\theta,p}$ is an interpolation space with respect to E_0, E_1. All the above mentioned properties of r.i. spaces and the K-method can be found in [**BS**], [**BK**], [**KPS**], [**LT**].

2. Hardy-Littlewood semiordering

The semiordering $\prec$ can be applied to vectors. The definition is completely analogous. In this section, we shall establish some preliminary statements about this semiordering.

If $x, y \in \mathbb{R}^n$, $x, y \geq 0$,

$$X(t) = \sum_{k=1}^n x_k \chi_{((k-1)/n, k/n)}, \quad Y(t) = \sum_{k=1}^n y_k \chi_{((k-1)/n, k/n)},$$

then the correlations $x \prec y$ and $X \prec Y$ are equivalent.

In general, if $a, b, c \in \mathbb{R}^n$, $a, b, c \geq 0$, then it does not follow from $a \prec b$ that $a + c \prec b + c$.

Given $x \in \mathbb{R}^n$, $I \subseteq \{1, 2, \ldots, n\}$, denote

$$(x\chi_I)_k = \begin{cases} x_k & \text{if } k \in I \\ 0 & \text{if } k \notin I, \ 1 \leq k \leq n. \end{cases}$$

Lemma 1. *Let* $\{1, 2, \ldots, n\} = I_1 \cup I_2 \cup \cdots \cup I_s$, *where* $I_1, I_2, \ldots, I_s$ *are disjoint subsets. If* $x\chi_{I_k} \prec y\chi_{I_k}$ *for all* $k = 1, 2, \ldots, s$, *then* $x \prec y$.

The proof is obvious. This statement remains true if we consider $x\chi_{I_k}$ and $y\chi_{I_k}$ as elements of $\mathbb{R}^{|I_k|}$.

Let $u = (u_{i,j})_{1\le i,j,\le n}$, $u \ge 0$, $u_{11} > 0$, $u_{i,2} = 0$ for $i = 1, 2, \ldots, n$. Put

$$v_{ij} = \begin{cases} 0 & \text{if } i = j = 1 \\ u_{11} & \text{if } i = 1,\ j = 2 \\ u_{ij} & \text{otherwise.} \end{cases}$$

Given an $n \times n$ matrix x, consider the vector

$$(Tx) = \left(\sum_{k=1}^{n} x_{k,\pi(k)}, \ \ \pi \in S_n \right)$$

as an element of $\mathbb{R}^{n!}$.

Lemma 2. $Tu \prec Tv$.

Proof. Denote

$$\begin{aligned} S_0 &= \{\pi \in S_n : \pi(1) \ne 1,\ \pi(2) \ne 1\} \\ S(1,j) &= \{\pi \in S_n : \pi(1) = 1,\ \pi(2) = j\} \\ S(j,1) &= \{\pi \in S_n : \pi(1) = j,\ \pi(2) = 1\}, \end{aligned}$$

where $j = 2, 3, \ldots, n$. It is evident that

$$S_n = S_0 \cup \bigcup_{j=2}^{n} (S(1,j) \cup S(j,1)) \tag{5}$$

is a disjoint decomposition of S_n. Clearly

$$(Tu)\chi_{S_0} = (Tv)\chi_{S_0}. \tag{6}$$

Put

$$(\tilde{T}x)(\pi) = x_{1,\pi(1)} + x_{2,\pi(2)}.$$

If $\mu \in S(1,j)$ for some $j \in \{2, 3, \ldots, n\}$, then there exists a unique $\nu \in S(j,1)$ such that $\mu(k) = \nu(k)$ for $k = 3, 4, \ldots, n$. Therefore,

$$\begin{aligned} u_{k,\mu(k)} &= v_{k,\nu(k)} \\ u_{k,\nu(k)} &= v_{k,\mu(k)} \end{aligned} \tag{7}$$

for every $k = 3, \ldots, n$. For such μ, ν we have

$$((\tilde{T}u)(\mu), (\tilde{T}u)(\nu)) \prec ((\tilde{T}v)(\mu), (\tilde{T}v)(\nu)). \tag{8}$$

From (6) and (7) we get

$$((Tu)(\mu), (Tu)(\nu)) \prec ((Tv)(\mu), (Tv)(\nu)).$$

Given $j \in \{3, 4, \ldots, n\}$, there exist $\mu_i \in S(1,j)$ and $\nu_i \in S(j,1)$ such that

$$S(1,j) \cup S(j,1) = \bigcup_{i=1}^{(n-2)!} \{\mu_i, \nu_i\}$$

and
$$((Tu)(\mu_i),(Tu)(\nu_i)) \prec ((Tv)(\mu_i),(Tv)(\nu_i))$$
for each $i = 1, 2, \dots, (n-2)!$. By Lemma 1
$$(Tu)\chi_{S(1,j)} \prec (Tv)\chi_{S(j,1)} \tag{9}$$
for every $j = 2, 3, \dots, n$. Taking into account (5), (6), and (9), and applying Lemma 1 again, we get that $Tu \prec Tv$. □

Denote by Q_n the set of $(n \times n)$ matrices (x_{ij}) such that $x_{ij} = 0$ or 1 for every $1 \le i, j \le n$ and
$$|\{(i,j) : x_{ij} = 1\}| = n.$$
The identity matrix is denoted by I_n.

LEMMA 3. *If $x \in Q_n$, then $Tx \prec TI_n$.*

PROOF. Clearly Tx and TI_n are equidistributed for every permutation matrix x. If x is not a permutation matrix, then there exists a pair of columns or rows such that the first one contains two or more 1's, and the second one contains no 1's. Without loss of generality we may assume that $x_{11} = x_{21} = 1$, and that $x_{i,2} = 0$ for each $i = 1, 2, \dots, n$. Put
$$y_{ij} = \begin{cases} 0 & \text{if } i = j = 1 \\ 1 & \text{if } i = 1,\ j = 2 \\ x_{ij} & \text{otherwise.} \end{cases}$$
By Lemma 2, $Tx \prec Ty$. If y is a permutation matrix, then the lemma is proved. If y is not a permutation matrix, we can use this construction again. We obtain a permutation matrix after less than n iterations. □

LEMMA 4. *Let x be a $(n \times n)$ matrix such that*
$$\sum_{i,j=1}^{n} x_{ij} \le n, \quad 0 \le x_{ij} \le 1. \tag{10}$$
Then
$$Tx \prec TI_n. \tag{11}$$

PROOF. Denote by P_n the set of $(n \times n)$ matrices satisfying condition (10). It is evident that the set of extremal points of P_n coincides with Q_n. Given $j = 1, 2, \dots, n$, consider the functional
$$f_j(x) = \max \sum_{\pi \in R} \sum_{k=1}^{n} x_{k,\pi(k)},$$
where the maximum is taken over all subsets $R \subset S_n$ with $|R| = j$. The functional f_j is convex. Therefore
$$\max_{x \in P_n} f_j(x) = \max_{x \in \operatorname{ext} P_n} f_j(x).$$
If $x \in P_n$, we can find $y \in Q_n$ such that $f_j(x) \le f_j(y)$. By Lemma 3, $f_j(y) \le f_j(I_n)$, and consequently, $f_j(x) \le f_j(I_n)$. Thus (11) is proved. □

Let $x, f \in L_1$, and $x, f \geq 0$. It is well known ([**KPS**], II.2.2) that $x \prec f$ implies $x^\alpha \prec f^\alpha$ for $\alpha \geq 1$. This is not true in general if $0 < \alpha < 1$. However, under some additional assumptions on f the implication $x \prec f \Rightarrow x^\alpha \prec C f^\alpha$ is true. Denote by O_f the set of $x \geq 0$ such that $x \prec f$.

LEMMA 5. *Let $f \geq 0$, $f \in L_1$, $0 < \alpha < 1$, $C > 1$. Then*

$$x^\alpha \prec C f^\alpha \quad \forall x \in O_f \tag{12}$$

if and only if

$$\tau^{1-\alpha} \left(\int_0^\tau f(t)\, dt \right)^\alpha \leq C \int_0^\tau f(t)^\alpha \, dt \quad \forall \tau \in [0,1]. \tag{13}$$

PROOF. We may assume that $f = f^*$. Given $\tau \in (0,1]$, consider the function

$$x_\tau(t) = \chi_{(0,\tau)}(t) \frac{1}{\tau} \int_0^\tau f(s)\, ds.$$

Since $x_\tau \in O_f$, then (12) implies

$$\int_0^\tau x_\tau(t)^\alpha \, dt = \left(\frac{1}{\tau} \int_0^\tau f(s)\, ds \right)^\alpha \tau \leq C \int_0^\tau f(t)^\alpha \, dt.$$

It is equivalent to (13).

Let inequality (13) be valid. By Hölder's inequality

$$\int_0^\tau x^*(t)^\alpha \, dt \leq \left(\int_0^\tau x^*(t)\, dt \right)^\alpha \tau^{1-\alpha} \leq \left(\int_0^\tau f(t)\, dt \right)^\alpha \tau^{1-\alpha} \leq C \int_0^\tau f(t)^\alpha \, dt.$$

□

LEMMA 6. *The function $f(t) = T_1 I_n(t)$ satisfies* (12) *for $\alpha > 0$, where $C = 6$.*

PROOF. There exists a sequence $1 > \tau_1 \geq \tau_2 \geq \cdots \geq \tau_{n+1} = 0$ such that

$$f^*(t) = \sum_{j=1}^n j \chi_{[\tau_{j+1}, \tau_j]}(t).$$

The function f is closely connected with the classical coincidence problem. It is well known (see [**W**], 4.9, 10) that

$$s_j := \tau_j - \tau_{j+1} = \frac{1}{j!} \sum_{k=0}^{n-j} \frac{(-1)^k}{k!}.$$

Denote

$$q_j = \sum_{k=0}^{n-j} \frac{(-1)^k}{k!}.$$

Then $q_{n-1} = 0$, $q_n = 1$ and $\frac{1}{3} \leq q_j \leq \frac{1}{2}$ for $j = 1, 2, \ldots, n-2$. Hence

$$\tau_j = \sum_{i=j}^n s_i \leq 3 s_j \tag{14}$$

for $j \neq n-1$. Since

$$js_j = \frac{1}{(j-1)!} \sum_{k=0}^{n-j} \frac{(-1)^k}{k!},$$

we get

$$\sum_{i=j}^{n} is_i \leq 3js_j \tag{15}$$

for $j \neq n-1$. Using (14) and (15), we have

$$\tau^{1-\alpha} \left(\sum_{i=j}^{n} is_i \right)^{\alpha} \leq (3s_j)^{1-\alpha}(3js_j)^{\alpha} = 3j^{\alpha}s_j < 3 \sum_{i=j}^{n} i^{\alpha}s_i$$

for $1 \leq j \leq n$. Consequently, the inequality

$$\tau^{1-\alpha} \left(\int_0^{\tau} f^*(t)\, dt \right)^{\alpha} \leq 3 \int_0^{\tau} f^*(t)^{\alpha}\, dt$$

is proved for $\tau = \tau_j$, $1 \leq j \leq n$.

If $\tau \in [0, \tau_1]$, we can find $1 \leq j \leq n$ and $\lambda \in [0,1]$ such that

$$\tau = \begin{cases} \tau_{j+1} + \lambda s_j & \text{if } \ \tau > \tau_{n-2} \ \text{ or } \ \tau \leq \tau_n \\ \tau_n + \lambda s_{n-2} & \text{if } \ \tau_n < \tau \leq \tau_{n-2}. \end{cases}$$

It is sufficient to consider only the case $\tau > \tau_{n-2}$. By (14) and (15),

$$\begin{aligned} \left(\sum_{i=j+1}^{n} s_i + \lambda s_j \right)^{1-\alpha} & \left(\sum_{i=j+1}^{k} is_i + \lambda js_j \right)^{\alpha} \leq (3s_{j+1} + \lambda s_j)^{1-\alpha}(3(j+1)s_{j+1} + \lambda js_j)^{\alpha} \\ & \leq 3(j+1)^{\alpha}(s_{j+1} + \lambda s_j) \\ & \leq 6j^{\alpha}(s_{j+1} + \lambda s_j) \\ & \leq 6 \left(\sum_{i=j+1}^{n} i^{\alpha}s_i + \lambda j^{\alpha}s_j \right). \end{aligned}$$

The obtained inequality shows that

$$\tau^{1-\alpha} \left(\int_0^{\tau} f^*(t)\, dt \right)^{\alpha} \leq 6 \int_0^{\tau} f^*(t)^{\alpha}\, dt.$$

$\square$

3. Reduction to diagonal matrices

Given an integer n, denote by D_n the set of diagonal matrices. It is evident that if $x \in D_n$, then $x_k^* = 0$ for $n < k \leq n^2$.

THEOREM 7. *Let $C, q > 1$, and let E be an r.i. space. If*

$$\|T_q y\|_E \leq C \ \|Uy\|_E$$

for any $y \in D_n$, then

$$\|T_q x\|_E \leq 7C \left(\|Ux\|_E + \left(\frac{1}{n} \sum_{k=n+1}^{n^2} x_k^{*q} \right)^{1/q} \right)$$

for any $(n \times n)$ matrix x.

Proof. Let

$$\|Ux\|_E \leq 1 \tag{16}$$

$$\frac{1}{n} \sum_{k=n+1}^{n^2} x_k^{*q} \leq 1. \tag{17}$$

We can find matrices $y = (y_{ij})$ and $z = (z_{ij})$ such that $x = y + z$, $|\operatorname{supp} y| \leq n$, and $Ux = Uy$. Denote by w a diagonal matrix such that $Uw = Uy$. By Lemma 2,

$$T_1 y \prec T_1 w.$$

Hence,

$$\|T_1 y\|_E \leq \|T_1 w\|_E \leq C\ \|Uw\|_E = C\ \|Ux\|_E$$

and

$$\|T_q y\|_E \leq \|T_1 y\|_E \leq C\ \|Ux\|_E \leq C. \tag{18}$$

If $x_{n+1}^* > 1$, then

$$\min_{0 \leq t \leq 1} Ux(t) = x_n^* \geq x_{n+1}^* > 1.$$

By (4),

$$\|Ux\|_E > \|1\|_E = 1.$$

The obtained inequality contradicts (16). Consequently, for every $1 \leq i, j \leq n$

$$|z_{ij}| \leq x_{n+1}^* \leq 1.$$

Denoting $|x_{ij}|^q$ by v_{ij}, we get

$$0 \leq v_{ij} \leq 1, \qquad 1 \leq i, j, \leq n$$

$$\sum_{i,j=1}^{n} v_{ij} \leq n$$

$$\|T_q z\|_E = \left\| (T_1 v)^{1/q} \right\|_E .$$

By Lemma 4,

$$T_1 v \prec T_1 I_n.$$

Applying Lemmas 5 and 6, we have

$$(T_1 v)^{1/q} \prec 6 (T_1 I_n)^{1/q}.$$

We have mentioned in Section 1 that this inequality implies

$$\left\| (T_1 v)^{1/q} \right\|_E \leq 6 \left\| (T_1 I_n)^{1/q} \right\|_E .$$

Since

$$\left\| (T_1 I_n)^{1/q} \right\|_E = \|T_q I_n\|_E \leq C\ \|U I_n\|_E = C,$$

then

$$\|T_q z\|_E \le 6C.$$

Using the obtained inequality and (18), we get

$$\|T_q x\|_E \le \|T_q y\|_E + \|T_q z\|_E \le 7C.$$

$\square$

Theorem 7 and (1) lead to the following.

COROLLARY 8. *Let E be an r.i. space, and $1 \le q < \infty$. The equivalence*

$$\|T_q x\|_E \approx \|Ux\|_E + \left(\frac{1}{n} \sum_{k=n+1}^{n^2} x_k^{*q} \right)^{1/q}, \tag{19}$$

where the equivalence constants depend neither upon the matrix x nor on n, takes place if and only if the estimate

$$\|T_q y\|_E \le C \ \|Uy\|_E$$

is valid for every diagonal matrix y.

Denote by $\|T_q\|_E$ the least C in the last inequality, and by F_q the set of r.i. spaces satisfying condition (19). Given an r.i. space E, denote by $\omega(E)$ the set of $q \in [1, \infty]$ such that equivalence (19) takes place, and put

$$\tau(E) = \inf \omega(E).$$

The monotonicity of the function $q \mapsto \|T_q x\|_E$, (21), and Corollary 8 imply that $w(E) = [\tau(E), \infty]$ or $\omega(E) = (\tau(E), \infty]$. Some examples show that both of these possibilities may be realized.

4. Lorentz spaces

Given $0 < j \le k < n$, denote

$$I_{n,k} = \operatorname{diag}(\underbrace{1, 1, \ldots, 1}_{k}, \underbrace{0, 0, \ldots, 0}_{n-k})$$

and put

$$\mu_{n,k,j} = \operatorname{mes}\{t \in [0,1] : T_1 I_{n,k}(t) = j\}.$$

LEMMA 9. *If $0 < s < 1$, then*

$$\frac{s^j}{ej!} \le \sup \mu_{n,k,j} \le \frac{s^j}{j!}, \tag{20}$$

where the supremum is taken over (n, k) such that $k \le ns$.

Proof. We have

$$\begin{aligned}\mu_{n,k,j} &\le \frac{C_k^j(n-j)!}{n!} = \frac{k!(n-j)!}{j!(k-j)!n!}\\ &= \frac{k(k-1)\dots(k-j+1)}{j!n(n-1)\dots(n-j+1)} \le \frac{1}{j!}\left(\frac{k}{n}\right)^j \le \frac{s^j}{j!}.\end{aligned} \tag{21}$$

Following ([**W**], 4.9.B), denote $B_{n,k,j} = n!\mu_{n,k,j}$. It is known that

$$B_{n,k,j} = \frac{j+1}{k+1}B_{n+1,k+1,j+1}.$$

Therefore,

$$\mu_{n,k,j} = \frac{1}{n!}B_{n,k,j} = \frac{k}{n!j}B_{n-1,k-1,j-1} = \frac{k}{nj}\mu_{n-1,k-1,j-1}.$$

Hence

$$\begin{aligned}\mu_{n,k,j} &= \frac{k(k-1)\dots(k-j+1)}{j!n(n-1)\dots(n-j+1)}\mu_{n-j,k-j,0}\\ &= \frac{k(k-1)\dots(k-j+1)}{j!n(n-1)\dots(n-j+1)}\left(1-\frac{1}{n-j}\right)^{k-j}.\end{aligned}$$

The last part tends to $s^je^{-s}/j!$ if $k=[ns]$ and n tends to infinity. This proves the left part of (20). $\square$

The above proved statement allows us to solve completely the problem on the validity of equivalence (19) in the class of Lorentz spaces. Recall that we denote by Φ the set of increasing concave functions on $[0,1]$ with $\varphi(0)=0$ and $\varphi(1)=1$.

Theorem 10. *Let $\varphi \in \Phi$, $1 \le q < \infty$. The equivalence*

$$\|T_qx\|_{\Lambda(\varphi)} \approx \|Ux\|_{\Lambda(\varphi)} + \left(\frac{1}{n}\sum_{k=n+1}^{n^2} x_k{}^q\right)^{1/q} \tag{22}$$

takes place if and only if

$$\Gamma_{\varphi,q} := \sup_{0<t\le 1}\frac{1}{\varphi(t)}\sum_{j=1}^{\infty} j^{(1/q)-1}\varphi(t^j/j!) < \infty. \tag{23}$$

Moreover,

$$\|T_q\|_{\Lambda(\varphi)} \le \Gamma_{\varphi,q} \le qe\|T_q\|_{\Lambda(\varphi)}.$$

Proof. We use the notation of Lemma 9. If (22) is fulfilled, then there exists a constant $C>0$ such that

$$\|T_qI_{n,k}\|_{\Lambda(\varphi)} \le C\|UI_{n,k}\|_{\Lambda(\varphi)} = C\varphi(k/n).$$

Since

$$\|T_q I_{n,k}\|_{\Lambda(\varphi)} = \left\| \sum_{j=1}^{k} (j^{1/q} - (j-1)^{1/q}) \chi_{(0,\mu_{n,k,j})} \right\|_{\Lambda(\varphi)}$$

$$= \sum_{j=1}^{k} (j^{1/q} - (j-1)^{1/q}) \varphi(\mu_{n,k,j}), \tag{24}$$

then it follows by Lemma 9 that for each $t \in (0,1)$ and for each integer m that we have

$$\sum_{j=1}^{m} (j^{1/q} - (j-1)^{1/q}) \varphi(t^j/ej!) \le C\varphi(t).$$

Hence,

$$\sum_{j=1}^{\infty} j^{(1/q)-1} \varphi(t^j/j!) \le Cqe\varphi(t).$$

This proves the first part of the theorem.

Now suppose that, for every $t \in (0,1]$,

$$\sum_{j=1}^{\infty} t^{(1/q)-1} \varphi(t^j/j!) \le C\varphi(t).$$

Applying the obvious inequality

$$j^{1/q} - (j-1)^{1/q} \le j^{(1/q)-1}$$

and (24), we get

$$\begin{aligned} \|T_q I_{n,k}\|_{\Lambda(\varphi)} &= \sum_{j=1}^{k} (j^{1/q} - (j-1)^{1/q} \varphi(\mu_{n,k,j}) \\ &\le \sum_{j=1}^{k} j^{(1/q)-1} \varphi(1/j!(k/n)^j) \\ &\le C\varphi(k/n) \\ &= C\|UI_{n,k}\|_{\Lambda(\varphi)}. \end{aligned} \tag{25}$$

All Lorentz spaces $\Lambda(\varphi)$ have the following property ([**KPS**], II.5.2). If a convex functional is uniformly bounded on the set of characteristic functions, then it is uniformly bounded on the set of step functions. We apply this property to matrices. Then (25) implies that

$$\|T_q y\|_{\Lambda(\varphi)} \le C\|Uy\|_{\Lambda(\varphi)}$$

for each $y \in D$. By Corollary 8, (22) is valid. □

In other words, $\Lambda(\varphi) \in F_q$ if and only if $\Gamma_{\varphi,q} < \infty$. We mention that

$$\|Ux\|_{\Lambda(\varphi)} = \sum_{k=1}^{n} x_k(\varphi(k/n) - \varphi((k-1)/n)).$$

Let us study condition (23) in detail.

LEMMA 11. *Let $0 < \alpha \le 1$, $a, q \ge 1$, and $\varphi \in \Phi$ with $\varphi(t) \le at^\alpha$ for every $t \in [0,1]$. Then $\Lambda(\varphi) \in F_q$ and $\Gamma_{\varphi,q} \le 5a/\alpha$.*

PROOF. Since $\Gamma_{\varphi,q} \le \Gamma_{\varphi,1}$, we shall estimate only $\Gamma_{\varphi,1}$. Given $s \in (0, a^{-1/\alpha})$, we construct the function

$$\varphi_s(t) = \begin{cases} at^\alpha & \text{if } \ 0 \le t \le s, \\ as^\alpha & \text{if } \ s \le t \le as^\alpha, \\ t & \text{if } \ as^\alpha \le t \le 1. \end{cases}$$

The set of the quasi-concave functions φ_s possesses the following property. If $\varphi \in \Phi$ and $t_1 \in (0,1)$, then we can find $s \in (0, a^{-1/\alpha})$ such that $\varphi_s(t) \le \varphi(t)$ for $t \in [t_1, 1]$, and $\varphi_s(t) \ge \varphi(t)$ for $t \in [0, t_1]$. Therefore it is sufficient to obtain the needed estimate only for the function φ_s and $t = as^\alpha$. Put

$$N = \max\left\{ j : \ \frac{(as^\alpha)^j}{j!} \ge s \right\} = \max\{ j : \ (j!)^{1/j} \le as^{\alpha - 1/j} \}.$$

Then

$$\sum_{j=1}^{\infty} \varphi_s\left(\frac{(as^\alpha)^j}{j!}\right) = Nas^\alpha + a \sum_{j=N+1}^{\infty} \left(\frac{(as^\alpha)^j}{j!}\right)^\alpha \le as^\alpha \left(N + a \sum_{j=1}^{\infty} (j!)^{-\alpha} \right).$$

Since $(j/3)^j \le j!$, it follows that

$$N \le \max\{ j : \ j \le 3as^{\alpha - 1/j} \} \le \max(3a, 1/\alpha).$$

Hence,

$$\Gamma_{\varphi,1} \le \max(3a, 1/\alpha) + a \sum_{j=1}^{\infty} (j!)^{-\alpha} \le 3a/\alpha + a \sum_{j=1}^{\infty} 2^{-j\alpha} \le 5a/\alpha.$$

□

The assumption $\varphi(1) = 1$ is essential in Lemma 11. Indeed, if we let $\psi_\epsilon(t) = \min(\sqrt{t}, \epsilon)$, then $\Gamma_{\psi_\epsilon,1}$ tends to ∞ when ϵ tends to 0.

5. Interpolation Spaces

Theorem 10 may be extended on a wider class of r.i. spaces. Given numbers $\alpha \in (0,1]$, $a \ge 1$, denote by $\Phi(a, \alpha)$ the set of functions $\varphi \in \Phi$ such that $\varphi(t) \le at^\alpha$ for each $t \in [0,1]$. Let E be an r.i. space. Given $y \in E'$, $\|y\|_{E'} = 1$, we put

$$\psi_y(t) = \frac{\int_0^t y^*(s)ds + t}{\|y\|_{L_1} + 1}$$

and

$$\|x\|_1 = \sup_{\|y\|_{E'}=1} \|x\|_{\Lambda(\psi_y)}.$$

We mention that $\psi_y \in \Phi$.

LEMMA 12.
i) *The norms* $\|\cdot\|_{E''}$ *and* $\|\cdot\|_1$ *are equivalent and*

$$\frac{1}{2}\|x\|_E \le \|x\|_1 \le 2\|x\|_E \qquad \textit{for all } x \in L_\infty. \tag{26}$$

ii) *If* $L_p \subset E$ *for some* $p < \infty$ *and*

$$\|x\|_E \le a\|x\|_{L_p} \qquad \textit{for all } x \in L_p, \tag{27}$$

then $\psi_y \in \Phi(a+1, 1/p)$ *for every* $y \in E'$ *with* $\|y\|_{E'} = 1$.

PROOF. i) By the Hardy-Littlewood theorem on rearrangements (see [**KPS**], II.2.2.17) it follows that for $x \in E''$ we have

$$\|x\|_{E''} = \sup_{\|y\|_{E'}=1} \int_0^1 x(t)y(t)dt = \sup_{\|y\|_{E'}=1} \int_0^1 x^*(t)y^*(t)dt = \sup_{\|y\|_{E'}=1} \|x\|_{\Lambda(\varphi_y)},$$

where $\varphi_y(t) = \int_0^t y^*(s)ds$. Since

$$\|y\|_{L_1} + 1 \le \|y\|_{E'} + 1 = 2,$$

then $\psi_y \ge \frac{1}{2}\varphi_y$ and

$$\|x\|_1 \ge \frac{1}{2}\|x\|_{E''} \qquad \text{for all } x \in E''. \tag{28}$$

On the other hand,

$$\|x\|_1 \le \|x\|_{E''} + \|x\|_{L_1} \le 2\|x\|_{E''} \qquad \text{for all } x \in E''. \tag{29}$$

The norms $\|\cdot\|_E$ and $\|x\|_{E''}$ coincide on L_∞ ([**BS**], 1.2.7). Therefore (28) and (29) imply (26).

ii) If $\frac{1}{p} + \frac{1}{p'} = 1$, then $E' \subset L_{p'}$ and

$$\|y\|_{L_{p'}} \le a\|y\|_{E'} = a.$$

By Hölder's inequality,

$$\psi_y(t) \le \int_0^t y^*(s)ds + t \le \|y\|_{E'}t^{1/p} + t \le at^{1/p} + t \le (a+1)t^{1/p}.$$

□

THEOREM 13. *Let* E *be an r.i. space,* $E \supset L_p$ *for some* $p < \infty$*, and* $1 \le q < \infty$*. Then (19) is fulfilled, that is,* $E \in F_q$.

PROOF. There is a constant $a > 1$ such that (27) is valid. By Lemma 12 (ii), $\psi_y \in \Phi(a+1, 1/p)$ for every $y \in E'$ with $\|y\|_{E'} = 1$. By Lemma 11, we know that

$$\Gamma_{\psi_y,q} \le 5(a+1)p.$$

Applying the second part of Theorem 10, we get that

$$\|T_q x\|_{\Lambda(\psi_y)} \le 5(a+1)p\|Ux\|_{\Lambda(\psi_y)}$$

for each $x \in D$, and $y \in E'$ with $\|y\|_{E'} = 1$. Hence,

$$\|T_q x\|_1 \le 5(a+1)p\|Ux\|_1.$$

This and (26) imply that

$$\|T_q x\|_E \le 20(a+1)p\|Ux\|_E.$$

By Corollary 8, it follows that (19) is fulfilled. $\square$

We mention that the conditions
1) $E \supset L_p$ for some $p < \infty$;
2) $t^{-\alpha} \in E$ for some $\alpha > 0$
are equivalent.

Let E_1, E_2 be r.i. spaces, $1 \le q < \infty$, $E_1 \supset E_2$ and $E_2 \in F_q$. Does it follow that $E_1 \in F_q$? Theorem 13 shows that the answer to this question is positive if $E_2 \supset L_p$ for some $p < \infty$. In general, the answer is negative. We now show this.

Theorem 14. *Let $\varphi \in \Phi$ and $1 \le q < \infty$. The following conditions are equivalent:*
i) *if $\psi \in \Phi$ and $\psi \le \varphi$, then $\Gamma_{\psi,q} < \infty$;*
ii) *there exists a constant $C > 0$ such that if $\psi \in \Phi$ and $\psi \le \varphi$, then $\Gamma_{\psi,q} \le C$.*
iii) *there exist numbers $\alpha \in (0,1]$ and $a \ge 1$ such that $\varphi(t) \le at^\alpha$ for every $t \in [0,1]$.*

Proof. The implication (iii) $\Rightarrow$ (ii) was proved in Lemma 11. The implication (ii) $\Rightarrow$ (i) is trivial. Therefore, we need prove only the implication (i) $\Rightarrow$ (iii). Let

$$\varphi \notin \bigcup_{0<\alpha\le 1\le a} \Phi(a,\alpha),$$

so that

$$\sup_{0<t\le 1} \varphi(t)t^{-1/n} = \infty \qquad n = 1,2,\dots \tag{30}$$

Using (30) we can find a sequence $t_n \downarrow 0$ such that $t_1 = 1$ and

$$\varphi(t_n) \ge \frac{n\varphi(t_{n-1})}{t_{n-1}} t_n^{1/n}$$

for every $n = 2, 3 \dots$. Then

$$\left(\frac{t_{n-1}}{\varphi(t_{n-1})}\varphi(t_n)\right)^n \ge n^n t_n > n! t_n. \tag{31}$$

If we put

$$s_n = \frac{t_{n-1}}{\varphi(t_{n-1})}\varphi(t_n),$$

then $t_n < s_n < t_{n-1}$ for every $n = 2, 3, \dots$. Define

$$\psi(t) = \begin{cases} \varphi(t_n) & \text{if } \ t_n \le t \le s_n, \\ t\varphi(t_{n-1})/t_{n-1} & \text{if } \ s_n \le t \le t_{n-1}, \ n = 2,3,\dots, \\ 0 & \text{if } \ t = 0. \end{cases}$$

The function $\psi(t)$ is quasi-concave on $[0,1]$, and $\psi \le \varphi$. By (31) we see that

$$\psi(s_n^n/n!) = \psi(t_n) = \varphi(t_n) = \psi(s_n).$$

For every integer n we get

$$\Gamma_{\psi,q} \geq \frac{1}{\psi(s_n)} \sum_{k=1}^{n} k^{(1/q)-1} \psi(s_n^k/k!) = \sum_{k=1}^{n} k^{(1/q)-1} > \sum_{k=1}^{n} k^{-1}.$$

Hence, $\Gamma_{\psi,q} = \infty$. Denote by $\nu(t)$ the concave majorant of $\psi(t)$. Then $\nu \in \Phi$, $\nu \leq \varphi$ and $\Gamma_{\nu,q} = \infty$. □

Theorem 13 is practically an interpolation theorem. It shows that if E is an r.i. space, $E = E''$ and $E \supset L_p$ for some $p < \infty$, then E is an interpolation space with respect to the set $\{\Lambda(\varphi),\ \Gamma_{\varphi,1} < \infty\}$. One can prove that the assumption $E = E''$ may be replaced with the separability of E. Using the K-method, we can obtain another sufficient condition for $E \in F_q$.

Theorem 15. *Let $\varphi_0, \varphi_1 \in \Phi$ and $0 < \gamma, \theta < 1$, and suppose that the function $\varphi_0^\gamma(t)/\varphi_1(t)$ increases on $(0,1]$. Then*

$$(\Lambda(\varphi_0), \Lambda(\varphi_1))_{\theta,\infty} \approx M(\tilde{\varphi}_\theta), \tag{32}$$

where

$$\tilde{\varphi}_\theta(t) = \frac{t}{\varphi_\theta(t)} = \frac{t}{\varphi_0^{1-\theta}(t)\varphi_1^\theta(t)}.$$

Theorem 15 is a special case of a more general result which is contained in [**S3**]. Using some results on the stability of the interpolation functions [**A**], one can obtain a similar statement.

Lemma 16. *Let E_0, $E_1 \in F_q$ be r.i. spaces, where $q \in [1,\infty)$, and suppose that E_2 is an interpolation space with respect to E_0, E_1. Then $E_2 \in F_q$.*

Proof. Given an integer n, we consider the operator

$$B_n x = \operatorname{diag}\left(n \int_{(k-1)/n}^{k/n} x(s)\, ds,\ 1 \leq k \leq n\right).$$

The operator B_n acts from L_1 into the set of diagonal matrices. The operator UB_n is an averaging operator, and $UB_n x$ is the conditional expectation of x with respect to the set of intervals $\{(\frac{k-1}{n}, \frac{k}{n}),\ 1 \leq k \leq n\}$. By Theorem 2.a.4 [**LT**], it follows that

$$\|UB_n\|_E = 1.$$

Corollary 8 shows that $E \in F_q$ if and only if

$$\sup_n \|T_q B_n\|_E < \infty.$$

This proves the Lemma. □

Lemma 17. *Suppose that $1 \leq q < \infty$, that $\varphi \in \Phi$ satisfies condition (23), that $1 < \mu < \lambda$, and that $\varphi^\lambda \in \Phi$. Then $M(\tilde{\varphi}^\mu) \in F_q$.*

PROOF. Applying (23) and Jensen's inequality, we get that

$$\sum_{j=1}^{\infty} j^{(1/q)-1}\varphi^{\lambda}(t^j/j!) \le \left(\sum_{j=1}^{\infty} j^{(1/q)-1}\varphi(t^j/j!)\right)^{\lambda} \le C^{\lambda}\varphi^{\lambda}(t)$$

for every $t \in (0,1]$. It means that φ^{λ} satisfies condition (23) with the constant C^{λ}. By Theorem 15, $M(\tilde{\varphi}^{\mu})$ is an interpolation space with respect to $\Lambda(\varphi)$ and $\Lambda(\varphi^{\lambda})$. The required statements now follow from Lemma 16. $\square$

Consider the following example. Given $p > 0$, we put

$$\varphi_p(t) = \left(\frac{\log(1+1/t)}{\log 2}\right)^{-1/p}. \tag{33}$$

If $p \ge 1$, then $\varphi_p \in \Phi$. If $p \in (0,1)$, then $\varphi_p(t)$ is concave in a sufficiently small neighborhood of the origin. Consequently, φ_p is concave up to equivalence.

LEMMA 18. *Let $p > 0$ and $1 \le q < \infty$. Then $\Gamma_{\varphi_p,q} < \infty$ if $p < q$. If $\varphi \in \Phi$ and $\varphi \ge C\varphi_q$ for some $C > 0$, then $\Gamma_{\varphi,q} = \infty$.*

PROOF. Let $p < q$ and $0 < t \le 1$. We have that

$$\begin{aligned}\frac{1}{\varphi_p(t)}\sum_{j=1}^{\infty} j^{(1/q)-1}\varphi_p(t^j/j!) &\le 1 + \sum_{j=2}^{\infty} j^{(1/q)-1}\left(\frac{\log(1+1/t)}{\log(1+j!/t^j)}\right)^{1/p}\\ &\le 1 + \sum_{j=2}^{\infty} j^{(1/q)-1}\left(\frac{1}{j-1}\right)^{1/p} < \infty.\end{aligned}$$

Therefore $\Gamma_{\varphi_p,q} < \infty$. If $\varphi \ge C\varphi_q$ and $0 < t \le 1$, then

$$\begin{aligned}\sum_{j=2}^{\infty} j^{(1/q)-1}\varphi(t^j/j!) &\ge C(\log 2)^{-1/q}\sum_{j=2}^{\infty} j^{(1/q)-1}\log^{-1/q}(1+j!/t^j)\\ &\ge C\sum_{j=2}^{\infty} j^{(1/q)-1}(j\,\log(j/t))^{-1/q}\\ &= C\sum_{j=2}^{\infty} j^{-1}(\log(j/t))^{-1/q} = \infty.\end{aligned}$$

$\square$

Let us consider the Orlicz space $\exp L_p$. It is generated by the function

$$M_p(u) = e^{|u|^p} - 1.$$

If $p \ge 1$, then $M_p(u)$ is convex and the fundamental function of $\exp L_p$ is equal to $(\log(1+1/t))^{-1/p}$. If $0 < p < 1$, then $M_p(u)$ is convex for sufficiently large u.

THEOREM 19. *Let $1 \le q < \infty$. If $p < q$, then $\exp L_p \in F_q$. If $p > q$, then $\exp L_p \notin F_q$.*

PROOF. Let $p \geq 1$. By [**Lo**], the spaces $\exp L_p$ and $M(\tilde{\varphi}_p)$ coincide, where φ_p is defined by (33). Applying Lemma 18, we get that $\exp L_p \in F_q$ if $p < q$. The first part of the theorem is proved.

If E is an r.i. space and $E \in F_q$, then

$$\sup_n \|T_q I_n\|_E < \infty.$$

In fact, let E be an Orlicz space L_M. Lemma 9 shows that

$$\sum_{j=1}^{\infty} \frac{e^{\epsilon^p j^{p/q}} - 1}{j!} < \infty$$

for some $\epsilon > 0$. This series diverges for any $p > q$ and $\epsilon > 0$. Hence, $p \leq q$.

So, the theorem has been proved for $p \geq 1$. If $0 < p < 1$, we can change $M_p(u)$ for a convex equivalent function. Therefore the theorem is valid for every $p > 0$. □

6. D and D^*-convex Spaces

The notion of D-convexity was introduced by Kalton [**K**] (Section 5). Indeed, much of the proof of this section is inspired by his proof of Lemma 5.5.

Given a function x on $[0,1]$, we will define its distribution function $d_x(t) = \text{mes}(\{|f| > t\})$. Thus the decreasing rearrangement $x^*(t)$ is essentially the inverse function of d_x. Given functions $x_1, x_2, \dots, x_n$ on $[0,1]$, we define their dilated disjoint sum to be the function on $[0,1]$:

$$C(x_1, \dots, x_n)(t) = |x_k(nt - k + 1)| \quad ((k-1)/n < t < k/n).$$

Thus

$$d_{C(x_1,\dots,x_n)}(t) = \frac{1}{n} \sum_{k=1}^{n} d_{x_k}(t).$$

We will say that an r.i. space E is D-convex if there is a constant $c > 0$ such that

$$\|C(x_1, \dots, x_n)\|_E \leq c \sup_{1 \leq k \leq n} \|x_k\|_E \,,$$

and that E is D^*-convex if there is a constant $c > 0$ such that

$$\|C(x_1, \dots, x_n)\|_E \geq c^{-1} \inf_{1 \leq k \leq n} \|x_k\|_E \,.$$

There is another way to define these notions. Let us consider the vector space V of right continuous functions from $[0,\infty)$ to $\mathbb{R}$ of bounded variation. Define the subsets

$$B_c^{\leq} = \{d_x : \|x\|_E \leq c\},$$
$$B_c^{\geq} = \{d_x : \|x\|_E \geq c\},$$
$$B_c^{=} = \{d_x : \|x\|_E = c\}.$$

Then E is D-convex if and only if there exists a constant $c > 0$ such that $\operatorname{conv} B_1^{\leq}$ is contained in $B_c^{\leq}$, and E is D^*-convex if and only if there exists a constant $c > 0$ such that $\operatorname{conv} B_1^{\geq}$ is contained in $B_c^{\geq}$.

Note that Lorentz spaces as defined in Section 1 are all D^*-convex, Marcinkiewicz spaces are all D-convex, and Orlicz spaces are both D and D^*-convex. An easy argument shows that if E is D-convex, then E' is D^*-convex, and it follows from Corollary 24 below that if E is D^*-convex, then E' is D-convex.

Suppose that $M : [0,\infty) \to [0,\infty)$ is increasing. We will say that M is p-convex if $M(t^{1/p})$ is convex, and we will say that M is q-concave if $-M(t^{1/q})$ is convex. By convention, we will say that M is always ∞-concave. We have the following result.

LEMMA 20. *Suppose that $M : [0,\infty) \to [0,\infty)$ is such that there exist $1 \le p < q \le \infty$ and a constant $c > 0$ such that for all $0 < s < 1$ that*

$$c^{-1} s^q M(t) \le M(st) \le c s^p M(t),$$

(where we shall suppose that the first inequality is missing if $q = \infty$). Then there exists an increasing, p-convex, q-concave function M_1 such that there exists a constant $c_1 > 0$ with $c_1^{-1} M \le M_1 \le c_1 M$.

PROOF. Let $M_2(t) = \sup_{s<1} M(st)/s^p$, and let $M_3(t) = \inf_{s<1} M(st)/s^q$ ($M_3 = M_2$ if $q = \infty$). From now on, if $q = \infty$, we shall suppose that any inequality involving q is automatically true. Then $c^{-1} M \le M_3 \le cM$, and

$$s^q M_3(t) \le M_3(st) \le s^p M_3(t),$$

that is, $M_3(t^{1/p})/t$ is an increasing function, and $M_3(t^{1/q})/t$ is a decreasing function. Now set

$$\begin{aligned} M_1(t) &= \int_0^t \frac{M_3(s)}{s}\, ds \\ &= \int_0^{t^p} \frac{M_3(s^{1/p})}{ps}\, ds \\ &= \int_0^{t^q} \frac{M_3(s^{1/q})}{qs}\, ds, \end{aligned}$$

where the last equality holds only if $q < \infty$. Then M_1 is p-convex and q-concave. Further, $M_1 \le M_3/p$, and

$$M_1(t) \ge \int_{t^p/2}^{t^p} \frac{M_3(s^{1/p})}{ps}\, ds \ge 1/(2p) M_3(t/2^{1/p}) \ge 1/(4p) M_3(t).$$

□

If $1 \le p < q \le \infty$, we say that E is an interpolation space for (L_p, L_q) if there is a constant $c > 0$ such that whenever $T : L_p \cap L_q \to L_p \cap L_q$ is a linear operator, such that $\|T\|_{L_p \to L_p} \le 1$ and $\|T\|_{L_q \to L_q} \le 1$, then $\|T\|_{E \to E} \le c$.

The following result is an immediate consequence of results in [**HM**] and Lemma 20 (see also [**AC**]).

THEOREM C. *Suppose that E is an interpolation space for (L_p, L_q). Then there is a constant $c > 0$ such that whenever $\|x\|_{L_M} \le \|y\|_{L_M}$ for all increasing p-convex and q-concave functions $M : [0,\infty) \to [0,\infty)$ and if $y \in E$, then $x \in E$ and $\|x\|_E \le c\, \|y\|_E$.*

Now let us state the main results of this section.

THEOREM 21. *Suppose that E is a D-convex interpolation space for (L_p, L_q), where $1 \le p < q \le \infty$. Then there exists a constant $c > 0$ such that for every $x \in E$ with $\|x\|_E = 1$, there exists an increasing, p-convex, q-concave function $M : [0,\infty) \to [0,\infty)$ such that $\int M(|x|)\,ds \ge c^{-1}$, and $\int M(|y|)\,ds \le c$ whenever $\|y\| \le c^{-1}$.*

Thus there exists a family of increasing, p-convex, q-concave functions $M_\alpha : [0,\infty) \to [0,\infty)$ $(\alpha \in A)$ such that $\|\cdot\|_E$ is equivalent to $\sup_{\alpha\in A} \|\cdot\|_{L_{M_\alpha}}$.

THEOREM 22. *Suppose that E is a D^*-convex interpolation space for (L_p, L_q), where $1 \le p < q \le \infty$. Then there exists a constant $c > 0$ such that for every $x \in E$ with $\|x\|_E = 1$, there exists an increasing, p-convex, q-concave function $M : [0,\infty) \to [0,\infty)$ such that $\int M(|x|)\,ds \le c$, and $\int M(|y|)\,ds \ge c^{-1}$ whenever $\|y\| \ge c$.*

Thus there exists a family of increasing, p-convex, q-concave functions $M_\alpha : [0,\infty) \to [0,\infty)$ $(\alpha \in A)$ such that $\|\cdot\|_E$ is equivalent to $\inf_{\alpha\in A} \|\cdot\|_{L_{M_\alpha}}$.

THEOREM 23. *Suppose that E is D-convex and D^*-convex. Then there exists an increasing function $M : [0,\infty) \to [0,\infty)$ such that E is equivalent to L_M.*

COROLLARY 24. *Suppose that E is an interpolation space for (L_p, L_q), where $1 \le p < q \le \infty$. If E is D-convex, then E is p-convex, and if $q < \infty$ then there is a constant $c > 0$ such that given functions x_1, $x_2, \dots, x_n$ on $[0,1]$*

$$\|C(x_1,\dots,x_n)\|_E \le c \left(\frac{1}{n} \sum_{k=1}^n \|x_k\|_E^q \right)^{1/q} .$$

If E is D^*-convex, then E is q-concave, and there is a constant $c > 0$ such that given functions x_1, $x_2, \dots, x_n$ on $[0,1]$

$$\|C(x_1,\dots,x_n)\|_E \ge c^{-1} \left(\frac{1}{n} \sum_{k=1}^n \|x_k\|_E^p \right)^{1/p} .$$

PROOF. Let us provide the proof of the stated inequality in the case that E is D-convex. The other results have almost identical proofs.

From Theorem 21, we see that it is sufficient to show that if $M : [0,\infty) \to [0,\infty)$ is increasing, convex, and q-concave (with $q < \infty$), then given functions x_1, $x_2, \dots, x_n$ on $[0,1]$, we have that

$$\|C(x_1,\dots,x_n)\|_{L_M} \le \left(\frac{1}{n} \sum_{k=1}^n \|x_k\|_{L_M}^q \right)^{1/q} .$$

Let us suppose that the left hand side is bounded below by 1. Thus

$$\int M(|C(x_1,\dots,x_n)|)\,ds = \frac{1}{n} \sum_{k=1}^n \int M(|x_k|)\,ds \ge 1.$$

Thus there exists a sequence $c_k \ge 0$ with $\sum_{k=1}^n c_k = n$ such that

$$\int M(|x_k|)\,ds \ge c_k.$$

Since M is q-concave, it follows that

$$\int M(|x_k|/c_k^{1/q})\,ds \geq 1,$$

that is, $\|x_k\|_{L_M} \geq c_k^{1/q}$. The result follows. $\square$

Let us now proceed with the proofs of the main theorems.

LEMMA 25. *If $E \neq L_\infty$, then for each $\epsilon > 0$, there exists a strictly increasing function $N : [0,\infty) \to [0,\infty)$ with $N(0) = 0$, such that if $\int_0^1 N(x^*(t))\,dt \leq 1$, then $\|x\|_E < \epsilon$.*

PROOF. Let $k \in E \setminus L_\infty$ such that $k^*(1) \leq 1$, $\|k\|_E < \epsilon$, and k^* is strictly decreasing. Define

$$N(t) = \begin{cases} 1/(k^*)^{-1}(t) & \text{if } t \geq 1 \\ t/(k^*)^{-1}(1) & \text{if } t \leq 1. \end{cases}$$

Now suppose that $\|f\|_{L_N} \leq 1$. Then

$$\int_0^1 N(f^*(t))\,dt \leq 1,$$

which implies that $tN(f^*(t)) \leq 1$, that is $f^*(t) \leq k^*(t)$. Therefore $\|f\|_E \leq \|k\|_E < \epsilon$. $\square$

For any $L > 0$, we will write

$$V_L = \{f \in V : f(t) = 0 \text{ for } t > L\}.$$

Note that $d_x \in V_L$ if and only if $\|x\|_\infty \leq L$. Notice also that V_L has a predual, $C([0,L])$, defined by the pairing

$$\langle f, N\rangle = -\int_{[0,L]} N(s)df(s).$$

Notice that if $\|x\|_\infty \leq L$, then

$$\langle d_x, N\rangle = \int_0^1 N(|x(s)|)\,ds.$$

LEMMA 26. *Suppose that $E \neq L_\infty$, and that $L > 0$. Then the set $B_c^= \cap V_L$ is weak* compact in V_L.*

PROOF. It is clear that $B_c^=$ is a bounded set in V, and hence it is sufficient to show that $B_c^\leq \cap V_L$ is weak* closed in V_L. Suppose that $\|x_n\|_E = c$, that $d_{x_n} \in V_L$, and that $d_{x_n} \to g$ weak*. Then it is easy to see that g is decreasing with $g(0) \leq 1$, and that $d_{x_n} \to g$ pointwise except possibly at discontinuities of g. Therefore $g = d_y$ for some $y \in V_L$, and $x_n^* \to y^*$ pointwise except possibly at points of discontinuity of y^*, of which there are only countably many. Hence by Lebesgue's Dominated Convergence Theorem, it follows that for any continuous function N that $\int_0^1 N(x_n^* - y^*)\,dt \to 0$, and hence by Lemma 1, it follows that $\|x_n^* - y^*\|_E \to 0$. $\square$

Now, if $1 \le p < q \le \infty$, and $L > 0$, we define the subset $C_{p,q,L}$ of V_L to be the set of all those $f \in V_L$ such that

$$-\int_{[0,L]} N\,df \ge 0$$

for all increasing $N : [0,\infty) \to [0,\infty)$ that are p-convex and q-concave. Notice that $C_{p,q,L}$ is weak* closed in V_L for all $L > 0$.

PROOF OF THEOREM 21. We may suppose that $E \ne L_\infty$. Since we have that $\|\min\{|x|\,,t\}\|_E \to \|x\|_E$ as $t \to \infty$, we may suppose without loss of generality that $x \in L_\infty$. By a further slight approximation, we may suppose that x^* is strictly decreasing and $x^*(1) = 0$, that is, we may suppose that d_x is absolutely continuous.

By the definition of D-convexity, and Theorem C, we know that there exists a constant $c > 0$ such that for all $L > 0$ we have that $\operatorname{conv}(B^{=}_{c^{-1}})$ does not intersect with $\{d_x\} + C_{p,q,L}$. Hence, by the Hahn-Banach Theorem, for each $L > \|x\|_\infty$, there exists $M_L \in C_0([0,L])$ such that for some constant $S = \pm 1$

$$\int_0^1 M_L(|y|)\,ds = -\int_0^L M_L\,d(d_y) \le S \qquad \text{for} \quad d_y \in B^{\le}_{c^{-1}} \cap V_L,$$

and

$$-\int_0^L M_L\,d(d_{cx} + f) \ge S \qquad \text{for} \quad f \in C_{p,q,L}.$$

Hence

$$\int_0^1 M_L(|x|)\,ds = -\int_0^L M_L d(d_x) \ge S.$$

Further, since $C_{p,q,L}$ is a cone, it follows that

$$-\int_0^L M_L\,df \ge 0 \qquad \text{for} \quad f \in C_{p,q,L}.$$

For $0 \le a < b \le L$, consider the functions

$$f_1(s) = \begin{cases} 1 & \text{if } \ a \le s \le b \\ 0 & \text{otherwise,} \end{cases}$$

$$f_2(s) = \begin{cases} -1 & \text{if } \ a^{1/p} \le s \le ((a+b)/2)^{1/p} \\ 1 & \text{if } \ ((a+b)/2)^{1/p} \le s \le b^{1/p} \\ 0 & \text{otherwise,} \end{cases}$$

$$f_3(s) = \begin{cases} 1 & \text{if } \ a^{1/q} \le s \le ((a+b)/2)^{1/q} \\ -1 & \text{if } \ ((a+b)/2)^{1/q} \le s \le b^{1/q} \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that f_1, f_2 and f_3 are in $C_{p,q,L}$, and hence it may be seen that M_L is positive, increasing, p-convex, and q-concave on $[0,L]$. Hence $S = 1$.

By Lemma 25, there exists $N : [0,\infty) \to [0,\infty)$ with $N(0) = 0$ and that is strictly increasing, and such that if $\int_0^1 N(|z|)\,ds \le 1$, then $\|z\|_E \le 1$. Hence, if z with $\|z\|_\infty \le L$, and if $\int_0^1 N(|z|)\,ds \le 1$, then $\int M_L(|z|)\,ds \le 1$. Hence $M_L \le N$.

Notice that (M_L/N) is a bounded sequence in $L_\infty([0,\infty))$. Let M/N be a weak* limit point of this sequence. Since $N(s)\frac{d}{ds}(d_z(s))$ is in $L_1([0,\infty))$ whenever $z \in L_N$, and d_x is absolutely continuous, it is easy to see that M satisfies the requirements of Theorem 1. (Initially one would have to restrict to those $y \in L_\infty$, but an application of Lebesgue's Monotone Convergence Theorem will deal with this.) $\square$

Proof of Theorem 22. Following the first part of the proof of Theorem 21 with only minor modifications, we can show the following. If $x \in L_\infty$ with $\|x\|_\infty = L$, then there exists an increasing, p-convex, q-concave function $M_1 : [0,L] \to [0,\infty)$ such that

$$\int M_1(|x|)\,ds \le c,$$

and whenever $\|y\|_\infty \le L$ with $\|y\|_E \ge c$, then

$$\int M_1(|y|)\,ds \ge 1.$$

If $q = \infty$, consider the function N generated by Lemma 25 in the case when $\epsilon = 1$. Notice that if we set

$$N_1(t) = \begin{cases} N(t) & \text{if } t \ge 1 \\ N(1)t^p & \text{if } t < 1, \end{cases}$$

and

$$N_2(t) = \sup_{s<1} N_1(st)/s^p,$$

then N_2 still satisfies the conclusion of Lemma 25. Furthermore, if we set

$$M(t) = \begin{cases} M_1(t) & \text{if } t \le L \\ (M_1(L)/N_2(L))N_2(t) & \text{if } t > L, \end{cases}$$

then we see that M satisfies the hypotheses of Lemma 20. Also, if $\|y\|_E \ge 2c$, with $y \ge 0$, then either $\|\min\{y,L\}\|_E \ge c$, in which case $\int M(\min\{y,L\})\,ds = \int M_1(\min\{y,L\})\,ds \ge c^{-1}$, or $\|yI_{y>L}\|_E \ge c$, in which case $\int M(yI_{y>L})\,ds = (M_1(L)/N_2(L))\int N_2(yI_{y>L})\,ds \ge c^{-1}$. In either case, $\int M(y)\,ds \ge c^{-1}$.

If $q < \infty$, it is an easy matter to see that $L_q \subseteq E$, and that there exists a constant $c_1 > 0$ such that $\|z\|_E \le c_1 \|z\|_q$ for all $z \in L_q$. Set

$$M(t) = \begin{cases} M_1(t) & \text{if } t \le L \\ M_1(L)(t/L)^q & \text{if } t > L. \end{cases}$$

Thus M satisfies the hypotheses of Lemma 20. Furthermore, if $\|y\|_E \ge 2c$, then $\int M(|y|)\,ds \ge c^{-1}$, by the same argument as in the case when $q = \infty$.

Now let us consider the case for general x. Note that $E \subseteq L_p$, and that there is a constant $c_2 > 0$ such that $\|z\|_p \le c_2 \|z\|_E$ for all $z \in E$. Without loss of generality, $x \ge 0$. Write $x = x_1 + x_2$, where x_1 and x_2 have disjoint support, $x_1 \in L_\infty$, $\|x_1\|_E \ge 1/2$, and $\|x_2\|_p \le 1$.

Let M_1 be the function described by Theorem 22 for x_1, and let $M(t) = \min\{M_1(t), t^p\}$. It is clear that M satisfies the hypotheses of Lemma 20, and also that $\int M(x)\,ds \le c$. Now suppose that $\|y\| \ge 4c$. We may suppose that $y \ge 0$. Write $y = y_1 + y_2$, where $y_1 = yI_{M_1(y) \le y^p}$. Then either $\|y_1\|_E \ge 2c$, in which case $\int M(y)\,ds \ge \int y_1^p\,ds \ge 2c_2^{-1}$, or $\|y_2\|_E \ge 2c$, in which case $\int M(y)\,ds \ge \int M_1(y_2)\,ds \ge c^{-1}$. $\square$

We will leave the proof of Theorem 23 to the reader, as it follows the ideas of the previous proofs.

We also leave with a problem that was given to the first named author by Carsten Schütt. If E is D^*-convex, does there exist an appropriately measurable family of increasing, convex functions $M_\alpha : [0,\infty) \to [0,\infty)$ $(\alpha \in A)$, where A is a measurable space with measure μ, such that $\|\cdot\|_E$ is equivalent to $\int \|\cdot\|_{M_\alpha} \, d\mu(\alpha)$?

7. Another generalization of Theorem B

In this section we will consider another generalization of Theorem B. Suppose that X is a symmetric sequence space on sequences $x = (x_i)_{1\le i\le n} \in \mathbb{R}^n$. Let us suppose that $\|(1,0,\dots,0)\|_X = 1$. Then we define its associated r.i. space, E_X by the following formula:

$$\|x\|_{E_X} = \left\| \left(\frac{1}{n} \int_{(i-1)/n}^{i/n} x^*(s)\, ds \right)_{1\le i\le n} \right\|_X .$$

Let us show that E_X really does satisfy the triangle inequality. It is clear that $\|x^* + y^*\|_{E_X} \le \|x\|_{E_X} + \|y\|_{E_X}$. It is also easy to see that if $x \prec y$, then $\|x\|_{E_X} \le \|y\|_{E_X}$. Since $x + y \prec x^* + y^*$, we are done.

To save space, if A and B are two quantities depending upon certain parameters, we will write $A \approx B$ if there exists a constant $c > 0$, independent of the parameters, such that $c^{-1}A \le B \le cA$. If t is a real number, we will write $[t]$ for the greatest integer less than t.

Theorem 27. *There exists a constant $c > 0$ such that if $(x_{i,j})_{1\le i,j\le n}$ is an $n \times n$ matrix, then*

$$\frac{1}{n!} \sum_{\pi\in S_n} \left\|(x_{i,\pi(i)})_{1\le i\le n}\right\|_X \ge c^{-1} \left(\frac{1}{n} \sum_{k=1}^n x_k^* + \|(x_{kn}^*)_{1\le k\le n})\|_X \right).$$

Furthermore, if the associated r.i. space is D^*-convex, then there exists a constant $c > 0$ such that if $(x_{i,j})_{1\le i,j\le n}$ is an $n \times n$ matrix, then

$$\frac{1}{n!} \sum_{\pi\in S_n} \left\|(x_{i,\pi(i)})_{1\le i\le n}\right\|_X \le c \left(\frac{1}{n} \sum_{k=1}^n x_k^* + \|(x_{kn}^*)_{1\le k\le n})\|_X \right).$$

We do not know whether the condition that the associated space be D^*-convex is necessary in order for the second inequality to hold. In order to show this result, we will use the following result due to Kwapień and Schütt.

Theorem D [**KS2**]. *There exist a constant $c > 0$ such that for any $n \times n \times n$ array $y = (y_{i,j,k})_{1\le i,j,k\le n}$, we have that*

$$\frac{1}{(n!)^2} \sum_{\pi,\sigma\in S_n} \max_{1\le i\le n} \left|y_{i,\pi(i),\sigma(i)}\right| \approx \frac{1}{n^2} \sum_{k=1}^{n^2} y_k^*.$$

Proof of Theorem 27. Let us first consider the case when X_m is the symmetric sequence space given by

$$\|z\|_{X_m} = \sum_{k=1}^m z_k^*.$$

Suppose that given z, one forms the array

$$y_{i,j} = \begin{cases} z_i & \text{if } j \le n/m \\ 0 & \text{otherwise.} \end{cases}$$

Then by Theorem A, it may be seen that

$$\begin{aligned}
\frac{1}{n!} \sum_{\pi \in S_n} \max_{1 \le i \le n} \left| y_{i,\pi(i)} \right| &\approx \frac{1}{n} \sum_{k=1}^n y_k^* \\
&\approx \frac{[n/m]}{n} \sum_{k=1}^{n/[n/m]} z_k^* \\
&\approx \frac{1}{m} \|z\|_{X_m} .
\end{aligned}$$

Now, given x as in the hypothesis of the theorem, form the following array:

$$y_{i,j,k} = \begin{cases} x_{i,j} & \text{if } k \le n/m \\ 0 & \text{otherwise.} \end{cases}$$

In that case

$$\begin{aligned}
\frac{1}{n!} \sum_{\pi \in S_n} \left\| (x_{i,\pi(i)})_{1 \le i \le n} \right\|_{X_m} &\approx \frac{m}{(n!)^2} \sum_{\pi, \sigma \in S_n} \max_{1 \le i \le n} \left| y_{i,\pi(i),\sigma(i)} \right| \\
&\approx \frac{m}{n^2} \sum_{k=1}^{n^2} y_k^* \\
&\approx \frac{m[n/m]}{n^2} \sum_{k=1}^{n^2/[n/m]} x_k^* \\
&\approx \frac{1}{n} \sum_{k=1}^{nm} x_k^* \\
&\approx \frac{1}{n} \sum_{k=1}^{n} x_k^* + \|(x_{kn}^*)_{1 \le k \le n}\|_{X_m} .
\end{aligned}$$

Now let us consider more general symmetric sequence spaces X. We know that

$$\begin{aligned}
\|z\|_X &= \sup_{\|w\|_{X^*} \le 1} \sum_{k=1}^n z_k^* w_k^* \\
&= \sup_{\|w\|_{X^*} \le 1} \sum_{m=1}^n (w_m^* - w_{m+1}^*) \|z\|_{X_m} ,
\end{aligned}$$

where by convention $w^*_{n+1} = 0$. From this, we immediately see that for some constant $c > 0$

$$\frac{1}{n!} \sum_{\pi \in S_n} \left\| (x_{i,\pi(i)})_{1 \le i \le n} \right\|_X \ge c^{-1} \sup_{\|w\|_{X^*} \le 1} \sum_{m=1}^{n} (w^*_m - w^*_{m+1}) \left(\frac{1}{n} \sum_{k=1}^{n} x^*_k + \|(x^*_{kn})_{1 \le k \le n}\|_{X_m} \right)$$
$$\ge c^{-1} \left(\frac{1}{n} \sum_{k=1}^{n} x^*_k + \|(x^*_{kn})_{1 \le k \le n}\|_X \right).$$

since $w^*_1 \le 1$ whenever $\|w\|_{X^*} \le 1$.

Now let us show the second inequality when E_X is D^*-convex. Let us consider the following functions:

$$z_\pi(t) = x_{i,\pi(i)} \quad t \in [(i-1)/n, i/n),$$
$$w(t) = x^*_k \quad t \in [(k-1)/n^2, k/n^2).$$

It is an easy matter to see that $w^* = C(z_\pi : \pi \in S_n)^*$. Hence, by Corollary 24, we see that for some constant $c > 0$ depending only on X

$$\frac{1}{n!} \sum_{\pi \in S_n} \|z_\pi\|_{E_X} \le c \, \|w\|_{E_X} .$$

The result now follows after we notice that

$$\|x\|_{E_X} \approx \frac{1}{n} \int_0^{1/n} x^*(s) \, ds + \|(x^*(k/n))_{1 \le k \le n}\|_X .$$

□

8. D and D^*-convex Lorentz Spaces

Although the results in this section are primarily concerned with Lorentz spaces, in order to prove our results, we will need a wider class of spaces, known as Orlicz-Lorentz spaces. If $M, N : [0,\infty) \to [0,\infty)$ are strictly increasing bijections, then we define the space $L_{M,N}$ to be the set of those measurable functions $x : [0,1] \to \mathbb{R}$ such that

$$\|x\|_{L_{M,N}} = \left\| x^* \circ \tilde{M} \circ \tilde{N}^{-1} \right\|_{L_N} ,$$

where $\tilde{M} = 1/M(1/t)$, and $\circ$ denotes function composition. It is not clear what are necessary and sufficient conditions for $L_{M,N}$ to have an equivalent norm that satisfies the triangle inequality, but this will not be relevant to our discussion. It is clear that $L_M = L_{M,M}$, and that $\Lambda_r(\varphi) = L_{\tilde{\varphi}^{-1},t^r}$ with equality of norms.

Following [**M1**], we say that an increasing bijection $M : [0,\infty) \to [0,\infty)$ is almost convex if there are numbers $a > 1$, $b > 1$, and a positive integer p such that for all positive integers m, the cardinality of the set of integers n such that we do not have $M(a^{n+m}) \ge a^{m-p} M(a^n)$ is less than b^m. It is clear that this notion also can be made to make sense if M is only a bijection from $[0,1] \to [0,1]$, or a bijection from $[1,\infty) \to [1,\infty)$, by stating that the inequality is true whenever it is undefined.

The following result is essentially Theorem 4.2 from [**M1**]. The results from [**M1**] are concerned with function spaces on $\mathbb{R}$ rather than $[0,1]$, but the change is not too hard to do.

Theorem E. *Let M, N_1, N_2 be increasing bijections $[0,\infty) \to [0,\infty)$ that map 1 to 1, such that one of N_1 or N_2 is convex and q-concave for some $q < \infty$. Then the following are equivalent.*

i) *For some $c > 0$ we have that $\|x\|_{L_{M,N_1}} \le c\,\|x\|_{L_{M,N_2}}$ for all measurable $x : [0,1] \to \mathbb{R}$.*

ii) *$N_1 \circ N_2^{-1}$ restricted to $[1,\infty)$ is almost convex.*

Theorem 28. *If $\varphi : [0,1] \to [0,1]$ is an increasing bijection, and $1 \le r < \infty$, such that $\Lambda_r(\varphi)$ is equivalent to a norm, then $\Lambda_r(\varphi)$ is D-convex if and only if $(\tilde\varphi(t))^r$ is almost convex, and D^*-convex if and only if $\tilde\varphi^{-1}(t^{1/r})$ is almost convex.*

Proof. Suppose that $\Lambda_r(\varphi)$ is D-convex. Define

$$M(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ \tilde\varphi^{-1}(t) & \text{if } t \ge 1. \end{cases}$$

It is clear that $\left\|I_{[0,t]}\right\|_{\Lambda_r(\varphi)} = \left\|I_{[0,t]}\right\|_{L_M} = \varphi(t)$ for $0 \le t \le 1$, and that $\Lambda_r(\varphi) = L_{M,t^r}$.

We will show that there is a constant $c_1 > 0$ such that $\|x\|_{\Lambda_r(\varphi)} \le c_1\,\|x\|_{L_M}$. Then the result will follow easily from Theorem E.

For, by Theorem 21, there exists a constant $c_2 > 0$ so that the following holds. Suppose that $\|x\|_{\Lambda_r(\varphi)} = 1$. Then there exists an increasing convex bijection $N : [0,\infty) \to [0,\infty)$ such that $\|x\|_{L_N} \ge c_2^{-1}$, but that in general $\|y\|_{L_N} \le \|y\|_{\Lambda_r(\varphi)}$. By considering $y = I_{[0,t]}$, we see that $N(t) \le M(t)$ for $t \ge 1$. Now, for any $\lambda < c_2^{-1}$, we have that

$$\int_0^1 N(|x|/\lambda)\,ds > 1.$$

Since N is convex, for any $\lambda < c_2^{-1}/2$, we have that

$$\int_0^1 N(|x|/\lambda)\,ds > 2.$$

Further, for any $\lambda > 0$

$$\int_{|x|\le\lambda} N(|x|/\lambda)\,ds \le N(1) \le 1.$$

Hence for $\lambda < c_2^{-1}/2$, we have that

$$\int_0^1 M(|x|/\lambda)\,ds \ge \int_{|x|\ge\lambda} M(|x|/\lambda)\,ds \ge \int_{|x|\ge\lambda} N(|x|/\lambda)\,ds > 1,$$

and hence $\|x\|_{L_M} \ge c_2^{-1}/2$.

The case when $\Lambda_r(\varphi)$ is D^*-convex is almost identical. □

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