The $p^{\frac{1}{p}}$ in Pisier's Factorization Theorem

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Abstract

We show that the constants in Pisier's factorization theorem for $(p,1)$-summing operators from $C(\Omega)$ cannot be improved.

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A theorem of Pisier (see [3]) states the following.

Theorem 1. *Let $T : C(\Omega) \to X$ be a bounded linear operator, where Ω is a compact Hausdorff topological space and X is a Banach space. Then the following are equivalent.*

i) T is $(p,1)$-summing.

ii) There is a constant $C < \infty$ and a Radon probability measure μ on Ω such that for all $f \in C(\Omega)$ we have

$$\|Tf\| \le C \, \|f\|_{L_1(\mu)}^{\frac{1}{p}} \, \|f\|_\infty^{1-\frac{1}{p}} . \tag{1}$$

iii) There is a constant $C < \infty$ and a Radon probability measure μ on Ω such that for all $f \in C(\Omega)$ we have

$$\|Tf\| \le C \, \|f\|_{L_{p,1}(\mu)} . \tag{2}$$

If we examine the proof of this theorem carefully, we can deduce the following. Let $\pi_{p,1}(T)$ denote the $(p,1)$-summing norm of T.

Theorem 2. *Let $T : C(\Omega) \to X$ be a bounded linear operator as in Theorem 1 that is $(p,1)$-summing. Then we can say the following.*

i) The set of C and μ that satisfy (1) coincide with the set of C and μ that satisfy (2).

ii) $\pi_{p,1}(T) \le C$ for all C satisfying (1) or (2).

iii) We can choose $C = p^{\frac{1}{p}} \, \pi_{p,1}(T)$ in (1) and (2).

Part (i) of this theorem is straight forward to demonstrate (we merely note that one needs the inequality: $\|f\|_{p,1} \le \|f\|_1^{\frac{1}{p}} \, \|f\|_\infty^{1-\frac{1}{p}}$). Part (ii) is also easy to verify. However, to show part (iii), the main part of Pisier's theorem, is much harder. Furthermore, the methods do not indicate whether the $p^{\frac{1}{p}}$ factor can be made smaller, or even completely removed (that is, replaced by 1).

The purpose of this paper is to show the rather surprising fact that the $p^{\frac{1}{p}}$ factor cannot be reduced at all.

Theorem 3. *Given $\epsilon > 0$, there is an operator $T : C(\Omega) \to X$ such that for any Radon probability measure μ on Ω, if C is the least number satisfying (1) or (2), then $C \geq p^{\frac{1}{p}}\, \pi_{p,1}(T)\,(1-\epsilon)$.*

Construction. Let $1 \leq S \leq N$ be integers, and let Ω be the collection of S-subsets of $\{1,\, 2, \dots,\, N\}$. For each $1 \leq n \leq N$, let $\Omega_n = \{\,\omega \in \Omega : n \in \omega\}$. We note the following facts for later on:

$$|\Omega_n| = \binom{N-1}{S-1}, \tag{3}$$

$$\frac{1}{N}\sum_{n=1}^{N} \chi_{\Omega_n} = \frac{S}{N}\,\chi_\Omega. \tag{4}$$

We give Ω the discrete topology, and define a norm $\|\cdot\|_*$ on $C(\Omega)$ by

$$\|f\|_* = \sup_{1\leq n\leq N} \sum_{\omega\in\Omega_n} |f(\omega)|\,.$$

We let T be the canonical embedding

$$T : \big(C(\Omega), \|\cdot\|_\infty\big) \to \big(C(\Omega), \|\cdot\|_*\big).$$

Lemma 4. *For $1 \leq p < \infty$, the $(p,1)$-summing norm of T may be estimated by*

$$\pi_{p,1}(T) \leq \frac{N^{S-1+\frac{1}{p}}}{(S-1)!\,(Sp-p+1)^{\frac{1}{p}}}.$$

In order to show Lemma 4, we will need two more lemmas.

Lemma 5. *If $1 \leq p < \infty$, and $T : C(\Omega) \to X$ is a bounded linear operator, where X is a Banach space, then the $(p,1)$-summing norm of T may be calculated by the formula*

$$\pi_{p,1}(T) = \sup\left\{\left(\sum_{s=1}^{S} \|Tf_s\|^p\right)^{\frac{1}{p}}\right\},$$

where the supremum is over all sequences $f_1,\ f_2, \dots,\ f_S$ of disjoint elements of the unit ball of $C(\Omega)$.

Proof. See [2], Lemma 6 or [1], Proposition 14.4. $\square$

Lemma 6. *If $1 \leq n \leq N$, then*

$$|\Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_n| \leq \frac{1}{(S-1)!} \sum_{m=1}^{n} (N-m)^{S-1}.$$

Proof. A simple counting argument shows that

$$|\Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_m \setminus \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_{m-1}| = \binom{N-m}{S-1},$$

and this is bounded by $(N-m)^{S-1}/(S-1)!$. □

Proof of Lemma 4. By Lemma 5, it is easy to see that

$$\pi_{p,1}(T) = \sup \left\{ \left(\sum_{n=1}^{N} \|\chi_{B_n}\|_*^p \right)^{\frac{1}{p}} \right\} = \sup \left\{ \left(\sum_{n=1}^{N} |B_n|^p \right)^{\frac{1}{p}} \right\},$$

where the supremum is over disjoint sets $B_1, B_2, \ldots, B_N \subseteq \Omega$ such that $B_n \subseteq \Omega_n$ for each $1 \leq n \leq N$. Since $\Omega_1, \Omega_2, \ldots, \Omega_N$ interact with one another in a completely symmetric fashion, we may assume, without loss of generality, that $|B_1| \geq |B_2| \geq \ldots \geq |B_N|$.

Now

$$\sum_{n=1}^{N} |B_n|^p = \sum_{n=1}^{N} \left(\sum_{m=1}^{n} |B_m| \right) \left(|B_n|^{p-1} - |B_{n+1}|^{p-1} \right).$$

(We take $B_{N+1} = \emptyset$.) Since $|B_n|^{p-1} - |B_{n+1}|^{p-1} \geq 0$, and $B_1 \cup B_2 \cup \ldots \cup B_n \subseteq \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_n$, we have, by Lemma 6, that

$$\begin{aligned} \sum_{n=1}^{N} |B_n|^p &\leq \frac{1}{(S-1)!} \sum_{n=1}^{N} \left(\sum_{m=1}^{n} (N-m)^{S-1} \right) \left(|B_n|^{p-1} - |B_{n+1}|^{p-1} \right) \\ &= \frac{1}{(S-1)!} \sum_{n=1}^{N} (N-n)^{S-1} |B_n|^{p-1}. \end{aligned}$$

Now, applying Hölder's inequality and dividing, we deduce

$$\left(\sum_{n=1}^{N} |B_n|^p \right)^{\frac{1}{p}} \leq \frac{1}{(S-1)!} \left(\sum_{n=1}^{N} (N-n)^{Sp-p} \right)^{\frac{1}{p}}.$$

Finally, we estimate the last quantity by an integral, and derive

$$\left(\sum_{n=1}^{N} |B_n|^p\right)^{\frac{1}{p}} \leq \frac{1}{(S-1)!}\left(\int_0^N x^{Sp-p}\,dx\right)^{\frac{1}{p}} \leq \frac{N^{S-1+\frac{1}{p}}}{(S-1)!\,(Sp-p+1)^{\frac{1}{p}}},$$

as desired. □

Proof of Theorem 3. By the hypothesis on C, there is a probability measure μ on Ω such that inequality (1) holds. In particular, if we substitute $f = \chi_{\Omega_n}$, we deduce that

$$|\Omega_n|^p = \|\chi_{\Omega_n}\|_*^p \leq C^p \int \chi_{\Omega_n}\,d\mu.$$

Hence

$$\frac{1}{N}\sum_{n=1}^{N} |\Omega_n|^p \leq C^p \int \frac{1}{N}\sum_{n=1}^{N} \chi_{\Omega_n}\,d\mu,$$

and so by equalities (3) and (4) we have

$$C \geq \frac{N^{\frac{1}{p}}\binom{N-1}{S-1}}{S^{\frac{1}{p}}}.$$

Thus, by Lemma 4, we deduce

$$C \geq \frac{N^{\frac{1}{p}}\binom{N-1}{S-1}(S-1)!\,(Sp-p+1)^{\frac{1}{p}}}{S^{\frac{1}{p}}N^{S-1+\frac{1}{p}}}\,\pi_{p,1}(T).$$

Choosing N much larger than S, we find that

$$C \geq \left(\frac{Sp-p+1}{S}\right)^{\frac{1}{p}}\,\pi_{p,1}(T)\,(1-\tfrac{1}{2}\epsilon).$$

Finally, choosing S large, we have the desired result, that is, $C \geq p^{\frac{1}{p}}\,\pi_{p,1}(T)\,(1-\epsilon)$. □

References

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