THE DISTRIBUTION OF VECTOR-VALUED RADEMACHER SERIES

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Abstract. Let $X = \sum \varepsilon_n x_n$ be a Rademacher series with vector-valued coefficients. We obtain an approximate formula for the distribution of the random variable $||X||$ in terms of its mean and a certain quantity derived from the K-functional of interpolation theory. Several applications of the formula are given.

1. Results

In [6] the second-named author calculated the distribution of a scalar Rademacher series $\sum \varepsilon_n a_n$. The principal result of the present paper extends the results of [6] to the case of a Rademacher series $\sum \varepsilon_n x_n$ with coefficients (x_n) belonging to an arbitrary Banach space E. Its proof relies on a deviation inequality for Rademacher series obtained by Talagrand [9]. A somewhat curious feature of the proof is that it appears to exploit in a non-trivial way (see Lemma 2) the platitude that every separable Banach space is isometric to a closed subspace of ℓ_∞. The principal result is applied to yield a precise form of the Kahane-Khintchine inequalities and to compute certain Orlicz norms for Rademacher series.

First we recall some notation and terminology from interpolation theory (see e.g. [1]). Let $(E_1, ||.||_1)$ and $(E_2, ||.||_2)$ be two Banach spaces which are continuously embedded into some larger topological vector space. For $t > 0$, the K-functional $K(x, t; E_1, E_2)$ is the norm on $E_1 + E_2$ defined by

$$K(x, t; E_1, E_2) = \inf\{||x_1||_1 + t||x_2||_2 : x = x_1 + x_2, \quad x_i \in E_i\}.$$

For a sequence $(a_n) \in \ell_2$, we shall denote the K-functional $K((a_n), t; \ell_1, \ell_2)$ by $K_{1,2}((a_n), t)$ for short. For $1 \le p < \infty$, a sequence (x_n) in a Banach space $(E, ||.||)$ is said to be weakly-ℓ_p if the scalar sequence $(x^*(x_n))$ belongs to ℓ_p for every $x^* \in E^*$. The collection of all weakly-ℓ_p sequences is a Banach space, denoted $\ell_p^w(E)$, with the norm given by $l_p^w((x_n)) = \sup_{||x^*|| \le 1} ||(x^*(x_n))||_p$ (where $||(a_n)||_p = (\sum |a_n|^p)^{1/p}$). If $(x_n) \in \ell_2^w(E)$, we make the following definition:

$$K_{1,2}^w((x_n), t) = \sup_{||x^*|| \le 1} K_{1,2}((x^*(x_n)), t).$$

Observe that $K_{1,2}^w((x_n), t)$ is a continuous increasing function of t. In fact, it is a Lipschitz function with Lipschitz constant at most $\ell_2^w((x_n))$.

1991 *Mathematics Subject Classification.* Primary 46B20; Secondary 60B11, 60G50.

Key words and phrases. Rademacher series, K-functional, Banach space.

The second author was supported in part by NSF DMS-9001796

Next we set up some function space notation. Let (Ω, Σ, P) be a probability space. A Rademacher (or Bernoulli) sequence (ε_n) is a sequence of independent identically distributed random variables such that $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = \frac{1}{2}$. For a random variable Y defined on Ω, its decreasing rearrangement, Y^*, is the function on $[0,1]$ defined by $Y^*(t) = \inf\{s > 0 : P(|Y| > s) \le t\}$. For $0 < p < \infty$, the weak-L_p norm of Y, denoted $||Y||_{p,\infty}$, is given by $||Y||_{p,\infty} = \sup_{0<t<1} t^{\frac{1}{p}} Y^*(t)$. As usual, $||Y||_p$ denotes $(\mathbb{E}|Y|^p)^{1/p}$. Let Ψ be an Orlicz function on $[0, \infty)$. The Orlicz norm, $||Y||_\Psi$, is given by $||Y||_\Psi = \inf\{c > 0 : \mathbb{E}\Psi(|Y|/c) \le 1\}$. We shall be particularly interested in the Orlicz functions $\Psi_q(t) = e^{t^q} - 1$ for $2 < q < \infty$. The weak-ℓ_p norm of the scalar sequence (a_n) is defined by $||(a_n)||_{p,\infty} = \sup n^{\frac{1}{p}} a_n^*$, where (a_n^*) is the decreasing rearrangement of $(|a_n|)$.

Finally, we shall write $A \approx B$ to mean that there is a constant $C > 0$ such that $\frac{1}{C} A \le B \le CA$. We shall try to indicate in each case whether the implied constant is absolute or whether it depends on some parameter, typically $p \in [1, \infty)$, entering into the expressions for A and B.

Now we can state the principal result of the paper.

Main Theorem. *Let $X = \sum \varepsilon_n x_n$ be an almost surely convergent Rademacher series in a Banach space E. Then, for $t > 0$, we have*

$$P(||X|| > 2\mathbb{E}||X|| + 6K^w_{1,2}((x_n), t)) \le 4e^{-t^2/8}, \tag{1}$$

and, for some absolute constant c, we have

$$P\left(||X|| > \frac{1}{2}\mathbb{E}||X|| + cK^w_{1,2}((x_n), t)\right) \ge ce^{-t^2/c}. \tag{2}$$

The proof of the Main Theorem will be deferred until the end of the paper in order to proceed at once with the applications.

Corollary 1. *Let $X = \sum \varepsilon_n x_n$ be an almost surely convergent Rademacher series in a Banach space. Then, for $0 < t \le \frac{1}{10}$, we have*

$$S^*(t) \approx \mathbb{E}||X|| + K^w_{1,2}((x_n), \sqrt{\log(1/t)}), \tag{3}$$

where S denotes the real random variable $||X||$. The implied constant is absolute.

Proof. (1) and (2) give rise to the inequalities $S^*(4e^{-t^2/8}) \le 2\mathbb{E}||X|| + 6K^w_{1,2}((x_n), t)$ and $S^*(ce^{-t^2/c}) \ge \frac{1}{2}\mathbb{E}||X|| + cK^w_{1,2}((x_n), t)$, respectively, whence (3) follows for all sufficiently small t by an appropriate change of variable. To see that the lower estimate implicit in (3) is valid in the whole range $0 < t < \frac{1}{10}$, we recall from [2] that $\mathbb{E}||X||^2 \le 9\mathbb{E}^2||X||$. Hence, by the Paley-Zygmund inequality (see e.g. [4,p.8]), for $0 < \lambda < 1$, we have

$$\begin{aligned} P(||X|| > \lambda\mathbb{E}||X||) &\ge (1-\lambda)^2 \frac{\mathbb{E}^2 X}{\mathbb{E}X^2} \\ &\ge \frac{1}{9}(1-\lambda)^2, \end{aligned}$$

whence $P(||X|| > (1 - \frac{3}{\sqrt{10}})\mathbb{E}||X||) \ge \frac{1}{10}$, which easily implies (3). □

In [4] Kahane proved that if $P(||X|| > t) = \alpha$, where X is a Rademacher series in a Banach space, then $P(||X|| > 2t) \le 4\alpha^2$. By iteration this implies $P(||X|| > st) \le \frac{1}{4}(4\alpha)^s$ for $s = 2^n$. According to our next corollary the exponent s in the latter result may be improved to be a certain multiple of s^2.

Corollary 2. *Let $X = \sum \varepsilon_n x_n$ be an almost surely convergent Rademacher series in a Banach space. Then, for $t > 0$ and $s \geq 1$, we have*

$$P(||X|| > st) \leq \left(\frac{1}{c_1} P(||X|| > t) \right)^{c_1 s^2}$$

for some absolute constant c_1.

Proof. By choosing $c_1 < c$, where c is the constant which appears in (2), the result becomes trivial whenever $P(||X|| > t) \geq c$. Hence we may assume that $P(||X|| > t) < c$. Choose $\alpha > 0$ such that $P(||X|| > t) = ce^{-\alpha^2/c}$. Then (2) gives $t \geq \frac{1}{2}\mathbb{E}||X|| + cK^w_{1,2}((x_n), \alpha)$. Thus,

$$\begin{aligned} st &\geq \frac{s}{2}\mathbb{E}||X|| + scK^w_{1,2}((x_n), \alpha) \\ &\geq 2\mathbb{E}||X|| + K^w_{1,2}((x_n), cs\alpha) \end{aligned}$$

provided $s \geq \max(4, 1/c)$. Now (1) gives

$$\begin{aligned} P(||X|| > st) &\leq 4e^{-(cs\alpha)^2/8} \\ &= 4\left(\frac{1}{c}(ce^{-\alpha^2/c}) \right)^{c^3 s^2/8} \\ &= 4\left(\frac{1}{c}(P(||X|| > t) \right)^{c^3 s^2/8}, \end{aligned}$$

which gives the result. □

Our next corollary, which is the vector-valued version of a recent result of Hitczenko [3], is a rather precise form of the Kahane-Khintchine inequalities.

Corollary 3. *Let $X = \sum \varepsilon_n x_n$ be a Rademacher series in a Banach space. Then, for $1 \leq p < \infty$, we have*

$$(\mathbb{E}||X||^p)^{1/p} \approx \mathbb{E}||X|| + K^w_{1,2}((x_n), \sqrt{p}).$$

The implied constant is absolute.

Proof. We may assume that $p \geq 2$. It follows from a result of Borell [2] that $(\mathbb{E}||X||^{2p})^{1/2p} \leq \sqrt{3}(\mathbb{E}||X||^p)^{1/p}$. Since $\frac{1}{2}||Y||_p \leq ||Y||_{2p,\infty} \leq ||Y||_{2p}$ for every random variable Y (as is easily verified), it follows (letting S denote the random variable $||X||$) that $\frac{1}{2}||S||_p \leq ||S||_{2p,\infty} \leq \sqrt{3}||S||_p$. So it suffices to prove that $||S||_{p,\infty} \approx \mathbb{E}S + K^w_{1,2}((x_n), \sqrt{p})$ to obtain the desired conclusion. By Corollary 1, we have

$$\begin{aligned} ||S||_{p,\infty} &\approx \mathbb{E}S + \sup_{0<t<1} t^{1/p} K^w_{1,2}((x_n), \sqrt{\log(1/t)}) \\ &= \mathbb{E}S + \sup_{0<t<1} \left\{ t^{1/p} \sup_{||x^*|| \leq 1} K_{1,2}((x^*(x_n)), \sqrt{\log(1/t)}) \right\} \\ &= \mathbb{E}S + \sup_{||x^*|| \leq 1} \left\{ \sup_{0<t<1} t^{1/p} K_{1,2}((x^*(x_n)), \sqrt{\log(1/t)}) \right\}. \end{aligned}$$

To evaluate the expression in brackets we shall make use once more (see Corollary 2) of the elementary inequality $K_{1,2}((a_n),s) \le \max(1,s/t)K_{1,2}((a_n),t)$. Thus,

$$\begin{aligned}\sup_{0<t\le e^{-p}} t^{1/p} K_{1,2}((x^*(x_n)),\sqrt{\log(1/t)}) &\le \left(\sup_{0<t\le e^{-p}} t^{1/p}\sqrt{\frac{\log(1/t)}{p}}\right) K_{1,2}((x^*(x_n)),\sqrt{p}) \\ &= e^{-1}K_{1,2}((x^*(x_n)),\sqrt{p}).\end{aligned}$$

Moreover,

$$\sup_{e^{-p}<t<1} t^{1/p} K_{1,2}((x^*(x_n)),\sqrt{\log(1/t)}) \le K_{1,2}((x^*(x_n)),\sqrt{p}).$$

Finally, we obtain

$$\frac{1}{e}K^w_{1,2}((x_n),\sqrt{p}) \le \sup_{||x^*||\le 1}\left\{\sup_{0<t<1} K_{1,2}((x^*(x_n)),\sqrt{\log(1/t)})\right\} \le K^w_{1,2}((x_n),\sqrt{p}),$$

which gives the desired result. □

Our final application is to the calculation of the Orlicz norms $||S||_{\psi_q}$ for $2<q<\infty$. The proof will use the scalar version of the result, which was obtained by Rodin and Semyonov [8] (see also [7]). (Recall that by a result of Kwapien, [5], $||S||_{\psi_q} \approx \mathbb{E}||X||$ in the range $0<q\le 2$.)

Corollary 4. *Let $X=\sum \varepsilon_n x_n$ be an almost surely convergent Rademacher series in a Banach space. Then, for $2<q<\infty$, we have*

$$||S||_{\psi_q} \approx \mathbb{E}||X|| + \sup_{||x^*||\le 1} ||(x^*(x_n))||_{p,\infty},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and S denotes $||X||$. The implied constant depends only on q.

Proof. It is easily verified that $||f||_{\psi_q} \approx \sup_{0<t<1}(\log(1/t))^{-1/q}f^*(t)$. Hence, by Corollary 1, we have

$$\begin{aligned}||S||_{\psi_q} &\approx \mathbb{E}||X|| + \sup_{0<t<1}(\log(1/t))^{-1/q}K^w_{1,2}((x_n),t) \\ &\approx \mathbb{E}||X|| + \sup_{0<t<1}\left\{(\log(1/t))^{-1/q}\sup_{||x^*||\le 1}K_{1,2}((x^*(x_n)),t)\right\} \\ &\approx \mathbb{E}||X|| + \sup_{||x^*||\le 1}\left\{\sup_{0<t<1}(\log(1/t))^{-1/q}K_{1,2}((x^*(x_n)),t)\right\} \\ &\approx \mathbb{E}||X|| + \sup_{||x^*||\le 1}||\sum \varepsilon_n x^*(x_n)||_{\psi_q} \\ &\approx \mathbb{E}||X|| + \sup_{||x^*||\le 1}||(x^*(x_n))||_{p,\infty},\end{aligned}$$

where the last line follows from the result of Rodin and Semyonov. □

2. Proof of main result

The principal ingredient in the proof of the Main Theorem is the following deviation inequality of Talagrand [9].

Theorem A. *Let* $X = \sum_{n=1}^{N} \varepsilon_n x_n$ *be a finite Rademacher series in a Banach space and let* M *be a median of* $||X||$. *Then, for* $t > 0$, *we have*

$$P\left(|||\sum_{n=1}^{N} \varepsilon_n x_n|| - M| > t \right) \leq 4e^{-t^2/8\sigma^2},$$

where $\sigma = \ell_2^w((x_n)_{n=1}^N)$.

Lemma 1. *Let* $X = \sum_{n=1}^{N} \varepsilon_n x_n$ *be a finite Rademacher series in a Banach space* E. *Then, for* $t > 0$, *we have*

$$P(||X|| > 2\mathbb{E}||X|| + 3K((x_n)_{n=1}^N, t; \ell_1^w(E), \ell_2^w(E))) \leq 4e^{-t^2/8}.$$

Proof. It follows from Theorem A that for all $y_1, \dots, y_N$ in E, we have

$$P(||\sum \varepsilon_n y_n|| > 2\mathbb{E}||\sum \varepsilon_n y_n|| + t\ell_2^w((y_n))) \leq 4e^{-t^2/8}. \tag{4}$$

On the other hand, since $\max ||\sum \varepsilon_n y_n|| = \ell_1^w((y_n))$, we have the trivial estimate

$$P(||\sum \varepsilon_n y_n|| > \ell_1^w((y_n))) = 0. \tag{5}$$

Let $x_n = x_n^{(1)} + x_n^{(2)}$ for $1 \leq n \leq N$, let $X^{(1)} = \sum \varepsilon_n x_n^{(1)}$, and let $X^{(2)} = \sum \varepsilon_n x_n^{(2)}$. Then

$$\begin{aligned} \ell_1^w((x_n^{(1)})) + t\ell_2^w((x_n^{(2)})) + 2\mathbb{E}||X^{(2)}|| &\leq \ell_1^w((x_n^{(1)})) + t\ell_2^w((x_n^{(2)})) + 2\mathbb{E}||X^{(1)}|| + 2\mathbb{E}||X|| \\ &\leq 3\ell_1^w((x_n^{(1)})) + t\ell_2^w((x_n^{(2)})) + 2\mathbb{E}||X|| \\ &\leq 2\mathbb{E}||X|| + 3(\ell_1^w((x_n^{(1)})) + t\ell_2^w((x_n^{(2)}))). \end{aligned}$$

Let Q denote $2\mathbb{E}||X|| + 3(\ell_1^w((x_n^{(1)})) + t\ell_2^w((x_n^{(2)})))$. Then, by (4) and (5) and by the above inequality, we have

$$\begin{aligned} P(||X|| > Q) &\leq P(||X^{(1)}|| + ||X^{(2)}|| > \ell_1^w((x_n^{(1)})) + t\ell_2^w((x_n^{(2)})) + 2\mathbb{E}||X^{(2)}||) \\ &\leq P(||X^{(1)}|| > \ell_1^w((x_n^{(1)}))) + P(||X^{(2)}|| > 2\mathbb{E}||X^{(2)}|| + t\ell_2^w((x_n^{(2)}))) \\ &< 0 + 4e^{-t^2/8} \end{aligned}$$

The desired conclusion now follows from the definition of the K-functional. □

Lemma 2. *Let* $x_1, \dots, x_N$ *be elements of the Banach space* ℓ_∞. *Then*

$$K((x_n)_{n=1}^N, t; \ell_1^w(\ell_\infty), \ell_2^w(\ell_\infty)) \leq 2K_{1,2}^w((x_n)_{n=1}^N, t).$$

Proof. For $1 \leq n \leq N$, let $x_n = (x_{n,j})_{j=1}^\infty \in \ell_\infty$. A simple convexity argument gives

$$||(x_n)||_{\ell_p^w(\ell_\infty)} = \sup_{1 \leq j \leq \infty} \left(\sum_{n=1}^N |x_{n,j}|^p \right)^{(1/p)}.$$

It follows that the mapping ϕ which associates an element $(y_n)_{n=1}^\infty \in \ell_p^w(\ell_\infty)$ with the element in $\ell_\infty(\ell_p)$ whose jth coordinate equals $(y_{n,j})_{n=1}^\infty$ is an isometry. Hence $K((x_n), t; \ell_1^w, \ell_2^w) = K(\phi((x_n)), t; \ell_\infty(\ell_1), \ell_\infty(\ell_2))$. Let $(y_n)_{n=1}^\infty \in \ell_\infty(\ell_2)$ and let $\varepsilon > 0$. For each n there exists a splitting $y_n = z_n^{(1)} + z_n^{(2)}$ such that

$$||(z_{n,j}^{(1)})_{j=1}^\infty||_1 + t||(z_{n,j}^{(2)})_{j=1}^\infty||_2 \leq K_{1,2}((y_{n,j})_{j=1}^\infty, t) + \varepsilon.$$

It follows that

$$\begin{aligned} ||(z_n^{(1)})||_{\ell_\infty(\ell_1)} + t||(z_n^{(2)})||_{\ell_\infty(\ell_2)} &= \sup_{1 \leq n < \infty} ||(z_{n,j}^{(1)})_{j=1}^\infty||_1 + t \sup_{1 \leq n < \infty} ||(z_{n,j}^{(2)})_{j=1}^\infty||_2 \\ &\leq 2 \sup_{1 \leq n < \infty} K_{1,2}((y_{n,j})_{j=1}^\infty, t) + 2\varepsilon \\ &\leq 2K_{1,2}^w((y_n), t) + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, the result now follows from the definition of the K-functional. $\square$

Proof of Main Theorem. First we prove (1) for a finite Rademacher series $\sum_{n=1}^N \varepsilon_n x_n$. Since every separable Banach space embeds isometrically into ℓ_∞, we may assume that E is a closed subspace of ℓ_∞. Recall that $K_{1,2}^w((x_n), t)$ was defined as $\sup_{||x^*|| \leq 1} K_{1,2}((x^*(x_n)), t)$. By the Hahn-Banach Theorem, the supremum is the same whether it is taken over elements of E^* or over elements of ℓ_∞^*. Hence (1) follows by combining Lemmas 1 and 2. The result for an infinite series follows from the result for $\sum_{n=1}^N \varepsilon_n x_n$ by taking the limit as $N \to \infty$. To prove (2), we use the result from [6] that there exists an absolute constant d such that $P(\sum \varepsilon_n a_n > dK_{1,2}((a_n), t)) \geq de^{-t^2/d}$ for every sequence $(a_n) \in \ell_2$. Hence

$$\begin{aligned} P\left(||\sum \varepsilon_n x_n|| > \frac{d}{2} K_{1,2}^w((x_n), t) \right) &\geq \inf_{||x^*|| \leq 1} P(||\sum \varepsilon_n x_n|| > dK_{1,2}((x^*(x_n)), t)) \\ &\geq \inf_{||x^*|| \leq 1} P(\sum \varepsilon_n x^*(x_n) > dK_{1,2}((x^*(x_n)), t)) \\ &\geq de^{-t^2/d}. \end{aligned}$$

The Paley-Zygmund inequality now gives

$$\begin{aligned} &P\left(||X|| > \frac{1}{2}\mathbb{E}||X|| + \frac{d}{6} K_{1,2}^w((x_n), t) \right) \\ &\qquad \geq \min\left(P\left(||X|| > \frac{3}{4}\mathbb{E}||X|| \right), P\left(||X|| > \frac{d}{2} K_{1,2}^w((x_n), t) \right) \right) \\ &\qquad \geq \min\left(\frac{1}{144}, de^{-t^2/d}. \right). \quad \square \end{aligned}$$

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