The Rademacher Cotype of Operators from l_∞^N

S.J. Montgomery-Smith
Department of Mathematics, University of Missouri,
Columbia, MO 65211.

M. Talagrand
Department of Mathematics, The Ohio State University,
231 W. 18th Avenue, Columbus, OH 43210.
Equipe d'Analyse – Tour 46, Université Paris VI,
4 Place Jussieu, 75230 Paris Cedex 05.

ABSTRACT: We show that for any operator $T : l_\infty^N \to Y$, where Y is a Banach space, that its cotype 2 constant, $K^{(2)}(T)$, is related to its $(2,1)$-summing norm, $\pi_{2,1}(T)$, by

$$K^{(2)}(T) \le c \log\log N \; \pi_{2,1}(T).$$

Thus, we can show that there is an operator $T : C(K) \to Y$ that has cotype 2, but is not 2-summing.

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Introduction

The notation we use in this paper is loosely based on that given in [L–T1], [L–T2] and [P1].

We let $\varepsilon_1, \varepsilon_2, \ldots$ be independent Rademacher random variables, that is, $\Pr(\varepsilon_s = 1) = \Pr(\varepsilon_s = -1) = \frac{1}{2}$. A linear operator $T : X \to Y$ is said to have *(Rademacher) cotype p* ($p \ge 2$) if there is a constant $C < \infty$ such that for all $x_1, x_2, \ldots, x_S$ in X we have

$$\left(\sum_{s=1}^{S} \|T(x_s)\|^p \right)^{\frac{1}{p}} \le C \; \mathbb{E} \left\| \sum_{s=1}^{S} \varepsilon_s x_s \right\| .$$

The smallest value of C is called the *(Rademacher) cotype p constant* of T, and is denoted by $K^{(p)}(T)$. These definitions extend to spaces in the obvious way; a space X has cotype p if its identity operator has cotype p.

We define the *(p,q)-summing norm* of a linear operator $T : X \to Y$, denoted by $\pi_{p,q}(T)$, to be the least number C such that for all $x_1, x_2, \dots, x_S$ in X we have

$$\left(\sum_{s=1}^S \|T(x_s)\|^p\right)^{\frac{1}{p}} \le C \sup\left\{\left(\sum_{s=1}^S |\langle x^*, x_s\rangle|^q\right)^{\frac{1}{q}}\right\},$$

where the supremum is taken over all x^* in the unit ball of X^*. We call a (p,p)-summing operator a *p-summing operator*, and write $\pi_p(T)$ for $\pi_{p,p}(T)$. We say that the operator is *(p,q)-summing* (*p-summing*) if $\pi_{p,q}(T) < \infty$ (respectively $\pi_p(T) < \infty$).

If $1 \le p < \infty$, and $1 \le q \le \infty$, then we let $L_{p,q}(\mu)$ denote the Lorentz space on the measure μ. We refer the reader to [H] or [L–T2] for details, but just note that the $L_{p,1}$ norm may be calculated using

$$\|f\|_{p,1} = \int_0^\infty \mu(|f| > t)^{\frac{1}{p}}\, dt = \tfrac{1}{p}\int_0^\infty s^{\frac{1}{p}-1} f^*(s)\, ds,$$

where f^* denotes the non-decreasing rearrangement of $|f|$.

The basic motivation behind this paper is in classifying operators from $C(K)$ that factor through a Hilbert space, where $C(K)$ denotes the continuous functions on the compact Hausdorff topological space, K. The first result in this direction is due to Grothendieck, which states that any bounded linear operator $C(K) \to L_1$ factors through Hilbert space. This was generalized by Maurey [Ma1], allowing L_1 to be replaced by any space of cotype 2, to give the following result (see also [P1]).

Theorem 1. *Let $T : C(K) \to Y$ be a linear operator, where Y is any Banach space. Then the following are equivalent:*

i) T is 2-summing;

ii) T factors through Hilbert space;

iii) T factors through a space of cotype 2.

However, we are still left with the following question: if the *operator* $T : C(K) \to Y$ has cotype 2, does it follow that it factors through Hilbert space?

One way one might tackle this problem is to consider the $(2,1)$-summing norms of such operators. Jameson [J] showed that there is an operator $T : l_\infty^N \to Y$ such that $\pi_2(T) \ge c^{-1}\sqrt{\log N}\,\pi_{2,1}(T)$. Hence, if we can establish a strong relationship between the cotype 2 constants and the $(2,1)$-summing norms of such operators, then we can answer the above question in the negative. To this end, we have the following — the main result of this paper.

Theorem 2. *There is a constant c such that for any operator $T : l_\infty^N \to Y$, where Y is a Banach space, then the cotype 2 constant is bounded according to the relation:*

$$K^{(2)}(T) \le c \log\log N \, \pi_{2,1}(T).$$

Corollary. *There is an operator $T : C(K) \to Y$, where Y is a Banach space, that has cotype 2, but does not factor through Hilbert space.*

Finally, before embarking on the proof of this result, we point out that for $p > 2$, the above problems have been completely answered.

Theorem 3. *Let $T : C(K) \to Y$ be a bounded linear operator, where Y is a Banach space. Then for all $p > 2$, the following are equivalent:*

i) T is $(p, 1)$-summing;

ii) T has Rademacher cotype p;

iii) T factors through a space with Rademacher cotype p.

The implication (i) $\Leftrightarrow$ (ii) is due to Maurey [Ma2]. The third equivalence follows from the fact that any $(p, 1)$-summing operator from $C(K)$ factors through $L_{p,1}$ (see [P2] or Theorem 5 below), and that $L_{p,1}$ has Rademacher cotype p, (see [C]).

Theorem 4. *If $p > 2$, then there is a bounded linear operator $C(K) \to L_p$ that is not p-summing.*

We refer the reader to [K].

Proof of the Main Result

To prove Theorem 2, we need the following two results. The first allows us to reduce questions about $(p,1)$-summing operators from $C(K)$ to the canonical embedding $C(K) \to L_{2,1}(K,\mu)$ (μ a probability measure), and is due to Pisier (see [P2]).

Theorem 5. *Let $T : C(K) \to Y$ be a $(p,1)$-summing operator, where Y is a Banach space, and $p \geq 1$. Then there is a Radon probability measure μ on K and a constant $C \leq p^{\frac{1}{p}} \pi_{p,1}(T)$ such that for all $x \in C(K)$ we have $\|Tx\| \leq C \, \|x\|_{L_{p,1}(K,\mu)}$.*

The second result is about Rademacher processes, and is due to the second named author (for the proof, see [Ld–T]). First we establish some more notation. If T is a bounded subset of $\mathbb{R}^S$, we write

$$r(T) = \mathbb{E} \sup_{t \in T} \left| \sum_{s=1}^{S} \varepsilon_s t(s) \right| .$$

If B is a subset of $\mathbb{R}^S$, we write $\mathcal{N}(T,B)$ for the minimal number of translates of B required to cover D. We write B_1^S for the unit ball of l_1^S, and B_2^S for the unit ball of l_2^S. From now on, we take all logarithms to base 2.

Theorem 6. *There is a constant c_1 such that if T is a bounded subset of $\mathbb{R}^S$, and $\epsilon > 0$, then letting $D = c_1 \, r(T) \, B_1^S + \epsilon \, B_2^S$, we have*

$$r(T) \geq c_1^{-1} \epsilon \sqrt{\log \mathcal{N}(T,D)}.$$

Now we will state the main result towards proving Theorem 2.

Proposition 7. *There is a constant c_2 such that if $(\Omega, \mathcal{F}, \mu)$ is a probability space with N atoms, and $x_1, x_2, \ldots, x_S \in L_\infty(\mu)$ are such that*

$$\mathbb{E} \left\| \sum_{s=1}^{S} \varepsilon_s x_s \right\|_\infty \leq 1,$$

then

$$\left(\sum_{s=1}^{S} \|x_s\|_{L_{2,1}(\mu)}^2 \right)^{\frac{1}{2}} \leq c_2 \, \log\log N.$$

Our first step in establishing this result is to restate Theorem 6 in a more suitable form.

Lemma 8. *There is a constant c_1 (the same one as in Theorem 6) such that the following holds. Suppose that $(\Omega, \mathcal{F}, \mu)$ is a measure space, with Ω finite, and $x_1, x_2, \ldots, x_S \in L_\infty(\mu)$ with*

$$\mathbb{E}\left\|\sum_{s=1}^S \varepsilon_s x_s\right\|_\infty \le 1.$$

Then for all integers k, we may partition Ω into at most 2^{2^k} measurable sets, find y_1, $y_2, \ldots, y_S$, z_1, $z_2, \ldots, z_S \in L_\infty(\mu)$, and find $\bar{x}_1$, $\bar{x}_2, \ldots, \bar{x}_S \in L_\infty(\Omega, \mathcal{F}', \mu)$ (where $\mathcal{F}'$ denotes the algebra generated by the partition), such that $x_s = \bar{x}_s + y_s + z_s$,

$$\mathbb{E}\left\|\sum_{s=1}^S \varepsilon_s \bar{x}_s\right\|_\infty \le 1, \quad \left\|\sum_{s=1}^S |y_s|\right\|_\infty \le c_1 \quad \text{and} \quad \left\|\left(\sum_{s=1}^S |z_s|^2\right)^{\frac12}\right\|_\infty \le c_1\, 2^{-\frac{k}{2}}.$$

Proof: Let $T = \left\{ \left(x_s(\omega)\right)_{s=1}^S : \omega \in \Omega \right\}$, and let $\epsilon = c_1^{-1} 2^{-\frac{k}{2}}$. If we apply Theorem 6, we see that there are 2^{2^k} translates, $t_l + c_1(B_1^S + 2^{-\frac{k}{2}} B_2^S)$ $(1 \le l \le 2^{2^k})$, that cover T. We let the covering of Ω be the sets

$$A_l = \left\{ \omega : \left(x_s(\omega)\right)_{s=1}^S \in t_l + c_1(B_1^S + 2^{-\frac{k}{2}} B_2^S) \right\},$$

and if A_l is non-empty, we choose $\omega_l \in A_l$. Define $\bar{x}_s(\omega) = x_s(\omega_l)$ if $\omega \in A_l$. Now, if $\omega \in A_l$, we know that $\left(x_s(\omega) - \bar{x}_s(\omega)\right)_{s=1}^S \in c_1(B_1^S + 2^{-\frac{k}{2}} B_2^S)$, that is, there are $\left(y_s(\omega)\right)_{s=1}^S \in c_1\, B_1^S$ and $\left(z_s(\omega)\right)_{s=1}^S \in c_1\, 2^{-\frac{k}{2}} B_2^S$, with $x_s(\omega) = \bar{x}_s(\omega) + y_s(\omega) + z_s(\omega)$. ☐

Lemma 9. *There is a constant c_3 such that if $(\Omega, \mathcal{F}, \mu)$ is a measure space with Ω finite, then the following hold.*

i) If $y \in L_\infty(\mu)$, then $\|y\|_{2,1} \le \|y\|_\infty^{\frac12} \|y\|_1^{\frac12}$.

ii) If the smallest atom is of size a, then for all $z \in L_\infty(\mu)$ we have

$$\|z\|_{2,1} \le c_3 \left(1 + \sqrt{\log(\mu(\Omega)/a)}\right) \|z\|_2 .$$

iii) If there are N atoms, then for all $z \in L_\infty(\mu)$ we have $\|z\|_{2,1} \le \sqrt{N}\, \|z\|_2$.

Proof: i) We have that

$$\begin{aligned}
\|y\|_{2,1} &= \int_0^{\|y\|_\infty} \sqrt{\mu(|y| > t)}\, dt \\
&\le \left(\int_0^{\|y\|_\infty} dt\right)^{\frac12} \left(\int_0^{\|y\|_\infty} \mu(|y| > t)\, dt\right)^{\frac12} \\
&= \|y\|_\infty^{\frac12} \|y\|_1^{\frac12} .
\end{aligned}$$

ii) We have

$$
\begin{aligned}
\|z\|_{2,1} &= \tfrac{1}{2} \int_0^\infty \frac{z^*(s)}{\sqrt{s}}\, ds \\
&\le \sqrt{a}\, \|z\|_\infty + \tfrac{1}{2} \int_a^{\mu(\Omega)} \frac{z^*(s)}{\sqrt{s}}\, ds \\
&\le \|z\|_2 + \tfrac{1}{2} \left(\int_a^{\mu(\Omega)} \frac{ds}{s} \right)^{\frac{1}{2}} \left(\int_a^{\mu(\Omega)} \bigl(z^*(s)\bigr)^2\, ds \right)^{\frac{1}{2}} \\
&\le c_3 \left(1 + \sqrt{\log(\mu(\Omega)/a)} \right) \|z\|_2 .
\end{aligned}
$$

iii) Let $B_1, B_2, \ldots, B_N$ be the atoms of Ω arranged so that $z^*(n)$, the value of $|z|$ on B_n, is in non-increasing order. Also, let $z^*(N+1) = 0$. Then

$$
\begin{aligned}
\|z\|_{2,1} &= \sum_{n=1}^N \left(\sum_{m=1}^n \mu(B_m) \right)^{\frac{1}{2}} \bigl(z^*(n) - z^*(n+1) \bigr) \\
&\le \sqrt{N} \left(\sum_{n=1}^N \sum_{m=1}^n \mu(B_m) \bigl(z^*(n) - z^*(n+1) \bigr)^2 \right)^{\frac{1}{2}} \\
&\le \sqrt{N} \left(\sum_{n=1}^N \mu(B_n) \bigl(z^*(n) \bigr)^2 \right)^{\frac{1}{2}} \\
&= \sqrt{N}\, \|z\|_2 .
\end{aligned}
$$

as desired. □

We remark that Lemma 9(i) is a well known interpolation result, and is true for all measure spaces.

Lemma 10. *If $(\Omega, \mathcal{F}, \mu)$ is a probability space with Ω finite, then*

$$
\left(\sum_{s=1}^S \|y_s\|_{2,1}^2 \right)^{\frac{1}{2}} \le \left\| \sum_{s=1}^S |y_s| \right\|_\infty .
$$

Proof: This follows straight away from Lemma 9(i). □

Lemma 10 is also well known (and true for all probability spaces). In fact it is a reformulation of the statement that the canonical embedding $C(\Omega) \to L_{2,1}(\mu)$ has $(2,1)$-summing norm equal to 1.

Lemma 11. *There is a constant c_4 such that, if $(\Omega, \mathcal{F}, \mu)$ is a probability space with at most N atoms, then*

$$\left(\sum_{s=1}^{S} \|z_s\|_{2,1}^2 \right)^{\frac{1}{2}} \le c_4 \sqrt{\log N} \left\| \left(\sum_{s=1}^{S} |z_s|^2 \right)^{\frac{1}{2}} \right\|_\infty .$$

Proof: Let $A \subset \Omega$ be the union of those atoms of measure less than $\frac{1}{N^2}$, so that $\mu(A) \le \frac{1}{N}$. By Lemma 9(ii), we have that $\left\| z_s \chi_{\Omega \setminus A} \right\|_{2,1} \le c_3 \sqrt{\log N} \, \|z_s\|_2$, and by Lemma 9(iii), we have that $\|z_s \chi_A\|_{2,1} \le \sqrt{N} \, \|z_s \chi_A\|_2$. Thus, we have that

$$\begin{aligned} \left(\sum_{s=1}^{S} \|z_s\|_{2,1}^2 \right)^{\frac{1}{2}} &\le \left(\sum_{s=1}^{S} \left\| z_s \chi_{\Omega \setminus A} \right\|_{2,1}^2 \right)^{\frac{1}{2}} + \left(\sum_{s=1}^{S} \|z_s \chi_A\|_{2,1}^2 \right)^{\frac{1}{2}} \\ &\le c_3 \sqrt{\log N} \left(\sum_{s=1}^{S} \|z_s\|_2^2 \right)^{\frac{1}{2}} + \sqrt{N \, \mu(A)} \left\| \left(\sum_{s=1}^{S} |z_s|^2 \right)^{\frac{1}{2}} \right\|_\infty \\ &\le c_4 \sqrt{\log N} \left\| \left(\sum_{s=1}^{S} |z_s|^2 \right)^{\frac{1}{2}} \right\|_\infty , \end{aligned}$$

as desired. □

Proof of Proposition 7: Without loss of generality, we may suppose that $N = 2^{2^k}$. We prove the result by induction over k. Suppose that Ω has $2^{2^{k+1}}$ atoms. Apply Lemma 8 to cover Ω by 2^{2^k} subsets, and to give $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_S, y_1, y_2, \ldots, y_S, z_1, z_2, \ldots, z_S$ as described in the lemma. Then, by the triangle inequality

$$\left(\sum_{s=1}^{S} \|x_s\|_{2,1}^2 \right)^{\frac{1}{2}} \le \left(\sum_{s=1}^{S} \|\bar{x}_s\|_{2,1}^2 \right)^{\frac{1}{2}} + \left(\sum_{s=1}^{S} \|y_s\|_{2,1}^2 \right)^{\frac{1}{2}} + \left(\sum_{s=1}^{S} \|z_s\|_{2,1}^2 \right)^{\frac{1}{2}} .$$

By the induction hypothesis,

$$\left(\sum_{s=1}^{S} \|\bar{x}_s\|_{2,1}^2 \right)^{\frac{1}{2}} \le c_2 k.$$

By Lemmas 10 and 11 we have that

$$\left(\sum_{s=1}^{S} \|y_s\|_{2,1}^2\right)^{\frac{1}{2}} \leq c_1$$

and

$$\left(\sum_{s=1}^{S} \|z_s\|_{2,1}^2\right)^{\frac{1}{2}} \leq c_1 c_4\, 2^{-\frac{k}{2}} \left(1 + \sqrt{\log 2^{2^k}}\right) \leq 2c_1 c_4.$$

Hence

$$\left(\sum_{s=1}^{S} \|x_s\|_{2,1}^2\right)^{\frac{1}{2}} \leq c_2(k+1),$$

as required, taking $c_2 = 1 + 2c_1c_4$. □

To prove the main result is now easy.

Proof of Theorem 2: By Theorem 5, it is sufficient to show that for any probability measure μ on $\{1,\ 2, \ldots,\ N\}$, the cotype 2 constant of the canonical embedding $l_\infty^N \to L_{2,1}(\mu)$ is bounded by some universal constant times $\log\log N$. But this is precisely what Proposition 7 says. □

Final Remarks

There is a similar result for Gaussian cotype (see [Mo2]).

Theorem 12. *There is a constant c such that, for any operator $T : l_\infty^N \to Y$, where Y is a Banach space, the Gaussian cotype 2 constant, $\beta^{(2)}(T)$, is bounded according to the relation:*

$$\beta^{(2)}(T) \leq c\sqrt{\log\log N}\, \pi_{2,1}(T).$$

This result is the best possible, as is implicitly shown in [T].

Theorem 13. *There is a constant c such that for any integer N, there is an operator $T : l_\infty^N \to Y$, where Y is a Banach space, such that*

$$\beta^{(2)}(T) \geq c^{-1}\sqrt{\log\log N}\, \pi_{2,1}(T).$$

Since the Rademacher cotype 2 constant is greater than a constant times the Gaussian cotype 2 constant, we have the following corollary.

Corollary. *There is a constant c such that for any integer N, there is an operator $T : l_\infty^N \to Y$, where Y is a Banach space, such that*

$$K^{(2)}(T) \geq c^{-1} \sqrt{\log \log N}\, \pi_{2,1}(T).$$

We also have the following, the result originally stated in [T].

Corollary. *There is an operator $T : C(K) \to Y$, where Y is a Banach space, that is $(2,1)$-summing, but does not have Rademacher cotype 2.*

If we write R_N for the supremum of $K^{(2)}(T)/\pi_{2,1}(T)$ over all $T : l_\infty^N \to Y$, then we have shown that $c^{-1}\sqrt{\log \log N} \leq R_N \leq c \log \log N$. Clearly, we are left with the following problem.

Open Question. *What is the asymptotic behavior of R_N?*

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