# p-Summing Operators on Injective Tensor Products of Spaces

by

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**Abstract** Let $X, Y$ and $Z$ be Banach spaces, and let $\prod_p(Y,Z)$ $(1 \le p < \infty)$ denote the space of $p$-summing operators from $Y$ to $Z$. We show that, if $X$ is a $\mathcal{L}_\infty$-space, then a bounded linear operator $T:\ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is 1-summing if and only if a naturally associated operator $T^\#:\ X \longrightarrow \prod_1(Y,Z)$ is 1-summing. This result need not be true if $X$ is not a $\mathcal{L}_\infty$-space. For $p>1$, several examples are given with $X = C[0,1]$ to show that $T^\#$ can be $p$-summing without $T$ being $p$-summing. Indeed, there is an operator $T$ on $C[0,1]\hat{\otimes}_\epsilon \ell_1$ whose associated operator $T^\#$ is 2-summing, but for all $N \in \mathbf{N}$, there exists an $N$-dimensional subspace $U$ of $C[0,1]\hat{\otimes}_\epsilon \ell_1$ such that $T$ restricted to $U$ is equivalent to the identity operator on $\ell_\infty^N$. Finally, we show that there is a compact Hausdorff space $K$ and a bounded linear operator $T:\ C(K)\hat{\otimes}_\epsilon \ell_1 \longrightarrow \ell_2$ for which $T^\#:\ C(K) \longrightarrow \prod_1(\ell_1,\ell_2)$ is not 2-summing.

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**Introduction** Let $X$ and $Y$ be Banach spaces, and let $X\hat{\otimes}_\epsilon Y$ denote their injective tensor product. In this paper, we shall study the behavior of those operators on $X\hat{\otimes}_\epsilon Y$ that are $p$-summing.

If $X$, $Y$ and $Z$ are Banach spaces, then every $p$-summing operator $T:\ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ induces a $p$-summing linear operator $T^\#:\ X \longrightarrow \prod_p(Y,Z)$. This raises the following question: given two Banach spaces $Y$ and $Z$, and $1 \le p < \infty$, for what Banach spaces $X$ is it true that a bounded linear operator $T:\ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is $p$-summing whenever $T^\#:\ X \longrightarrow \prod_p(Y,Z)$ is $p$-summing?

In [11], it was shown that whenever $X = C(\Omega)$ is a space of all continuous functions on a compact Hausdorff space $\Omega$, then $T:\ C(\Omega)\hat{\otimes}_\epsilon Y \longrightarrow Z$ is 1-summing if and only if $T^\#:\ C(\Omega) \longrightarrow \prod_1(Y,Z)$ is 1-summing. We will extend this result by showing that this result still remains true if $X$ is any $\mathcal{L}_\infty$-space. We will also give an example to show that the result need not be true if $X$ is not a $\mathcal{L}_\infty$-space. For this, we shall exhibit a 2-summing operator $T$ on $\ell_2\hat{\otimes}_\epsilon\ell_2$ that is not 1-summing, but such that the associated operator $T^\#$ is 1-summing.

The case $p > 1$ turns out to be quite different. Here, the $\mathcal{L}_\infty$-spaces do not seem to play any important role. We show that for each $1 < p < \infty$, there exists a bounded linear operator $T:\ C[0,1]\hat{\otimes}_\epsilon\ell_2 \longrightarrow \ell_2$ such that $T^\# : C[0,1] \longrightarrow \prod_p(\ell_2,\ell_2)$ is $p$-summing, but such that $T$ is not $p$-summing. We will also give an example that shows that, in general, the condition on $T^\#$ to be 2-summing is too weak to imply any good properties for the operator $T$ at all. To illustrate this, we shall exhibit a bounded linear operator $T$ on $C[0,1]\hat{\otimes}_\epsilon\ell_1$ with values in a certain Banach space $Z$, such that $T^\#:\ C[0,1] \longrightarrow \prod_2(\ell_1,Z)$ is 2-summing, but for any given $N \in \mathbf{N}$, there exists a subspace $U$ of $C[0,1]\hat{\otimes}_\epsilon\ell_1$, with $\dim U = N$, such that $T$ restricted to $U$ is equivalent to the identity operator on $\ell_\infty^N$.

Finally, we show that there is a compact Hausdorff space $K$ and a bounded linear operator $T:\ C(K)\hat{\otimes}_\epsilon\ell_1 \longrightarrow \ell_2$ for which $T^\#:\ C(K) \longrightarrow \prod_1(\ell_1,\ell_2)$ is not 2-summing.

## I - Definitions and Preliminaries

Let $E$ and $F$ be Banach spaces, and let $1 \leq q \leq p < \infty$. An operator $T: \ E \longrightarrow F$ is said to be $(p,q)$-**summing** if there exists a constant $C \geq 0$ such that for any finite sequence $e_1, e_2, \dots, e_n$ in $E$, we have

$$\left(\sum_{i=1}^{n} \| T(e_i) \|^p\right)^{\frac{1}{p}} \leq C \sup \left\{ \left(\sum_{i=1}^{n} |e^*(e_i)|^q\right)^{\frac{1}{q}} : e^* \in E^*, \| e^* \| \leq 1 \right\}.$$

We let $\pi_{p,q}(T)$ denote the smallest constant $C$ such that the above inequality holds, and let $\prod_{p,q}(E,F)$ be the space of all $(p,q)$-summing operators from $E$ to $F$ with the norm $\pi_{p,q}$. It is easy to check that $\prod_{p,q}(E,F)$ is a Banach space. In the case $p=q$, we will simply write $\prod_{p}(E,F)$ and $\pi_p$. We will use the fact that $T \in \prod_{p,q}(E,F)$ if and only if $\sum\limits_{n} \|Te_n\|^p < \infty$ for every infinite sequence $(e_n)$ in $E$ with $\sum\limits_{n} |e^*(e_n)|^q < \infty$ for each $e^* \in E^*$. That is to say, $T$ is in $\prod_{p,q}(E,F)$ if and only if $T$ sends all weakly $\ell_q$-summable sequences into strongly $\ell_p$-summable sequences. In what follows we shall mainly be interested in the case where $p=q$ and $p=1$ or 2.

Given two Banach spaces $E$ and $F$, we will let $E\hat{\otimes}_\epsilon F$ denote their injective tensor product, that is, the completion of the algebraic tensor product $E \otimes F$ under the cross norm $\| \cdot \|_\epsilon$ given by the following formula. If $\sum\limits_{i=1}^{n} e_i \otimes x_i \in E \otimes F$, then

$$\| \sum_{i=1}^{n} e_i \otimes x_i \|_\epsilon = \sup \left\{ \left| \sum_{i=1}^{n} e^*(e_i) x^*(x_i) \right| : \| e^* \| \leq 1, \| x^* \| \leq 1, e^* \in E^*, x^* \in F^* \right\}.$$

We will say that a bounded linear operator $T$ between two Banach spaces $E$ and $F$ is called an **integral operator** if the bilinear form $\tau$ defines an element of $(E\hat{\otimes}_\epsilon F^*)^*$, where $\tau$ is induced by $T$ according to the formula $\tau(e,x^*) = x^*(Te)$ $(e \in E,\ x^* \in F^*)$. We will define the **integral norm** of $T$, denoted by $\| T \|_{\text{int}}$, by

$$\| T \|_{\text{int}} = \sup \left\{ \left| \sum_{i=1}^{n} x_i^*(Te_i) \right| : \| \sum_{i=1}^{n} e_i \otimes x_i^* \|_\epsilon \leq 1 \right\}.$$

The space of all integral operators from a Banach space $E$ into a Banach space $F$ will be denoted by $I(E, F)$. We note that $I(E, F)$ is a Banach space under the integral norm $\| \quad \|_{\text{int}}$.

We will say that a Banach space $X$ is a $\mathcal{L}_\infty$-**space** if, for some $\lambda > 1$, we have that for every finite dimensional subspace $B$ of $X$, there exists a finite dimensional subspace $E$ of $X$ containing $B$, and an invertible bounded linear operator $T: \ E \longrightarrow \ell_\infty^{\dim E}$ such that $\| T \| \| T^{-1} \| \leq \lambda$.

It is well known that for any Banach spaces $E$ and $F$, if $T$ is in $I(E, F)$, then it is also in $\prod_1(E, F)$, with $\pi_1(T) \leq \| T \|_{\text{int}}$. But $I(E, F)$ is strictly included in $\prod_1(E, F)$. It was shown in [12, p. 477] that a Banach space $E$ is a $\mathcal{L}_\infty$-space if and only if for any Banach space $F$, we have that $I(E, F) = \prod_1(E, F)$. We will use this characterization of $\mathcal{L}_\infty$-spaces in the sequel.

Finally, we note the following characterization of 1-summing operators (called right semi-integral by Grothendieck in [5]), which will be used later.

**Proposition 1** Let $E$ and $F$ be Banach spaces. Then the following properties about a bounded linear operator $T$ from $E$ to $F$ are equivalent:

(i) $T$ is 1-summing;

(ii) There exists a Banach space $F_1$, and an isometric injection $\varphi: \ F \longrightarrow F_1$, such that $\varphi \circ T: \ E \longrightarrow F_1$ is an integral operator.

For all other undefined notions we shall refer the reader to either [3], [7] or [10].

**II 1-Summing and Integral Operators**

Let $X$ and $Y$ be Banach spaces with injective tensor product $X\hat{\otimes}_\epsilon Y$. For a Banach space $Z$, any bounded linear operator $T:\ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ induces a linear operator $T^{\#}$ on $X$ by

$$T^{\#}x(y) = T(x \otimes y) \qquad (y \in Y).$$

It is clear that the range of $T^{\#}$ is the space $\pounds(Y, Z)$ of bounded linear operators from $Y$ into $Z$, and that $T^{\#}$ is a bounded linear operator.

In this section, we are going to investigate the 1-summing operators, and the integral operators, on $X\hat{\otimes}_\epsilon Y$. We will use Proposition 1 to relate these two ideas together. First of all, we have the following result.

**Theorem 2** Let $X, Y$ and $Z$ be Banach spaces, and let $T:\ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ be a bounded linear operator. Denote by $i:\ Z \longrightarrow Z^{**}$ the isometric embedding of $Z$ into $Z^{**}$. Then the following two properties are equivalent:

(i) $T \in I(X\hat{\otimes}_\epsilon Y, Z)$;

(ii) $\hat{i} \circ T \in I(X, I(Y, Z^{**}))$, where $\hat{i}:\ I(Y, Z) \longrightarrow I(Y, Z^{**})$ is defined by $\hat{i}(U) = i \circ U$ for each $U \in I(Y, Z)$.

In particular, if $T^{\#} \in I(X, I(Y, Z))$, then $T \in I(X\hat{\otimes}_\epsilon Y, Z)$.

**Proof:** First, we show that $(X\hat{\otimes}_\epsilon Y)\hat{\otimes}_\epsilon Z^*$ and $X\hat{\otimes}_\epsilon(Y\hat{\otimes}_\epsilon Z^*)$ are isometrically isomorphic to one another. To see this, note that the algebraic tensor product is an associative operation, that is, $(X \otimes Y) \otimes Z^*$ and $X \otimes (Y \otimes Z^*)$ are algebraically isomorphic. Also, they are both generated by elements of the form $\sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^*$, where $x_i \in X,\ y_i \in Y$ and $z_i^* \in Z^*$. Now, if we let $B(X^*),\ B(Y^*)$ and $B(Z^{**})$ denote the dual unit balls of $X^*,\ Y^*$ and $Z^{**}$ equipped with their respective weak$^*$ topologies, then the spaces $(X \otimes_\epsilon Y) \otimes_\epsilon Z^*$ and $X \otimes_\epsilon (Y \otimes_\epsilon Z^*)$ embed isometrically into $C\left(B(X^*) \times B(Y^*) \times B(Z^{**})\right)$ in a natural way, by

$$\langle \sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^*, \quad (x^*, y^*, z^{**})\rangle = \sum_{i=1}^{n} x^*(x_i) y^*(y_i) z^{**}(z_i^*),$$

where $\sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^*$ is in $(X \otimes_\epsilon Y) \otimes_\epsilon Z^*$ or $X \otimes_\epsilon (Y \otimes_\epsilon Z^*)$, and $(x^*, y^*, z^{**})$ is in the compact set $B(X^*) \times B(Y^*) \times B(Z^{**})$. Thus both spaces $(X \hat{\otimes}_\epsilon Y) \hat{\otimes}_\epsilon Z^*$ and $X \hat{\otimes}_\epsilon (Y \hat{\otimes}_\epsilon Z^*)$ can be thought of as the closure in $C\left(B(X^*) \times B(Y^*) \times B(Z^{**})\right)$ of the algebraic tensor product of $X$, $Y$ and $Z^*$.

Now let us assume that $T: \ X \hat{\otimes}_\epsilon Y \longrightarrow Z$ is an integral operator. Then the bilinear map $\tau$ on $X \hat{\otimes}_\epsilon Y \times Z^*$, given by $\tau(u, z^*) = z^*(Tu)$ for $u \in X \hat{\otimes}_\epsilon Y$ and $z^* \in Z^*$, defines an element of $\left(X \hat{\otimes}_\epsilon Y \hat{\otimes}_\epsilon Z^*\right)^*$, that is,

$$(*) \qquad \| T \|_{\text{int}} = \sup \left\{ |\sum_{i=1}^{n} z_i^* \left(T(x_i \otimes y_i)\right) : \ \| \sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^* \|_\epsilon \leq 1 \right\}.$$

To show that for every $x$ in $X$ the operator $T^{\#}x$ is in $I(Y, Z)$, with

$$\| T^{\#}x \|_{\text{int}} \leq \| x \| \, \| T \|_{\text{int}},$$

is easy. This is because, for each $x \in X$, the operator $T^{\#}x$ is the composition of $T$ with the bounded linear operator from $Y$ to $X \hat{\otimes}_\epsilon Y$, which to each $y$ in $Y$ gives the element $x \otimes y$.

If $i: \ Z \longrightarrow Z^{**}$ denotes the isometric embedding of $Z$ into $Z^{**}$, it induces a bounded linear operator $\hat{i}: \ I(Y, Z) \longrightarrow I(Y, Z^{**})$ given by $\hat{i}(U) = i \circ U$ for all $U \in I(Y, Z)$. It is immediate that $\hat{i}$ is an isometry. We will now show that the operator $\hat{i} \circ T^{\#}: X \longrightarrow I(Y, Z^{**})$ is integral. It is well known (see [3, p. 237]) that the space $I(Y, Z^{**})$ is isometrically isomorphic to the dual space $(Y \hat{\otimes}_\epsilon Z^*)^*$. Thus to show that $\hat{i} \circ T^{\#}: X \longrightarrow (Y \hat{\otimes}_\epsilon Z^*)^*$ is an integral operator, we need to show that it induces an element of $\left(X \hat{\otimes}_\epsilon (Y \hat{\otimes}_\epsilon Z^*)\right)^*$. For this, it is enough to note that, by our discussion concerning the isometry of $(X \hat{\otimes}_\epsilon Y) \hat{\otimes}_\epsilon Z^*$ and $X \hat{\otimes}_\epsilon (Y \hat{\otimes}_\epsilon Z^*)$, that

$$(**) \qquad \| \hat{i} \circ T^{\#} \|_{\text{int}} = \sup \left\{ |\sum_{i=1}^{n} \hat{i} \circ T^{\#} x_i, y_i \otimes z_i^* | : \| \sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^* \|_\epsilon \leq 1 \right\}.$$

But for each $x \in X$, $y \in Y$ and $z^* \in Z^*$, we have

$$\langle \hat{i} \circ T^{\#} x, y \otimes z^* \rangle = \langle T(x \otimes y), z^* \rangle.$$

Hence, from (*) and (**), it follows that

$$\| \hat{i} \circ T \|_{\text{int}} = \| T \|_{\text{int}} .$$

Thus we have shown that (i) $\Rightarrow$ (ii). The proof of (ii) $\Rightarrow$ (i) follows in a similar way. If $\hat{i} \circ T^{\#} : X \longrightarrow I(Y, Z^{**})$ is an integral operator, then one can show that $i \circ T : X \hat{\otimes}_\epsilon Y \longrightarrow Z^{**}$ is integral, which in turn implies that $T$ itself is integral (see [3, p. 233]).

Finally, the last assertion follows easily, since if $T^{\#} : X \longrightarrow I(Y, Z)$ is integral, then $\hat{i} \circ T$ is integral (see [3, p. 232]). □

Since the mapping $\hat{i} : I(Y, Z) \longrightarrow I(Y, Z^{**})$ is an isometry, Proposition 1 coupled with Theorem 2 implies that, if $T : X \hat{\otimes}_\epsilon Y \longrightarrow Z$ is an integral operator, then $T^{\#} : X \longrightarrow I(Y, Z)$ is 1-summing. This result can be shown directly from the definitions. In what follows we shall present a sketch of that alternative approach.

**Theorem 3** Let $X$, $Y$ and $Z$ be Banach spaces, and let $T : X \hat{\otimes}_\epsilon Y \longrightarrow Z$ be a bounded linear operator. If $T$ is integral, then $T^{\#} : X \longrightarrow I(Y, Z)$ is 1-summing. If in addition $X$ is a $\mathcal{L}_\infty$-space, then $T : X \hat{\otimes}_\epsilon Y \longrightarrow Z$ is integral if and only if $T^{\#} : X \longrightarrow I(Y, Z)$ is integral.

**Proof:** First, we will show that, if $T : X \hat{\otimes}_\epsilon Y \longrightarrow Z$ is an integral operator, then $T^{\#}$ is in $\prod_1 (X, I(Y, Z))$ with $\pi_1(T^{\#}) \leq \| T \|_{\text{int}}$. Let $x_1, x_2, \ldots, x_n$ be in $X$, and fix $\epsilon > 0$. For each $i \leq n$, there exists $n_i \in \mathbf{N}$, $(y_{ij})_{j \leq n_i}$ in $Y$, and $(z^*_{ij})_{j \leq n_i}$ in $Z^*$, such that $\| \sum_{j=1}^{ni} y_{ij} \otimes z^*_{ij} \|_\epsilon \leq 1$, and

$$\| T^{\#} x_i \|_{\text{int}} \leq \sum_{j=1}^{n_i} z^*_{ij} \left( T(x_i \otimes y_{ij}) \right) + \frac{\epsilon}{2^i}.$$

Since $T$ is an integral operator, and

$$\| \sum_{i=1}^{n}\sum_{j=1}^{n_i} x_i \otimes y_{ij} \otimes z_{ij}^* \|_\epsilon \leq \sup \left\{ \sum_{i=1}^{n} |x^*(x_i)| : \| x^* \| \leq 1, x^* \in X^* \right\},$$

it follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{n_i} z_{ij}^* \left(T(x_i \otimes y_{ij})\right) \leq \| T \|_{\text{int}} \ \sup \left\{ \sum_{i=1}^{n} |x^*(x_i)| : \| x^* \| \leq 1, \ x^* \in X^* \right\}.$$

Therefore

$$\sum_{i=1}^{n} \| T^{\#} x_i \|_{\text{int}} \leq \| T \|_{\text{int}} \ \sup \left\{ \sum_{i=1}^{n} |x^*(x_i)| : \ x^* \in X^*, \| x^* \| \leq 1 \right\} + \epsilon.$$

Now, if in addition $X$ is a $\pounds_\infty$-space, then by [12, p. 477], the operator $T^{\#}$ is indeed integral. □

**Remark 4** If $X = C(\Omega)$ is a space of continuous functions defined on a compact Hausdorff space $\Omega$, one can deduce a similar result to Theorem 3 from the main result of [13].

Our next result extends a result of [16] to $\pounds_\infty$-spaces, where it was shown that whenever $X = C(\Omega)$, a space of all continuous functions on a compact Hausdorff space $\Omega$, then a bounded linear operator $T : \ C(\Omega)\hat{\otimes}_\epsilon Y \longrightarrow Z$ is 1-summing if and only if $T^{\#} : \ C(\Omega) \longrightarrow \prod_1(Y, Z)$ is 1-summing. This also extends a result of [14] where similar conclusions were shown to be true for $X = A(K)$, a space of continuous affine functions on a Choquet simplex $K$ (see [2]).

We note that one implication follows with no restriction on $X$. If $X$, $Y$ and $Z$ are Banach spaces, and $T : \ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is a 1-summing operator, then $T^{\#}$ takes its values in $\prod_1(Y, Z)$. This follows from the fact that for each $x \in X$, the operator $T^{\#}x$ is the composition of $T$ with the bounded linear operator from $Y$ into $X\hat{\otimes}_\epsilon Y$ which to each $y$ in $Y$ gives the element $x \otimes y$ in $X\hat{\otimes}_\epsilon Y$, and hence

$$\pi_1(T^{\#}x) \leq \| x \| \ \pi_1(T).$$

Moreover, one can proceed as in [16] to show that $T^{\#}: X \longrightarrow \prod_1(Y,Z)$ is 1-summing.

**Theorem 5** If $X$ is a $\mathcal{L}_\infty$ space, then for any Banach spaces $Y$ and $Z$, a bounded linear operator $T: X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is 1-summing if and only if $T^{\#}: X \longrightarrow \prod_1(Y,Z)$ is 1-summing.

**Proof:** Let $T: X\hat{\otimes}_\epsilon Y \longrightarrow Z$ be such that $T^{\#}: X \longrightarrow \prod_1(Y,Z)$ is 1-summing. Since $X$ is a $\mathcal{L}_\infty$-space, it follows from [14, p. 477] that $T^{\#}: X \longrightarrow \prod_1(Y,Z)$ is an integral operator. Let $\varphi$ denote the isometric embedding of $Z$ into $C(B(Z^*))$, the space of all continuous scaler functions on the unit ball $B(Z^*)$ of $Z^*$ with its weak$^*$-topology. This induces an isometry

$$\hat{\varphi}: \prod_1(Y,Z) \longrightarrow \prod_1((Y, C(B(Z^*))),$$

$$\hat{\varphi}(U) = \varphi \circ U \qquad \text{for all } U \in \prod_1(Y,Z).$$

Now, it follows from [15, p. 301], that $\prod_1(Y, C(B(Z^*)))$ is isometric to $I(Y, C(B(Z^*)))$. Hence we may assume that $\hat{\varphi} \circ T^{\#}: X \longrightarrow I(Y, C(B(Z^*)))$ is an integral operator. Moreover, it is easy to check that $(\varphi \circ T)^{\#} = \hat{\varphi} \circ T^{\#}$. By Theorem 2 the operator $\varphi \circ T: X\hat{\otimes}_\epsilon Y \longrightarrow C(B(Z^*))$ is an integral operator, and hence $T$ is in $\prod_1\left(X\hat{\otimes}_\epsilon Y, Z\right)$ by Proposition 1. □

In the following section we shall, among other things, exhibit an example that illustrates that it is crucial for the space $X$ to be a $\mathcal{L}_\infty$-space if the conclusion of Theorem 5 is to be valid.

## III 2-summing Operators and some Counter-examples.

In this section we shall study the behavior of 2-summing operators on injective tensor product spaces. As we shall soon see, the behavior of such operators when $p = 2$ is quite different from when $p = 1$. For instance, unlike the case $p = 1$, the $\mathcal{L}_\infty$-spaces don't seem to play any particular role. In fact, we shall exhibit operators $T$ on $C[0,1]\hat{\otimes}_\epsilon \ell_2$ which are not 2-summing, yet their corresponding operators $T^{\#}$ are. We will also give other interesting examples that answer some other natural questions.

We will present the next theorem for $p = 2$, but the same result is true for any $1 \leq p < \infty$, with only minor changes.

**Theorem 6** Let $X, Y$ and $Z$ be Banach spaces. If $T: \ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is a 2-summing operator, then $T^\#: \ X \longrightarrow \prod_2(Y,Z)$ is a 2-summing operator.

**Proof:** If $T: \ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is 2-summing, then using the same kind of arguments that we have given above, it can easily be shown that for each $x \in X$, that $T^\# x \in \prod_2(Y,Z)$, with $\pi_2(T^\# x) \leq \pi_2(T) \parallel x \parallel$.

Now we will show that $T^\#: \ X \longrightarrow \prod_2(Y,Z)$ is 2-summing. Let $(x_n)$ be in $X$ such that $\sum_n |x^*(x_n)|^2 < \infty$ for each $x^*$ in $X^*$. Fix $\epsilon > 0$. For each $n \geq 1$, let $(y_{nm})$ be a sequence in $Y$ such that

$$\sup\left\{\left(\sum_{m=1}^{\infty} |y^*(y_{nm})|^2\right)^{1/2} : \parallel y^* \parallel \leq 1, y^* \in Y^*\right\} \leq 1,$$

and

$$\pi_2\left(T^\# x_n\right) \leq \left(\sum_{m=1}^{\infty} \parallel T(x_n \otimes y_{nm}) \parallel^2\right)^{1/2} + \frac{\epsilon}{2^n}.$$

Then

$$\left[\pi_2\left(T^\# x_n\right)\right]^2 \leq \sum_{m=1}^{\infty} \parallel T\left(x_n \otimes y_{nm}\right) \parallel^2 + \frac{\epsilon}{2^{n-1}}\left(\sum_{m=1}^{\infty} \parallel T\left(x_n \otimes y_{nm}\right) \parallel^2\right)^{1/2} + \frac{\epsilon^2}{2^{2n}}.$$

Now, consider the sequence $(x_n \otimes y_{nm})$ in $X\hat{\otimes}_\epsilon Y$. For each $\xi \in \left(X\hat{\otimes}_\epsilon Y\right)^* \simeq I(X, Y^*)$ we have that

$$\begin{aligned}\sum_{m,n} |\xi(x_n)(y_{nm})|^2 &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\xi(x_n)(y_{nm})|^2 \\ &\leq \sum_{n=1}^{\infty} \parallel \xi(x_n) \parallel^2 .\end{aligned}$$

Since $\xi \in I(X, Y^*)$, it follows that $\xi \in \prod_2(X, Y^*)$, and so

$$\sum_{n=1}^{\infty} \parallel \xi(x_n) \parallel^2 < \infty.$$

Hence we have shown that for all $\xi \in (X\hat{\otimes}_\epsilon Y)^*$,

$$\sum_{m,n} |\xi(x_n)(y_{nm})|^2 < \infty.$$

Since $T \in \prod_2 \left(X\hat{\otimes}_\epsilon Y, Z\right)$, we have that

$$\sum_{m,n} \| T(x_n \otimes y_{nm}) \|^2 < \infty,$$

and therefore

$$\sum_{n} \left[\pi_2 \left(T^{\#} x_n\right)\right]^2 < \infty.$$

□

**Remark 7** The above result extends a result of [1], where it was shown that if $T : X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is $p$-summing for $1 \leq p < \infty$, then $T^{\#} : X \longrightarrow \pounds(Y, Z)$ is $p$-summing.

Now we shall give the example that we promised at the end of section II.

**Theorem 8** There exists a bounded linear operator $T : \ell_2\hat{\otimes}_\epsilon\ell_2 \longrightarrow \ell_2$ such that $T$ is not 1-summing, yet $T^{\#} : \ell_2 \longrightarrow \pi_1(\ell_2, \ell_2)$ is 1-summing.

**Proof:** First, we note the well known fact that $\ell_2\hat{\otimes}_\epsilon\ell_2 = \mathcal{K}(\ell_2, \ell_2)$, the space of all compact operators from $\ell_2$ to $\ell_2$. Now we define $T$ as the composition of two operators.

Let $P : \mathcal{K}(\ell_2, \ell_2) \longrightarrow c_0$ be the operator defined so that for each $K \in \mathcal{K}(\ell_2, \ell_2)$,

$$P(K) = (K(e_n)(e_n)),$$

where $(e_n)$ is the standard basis of $\ell_2$. It is well known [10, p.145] that the sequence $(e_n \otimes e_n)$ in $\ell_2\hat{\otimes}_\epsilon\ell_2$ is equivalent to the $c_0$-basis, and that the operator $P$ defines a bounded linear projection of $\mathcal{K}(\ell_2, \ell_2)$ onto $c_0$.

Let $S : c_0 \longrightarrow \ell_2$ be the bounded linear operator such that for each $(\alpha_n) \in c_0$

$$S(\alpha_n) = \left(\frac{\alpha_n}{n}\right).$$

It is easily checked [7, p. 39] that $S$ is a 2-summing operator that is not 1-summing.

Now we define $T: \ \mathcal{K}(\ell_2, \ell_2) \longrightarrow \ell_2$ to be $T = S \circ P$. Thus $T$ is 2-summing but not 1-summing. It follows from Theorem 6 that the induced operator $T^{\#} : \ell_2 \longrightarrow \prod_2(\ell_2, \ell_2)$ is 2-summing. Since $\ell_2$ is of cotype 2, it follows from [10, p. 62], that for any Banach space $E$, we have $\prod_2(\ell_2, E) = \prod_1(\ell_2, E)$, and that there exists a constant $C > 0$ such that for all $U \in \prod_2(\ell_2, E)$ we have

$$\pi_1(U) \leq C\pi_2(U).$$

This implies that $T^{\#}$ is 1-summing as an operator taking its values in $\prod_1(\ell_2, \ell_2)$. □

**Remark 9** We do not need to use Theorem 6 to show that $T^{\#}$ is 1-summing in the example above. Instead, we can use the following argument. First note that $T^{\#}$ factors as follows:

$$\begin{array}{ccc} \ell_2 & \xrightarrow{T^{\#}} & \pi_1(\ell_2, \ell_2) \\ {\scriptstyle A}\downarrow & & \\ \ell_2 & \nearrow B & \end{array}$$

Here $A: \ \ell_2 \to \ell_2$ is the 1-summing operator defined by

$$A(\alpha_n) = \left(\frac{\alpha_n}{n}\right),$$

for each $(\alpha_n) \in \ell_2$, and $B: \ \ell_2 \longrightarrow \pi_1(\ell_2, \ell_2)$ is the natural embedding of $\ell_2$ into the space $\pi_1(\ell_2, \ell_2)$ defined by

$$B(\beta_n)(\gamma_n) = (\beta_n \gamma_n)$$

for each $(\beta_n)$, $(\gamma_n) \in \ell_2$.

Now we will give two examples concerning the case when $p > 1$. We will show that we do not have a converse to Theorem 8, even when the underlying space $X$ is a $\mathcal{L}_\infty$-space.

First, let us fix some notation. In what follows we shall denote the space $\ell_p(\mathbf{Z})$ by $\ell_p$, and call its standard basis $\{e_n : n \in \mathbf{Z}\}$. Thus if $x = (x(n)) \in \ell_p$, then $x(n) = \langle x, e_n \rangle$, and

$$\| x \|_{\ell_p} = \left( \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^p \rangle \right)^{\frac{1}{p}} .$$

For $f \in L_p[0,1]$, we let

$$\| f \|_{L_p} = \left( \int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}} .$$

If $\Omega$ is a compact Hausdorff space, and $Y$ is a Banach space, then $C(\Omega, Y) = C(\Omega) \hat{\otimes}_\epsilon Y$ will denote the Banach space of continuous $Y$-valued functions on $\Omega$ under the supremum norm.

We recall that since $\ell_2$ is of cotype 2, we have that $\prod_2(\ell_2, \ell_2) = \prod_1(\ell_2, \ell_2)$. We also recall that, if $u = \sum_{n=1}^{\infty} \alpha_n e_n \otimes e_n$ is a diagonal operator in $\prod_2(\ell_2, \ell_2)$, then

$$\pi_2(u) = \left( \sum_{n=1}^{\infty} |\alpha_n|^2 \right)^{\frac{1}{2}} = \text{ the Hilbert-Schmidt norm of } u.$$

**Theorem 10** For each $1 < p < \infty$, there is a bounded linear operator $T : C([0,1], \ell_2) \to \ell_2$ that is not $p$-summing, but such that $T^{\#} : C[0,1] \longrightarrow \Pi_p(\ell_2, \ell_2)$ is $p$-summing.

**Proof:** We present the proof for $p \leq 2$. The case where $p > 2$ follows by the same argument. For each $n \in \mathbf{Z}$, let $\epsilon_n(t) : [0,1] \to \mathbf{C}$, $\epsilon_n(t) = e^{2\pi \, \text{int}}$ denote the standard trigonometric basis of $L_2[0,1]$. If $f \in L_1[0,1]$, let $\hat{f}(n) = \int_0^1 f(t)\epsilon_n(t)dt$ denote the usual Fourier coefficient of $f$. For each $\lambda = (\lambda_n)$, where $|\lambda_n| \leq 1$ for all $n \in \mathbf{Z}$, define the operator

$$T_\lambda : C([0,1], \ell_2) \longrightarrow \ell_2$$

such that for $\varphi \in C([0,1], \ell_2)$ we have

$$T_\lambda \varphi = (\lambda_n \langle \hat{\varphi}(n), e_n \rangle ) .$$

Here $\hat{\varphi}(n) = $ Bochner $-\int_0^1 \varphi(t)\epsilon_n(t)dt$.

The operator $T_\lambda$ is a bounded linear operator, with $\| T_\lambda\varphi \|_{\ell_2} \leq \| \varphi \|$. To see this, note that for $\varphi \in C([0,1], \ell_2)$ we have

$$\begin{aligned}
\| T_\lambda\varphi \|_{\ell_2}^2 &= \sum_n |\lambda_n|^2 |\langle\hat{\varphi}(n), e_n\rangle|^2 \\
&\leq \sum_n |\langle\hat{\varphi}(n), e_n\rangle|^2 \\
&\leq \sum_n \int_0^1 |\langle\varphi(t), e_n\rangle|^2 dt \\
&= \int_0^1 \| \varphi(t) \|_{\ell_2}^2 \, dt \\
&\leq \sup_t \| \varphi(t) \|_{\ell_2}^2 .
\end{aligned}$$

Now, note that if $f \in C([0,1])$, and $x \in \ell_2$, then

$$T_\lambda(f \otimes x) = \left(\lambda_n \hat{f}(n)\langle x, e_n\rangle\right),$$

and hence the operator $T_\lambda^{\#} : C[0,1] \to \pounds(\ell_2, \ell_2)$ is such that

$$T_\lambda^{\#} f(x) = \left(\lambda_n \hat{f}(n)\langle x, e_n\rangle\right).$$

Thus

$$\pi_2(T_\lambda^{\#} f) = \left(\sum_n |\lambda_n|^2 |\hat{f}(n)|^2\right)^{\frac{1}{2}}.$$

Hence, by Hölder's inequality,

$$\pi_2(T_\lambda^{\#} f) \leq \| (\lambda_n) \|_{\ell_r} \| (\hat{f}(n)) \|_{\ell_q},$$

where $\dfrac{1}{r} + \dfrac{1}{q} = \dfrac{1}{2}$. By the Hausdorff-Young inequality, we have that

$$\| (\hat{f}(n)) \|_{\ell_q} \leq \| f \|_{L_p},$$

where $1 \leq p \leq 2$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Thus

$$\pi_2(T_\lambda^{\#} f) \leq \| (\lambda_n) \|_{\ell_r} \| f \|_{L_p},$$

for $1 \leq p \leq 2,\ 2 \leq r \leq \infty$ and $\dfrac{1}{p} = \dfrac{1}{r} + \dfrac{1}{2}$. This shows that if $\| (\lambda_n) \|_{\ell_r} < \infty$, then

(1) $T_\lambda^{\#}\,(C[0,1]) \subseteq \pi_2(\ell_2, \ell_2) = \pi_p(\ell_2, \ell_2)$;

(2) $T_\lambda^{\#} :\ C[0,1] \longrightarrow \pi_p(\ell_2, \ell_2)$ is $p$-summing.

Now, let $U \subset C\,([0,1], \ell_2)$ be the closed linear span of $\{\epsilon_i \otimes e_i,\ a_i \in \mathbf{Z}\}$. Then $U$ is isometrically isomorphic to $\ell_2$. This is because

$$\begin{aligned} \| \sum_i \mu_i \epsilon_i \otimes e_i \| &= \sup_{t \in [0,1]} \| (\mu_n \epsilon_n(t)) \|_{\ell_2} \\ &= \| (\mu_i \epsilon_i(t_0)) \|_{\ell_2}, \end{aligned}$$

for some $t_0 \in [0,1]$, and hence

$$\| \sum_i \mu_i \epsilon_i \otimes e_i \| = \left( \sum_i |\mu_i|^2 \right)^{\frac{1}{2}}.$$

Moreover

$$T_\lambda(\epsilon_i \otimes e_i) = \lambda_i e_i \qquad \text{for all } i \in \mathbf{Z},$$

Therefore, we have the following commuting diagram

$$\begin{array}{ccc} U & \xrightarrow{T_{\lambda|U}} & \ell_2 \\ {\scriptstyle Q}\big\downarrow & & \nearrow S_\lambda \\ \ell_2 & & \end{array}$$

where $Q :\ U \to \ell_2$ is the isomorphism from $U$ onto $\ell_2$ such that $Q(\epsilon_n \otimes e_n) = e_n$ for all $n \in \mathbf{Z}$, and $S_\lambda :\ \ell_2 \longrightarrow \ell_2$ is the operator given by $S_\lambda(e_n) = \lambda_n e_n$. So to show that $T_\lambda$ is not $p$-summing, it is sufficient to show that one can pick $\lambda = (\lambda_n)$ such that $S_\lambda$ is not $p$-summing. To do this, we consider two cases. If $p = 2$, we take $\lambda_n = 1$ for all $n \in \mathbf{Z}$. Then the map $S_\lambda$ induced on $\ell_2$ is the identity map which is not $s$-summing for any $s < \infty$. If $1 < p < 2$, let $\lambda_n = \dfrac{1}{|n+1|^{\frac{1}{r}} \log |n+1|}$, so that $\| (\lambda_n) \|_{\ell_r} < \infty$. Then the map $S_\lambda :\ \ell_2 \longrightarrow \ell_2$ is not $s$-summing for any $s < r$. To show this, we may assume, without loss of generality, that $s \geq 2$. Let $x_n = e_n$ for all $n \geq 1$, and note that

$$\sup_{x^* \in B(\ell_2)} \left( \sum_n |x^*(x_n)|^s \right)^{\frac{1}{s}} \leq \| x^* \|_{\ell_2} \leq 1,$$

whilst

$$\left(\sum_n \| \lambda_n x_n \|^s\right)^{\frac{1}{s}} = \infty.$$

□

While the operators $T_\lambda$ in the previous example failed to be $p$-summing, they were all (2,1)-summing. This suggests the following question: suppose $T:\ C([0,1],Y) \longrightarrow Z$ is a bounded linear operator such that $T^{\#}:\ C[0,1] \longrightarrow \prod_2(Y,Z)$ is 2-summing. What can we say about $T$? Is $T$ $(2,1)$-summing? The following example shows that $T$ can be very bad.

**Theorem 11** There exists a Banach space $Z$, and a bounded linear operator $T:\ C([0,1],\ell_1) \to Z$ such that $T^{\#}:\ C[0,1] \to \prod_2(\ell_1,Z)$ is 2-summing, with the property that, for any $N \in \mathbf{N}$, there exists a subspace $U$ of $C([0,1],\ell_1)$ with $\dim U = N$, such that $T$ restricted to $U$ behaves like the identity operator on $\ell_\infty^N$. In particular $T$ is not (2,1)-summing.

**Proof:** If $X$ and $Y$ are Banach spaces, we denote by $X\hat{\otimes}_\pi Y$ the projective tensor product, that is, the completion of the algebraic tensor product of $X$ and $Y$ under the norm

$$\| u \|_\pi = \inf\{\sum_{i=1}^n \| x_i \| \| y_i \|,\ u = \sum_{i=1}^n x_i \otimes y_i\}.$$

It is well known that $(X\hat{\otimes}_\pi Y)^*$ is isometrically isomorphic to the space $\pounds(X,Y^*)$ of all bounded linear operators from $X$ to $Y^*$.

Let $Z = C([0,1],\ell_1) + L_2[0,1]\hat{\otimes}_\pi \ell_2$ be the Banach space with the norm

$$\| x \|_Z = \inf\{\| x' \|_\epsilon + \| x'' \|_\pi :\ x = x' + x''\},$$

where $\| \|_\epsilon$ denotes the sup norm in $C([0,1],\ell_1)$, and $\| \|_\pi$ denotes the norm of the projective tensor product $L_2[0,1]\hat{\otimes}_\pi \ell_2$. Let

$$T:\ C([0,1],\ell_1) \longrightarrow Z$$

be the identity operator.

We first see that for each $f \in C[0,1]$, the operator $T^{\#}f:\ \ell_1 \to Z$ is 2-summing with

$$\pi_2(T^{\#}f) \leq \pi_2(I) \parallel T^{\#}f \parallel_{\mathcal{L}(\ell_2,Z)},$$

where $I: \ell_1 \longrightarrow \ell_2$ is the natural mapping. This is because, for each $f \in C[0,1]$, and each $x \in \ell_1$, we have that

$$\parallel T(f \otimes x) \parallel \leq \parallel f \otimes x \parallel_{L_2 \hat{\otimes}_\pi \ell_2} \leq \parallel f \parallel_{L_2} \parallel x \parallel_{\ell_2} .$$

To see that $T^{\#}:\ C[0,1] \longrightarrow \prod_2(\ell_1, X)$ is 2-summing, note that $\parallel T^{\#}f \parallel_{\mathcal{L}(\ell_2,Z)} \leq \parallel f \parallel_{L_2}$, and hence if $f_1, \ldots, f_n \in C[0,1]$, then

$$\begin{aligned}\left(\sum_{k=1}^{n} \left[\pi_2(T^{\#}f_k)\right]^2\right)^{\frac{1}{2}} &\leq \pi_2(I) \left(\sum_{k=1}^{n} \parallel f_k \parallel_{L_2}^2\right)^{\frac{1}{2}} \\ &\leq \pi_2(I)\pi_2(J) \sup_{t\in[0,1]} \left\| \left(\sum_{K=1}^{n} |f_k(t)|^2\right)^{\frac{1}{2}} \right\|.\end{aligned}$$

Here $J:\ C[0,1] \longrightarrow L_2[0,1]$ denotes the natural mapping.

Now we define the space $U$, a closed linear subspace of $C\left([0,1],\ell_1\right)$. Let $\{f_{ij}:\ 1 \leq i,j \leq N\}$ be disjoint functions in $C[0,1]$, for which $0 \leq f_{ij} \leq 1$, $\parallel f_{ij} \parallel = 1$, each $f_{ij}$ is supported in an interval of length $\dfrac{1}{N^2}$, and

$$\int_0^1 f_{ij}\,dt = \frac{1}{2N^2} \text{ and } \int_0^1 f_{ij}^2\,dt = \frac{1}{3N^2}.$$

Let $\{e_{ij}:\ 1 \leq i,j \leq N\}$ be distinct unit vectors in $\ell_1$. We let $U = \{\sum_{i,j} \lambda_i f_{ij} \otimes e_{ij},\ \lambda_i \in \mathbf{R}\}$.

Now we consider $T$ restricted to $U$. If $\sum_{i,j} \lambda_i f_{ij} \otimes e_{ij} \in U$, then

$$\parallel \sum_{i,j} \lambda_i f_{ij} \otimes e_{ij} \parallel_\epsilon \leq \sup_i |\lambda_i|,$$

and hence

$$\parallel \sum_{i,j} \lambda_i f_{ij} \otimes e_{ij} \parallel_Z \leq \sup_i |\lambda_i|.$$

Let $y_i^* = N\sum_j f_{ij}\otimes e_{ij}$, and set $x = \sum_{i,j}\lambda_i f_{ij}\otimes e_{ij}$. Then whenever $x = x' + x''$, with $x' \in C([0,1],\ell_1)$ and $x'' \in L_2[0,1]\hat{\otimes}_\pi \ell_2$, we know that

$$|y_i^*(x)| \leq |y_i^*(x')| + |y_i^*(x'')|.$$

Hence

$$|y_i^*(x)| \leq \| y_i^* \|_{C([0,1],\ell_1)^*} \| x' \|_\epsilon + \| y_i^* \|_{(L_2[0,1]\hat{\otimes}_\pi \ell_2)^*} \| x'' \|_\pi .$$

But

$$\begin{aligned}\| y_i^* \|_{C([0,1],\ell_1)^*} &= N\sum_{i=1}^N \int_{\text{supp } f_{ij}} |f_{ij}|dt \\ &= N\cdot\frac{N}{2N^2} = \frac{1}{2},\end{aligned}$$

and, since $\left(L_2[0,1]\hat{\otimes}_\pi \ell_2\right)^*$ is isometric to $\mathcal{L}(L_2[0,1],\ell_2)$,

$$\begin{aligned}\| y_i^* \|_{(L_2[0,1]\hat{\otimes}_\pi \ell_2)^*} &= \sup\left\{[\sum_{j=1}^N (N\int_0^1 f_{ij}g dt)^2]^{\frac{1}{2}} : \| g \|_{L_2} \leq 1\right\} \\ &\leq \sup\left\{N[\sum_{j=1}^N \int_0^1 f_{ij}^2 dt \cdot \int_{\text{supp } f_{ij}} |g|^2 dt]^{\frac{1}{2}} : \| g \|_{L_2} \leq 1\right\} \\ &= \frac{1}{\sqrt{3}}\left\{(\sum_{j=1}^N \int_{\text{supp } f_{ij}} |g|^2 dt)^{\frac{1}{2}} : \| g \|_2 \leq 1\right\} \\ &= \frac{1}{\sqrt{3}}.\end{aligned}$$

Therefore

$$|y_i^*(x)| \leq \frac{1}{2} \| x' \|_\epsilon + \frac{1}{\sqrt{3}} \| x'' \|_\pi, \leq \frac{1}{\sqrt{3}} \| x \| .$$

However,

$$\begin{aligned}y_i^*(x) &= N\sum_{j=1}^N \lambda_i \int_0^1 f_{ij}^2 dt \\ &= N^2\lambda_i \frac{1}{3N^2} = \frac{\lambda_i}{3}.\end{aligned}$$

Therefore

$$\begin{aligned}\| \sum_{i,j}\lambda_i f_{ij}\otimes e_{ij} \|_Z &\geq \sqrt{3}\sup_i |y_i^*(x)| \\ &\geq \frac{1}{\sqrt{3}}\sup|\lambda_i|.\end{aligned}$$

Thus the space $U$ is isomorphic to $\ell_\infty^N$, and we have the commuting diagram

$$\begin{array}{ccc} U & \xrightarrow{T_{|U}} & T(U) \\ {\scriptstyle A}\downarrow & & \uparrow{\scriptstyle A^{-1}} \\ \ell_\infty^N & \xrightarrow{id_{\ell_\infty^N}} & \ell_\infty^N \end{array}$$

where $A: \ U \to \ell_\infty^N$ is the isomorphism between $U$ and $\ell_\infty^N$. □

**IV Operators that factor through a Hilbert space**

It is well known that $\mathcal{L}(X, \ell_2) = \prod_2(X, \ell_2)$ whenever $X$ is $C(K)$ or $\ell_1$. One might ask whether this is true when $X = C(K, \ell_1)$. Indeed one could ask the weaker question: if $T: \ C(K, \ell_1) \longrightarrow \ell_2$ is bounded, does it follow that the induced operator $T^\#$ is 2-summing? We answer this question in the negative.

**Theorem 12** There is a compact Hausdorff space $K$ and a bounded linear operator $T: \ C(K, \ell_1) \longrightarrow \ell_2$ for which $T^\#: \ C(K) \longrightarrow \prod_1(\ell_1, \ell_2)$ is not 2-summing.

**Proof:** First, we show that there is a compact Hausdorff space $K$, and an operator $R: \ C(K) \longrightarrow \ell_\infty$ that is (2,1)-summing but not 2-summing. To see this, let $K = [0,1]$, and consider the natural embedding $C[0,1] \longrightarrow L_{2,1}[0,1]$, where $L_{2,1}[0,1]$ is the Lorentz space on $[0,1]$ with the Lebesque measure (see [6]). By [11], it follows that this map is (2,1)-summing. To show that this map is not 2-summing, we argue in a similar fashion to [8]. For $n \in \mathbf{N}$, consider the functions $e_i(t) = f(t + \frac{1}{i} \text{ mod } 1)$ $(1 \le i \le n)$, where $f(t) = \frac{1}{\sqrt{t}}$ if $t \ge \frac{1}{n}$ and $\sqrt{n}$ otherwise. Then it is an easy matter to verify that for some constant $C > 0$,

$$\left( \sum_{i=1}^{n} |e^*(e_i)|^2 \right)^{\frac{1}{2}} \le C \sqrt{\log n}$$

for every $e^*$ in the unit ball of $C[0,1]^*$, whereas

$$\left( \sum_{i=1}^{n} \|e_i\|_{L_{2,1}[0,1]}^2 \right)^{\frac{1}{2}} \ge C^{-1} \log n.$$

Finally, since $L_{2,1}[0,1]$ is separable, it embeds isometrically into $\ell_\infty$.

Define $T : C(K, \ell_1) \to \ell_2$ as follows: for $\varphi = (f_n) \in C(K, \ell_1)$, let

$$T(f_n) = \sum_n Rf_n(n) e_n.$$

Then $T$ is bounded, for

$$\begin{aligned} \| T(f_n) \|_2 &= \left( \sum_n |Rf_n(n)|^2 \right)^{\frac{1}{2}} \\ &\leq \left( \sum_n \| Rf_n \|_{\ell_\infty}^2 \right)^{\frac{1}{2}} \\ &\leq \pi_{2,1}(R) \sup_{t \in K} \sum_n |f_n(t)|. \end{aligned}$$

Thus

$$\| T \| \leq \pi_{2,1}(R).$$

But $T^{\#} : C(K) \longrightarrow \pounds(\ell_1, \ell_2)$ is not 2-summing, because for each $f \in C(K)$, the operator $T^{\#} f : \ell_1 \longrightarrow \ell_2$ is the diagonal operator $\sum_n Rf(n) e_n \otimes e_n$. Hence the strong operator norm of $T^{\#} f$ is

$$\| T^{\#} f \| = \sup_n |Rf(n)| = \| Rf \|_{\ell_\infty} .$$

Thus $T^{\#} : C(K) \longrightarrow \pounds(\ell_1, \ell_2)$ is not 2-summing, because $R : C(K) \longrightarrow \ell_\infty$ is not 2-summing. □

**Discussions and concluding remarks**

**Remark 13** Theorem 12 shows that if $X$ and $Y$ are Banach spaces such that $\pounds(X, \ell_2) = \prod_2(X, \ell_2)$ and $\pounds(Y, \ell_2) = \prod_2(X, \ell_2)$, then $X \hat{\otimes}_\epsilon Y$ need not share this property. This observation could also be deduced from arguments presented in [4] (use Example 3.5 and the proof of Proposition 3.6 to show that there is a bounded operator $T : (\ell_1 \oplus \ell_1 \oplus \ldots \oplus \ell_1)_{\ell_\infty} \longrightarrow \ell_2$ that is not $p$-summing for any $p < \infty$).

**Remark 14** In the proof of Theorem 2 we showed that the injective tensor product is an associative operation, that is, if $X, Y$ and $Z$ are Banach spaces, then $(X\hat{\otimes}_\epsilon Y)\hat{\otimes}_\epsilon Z$ is isometrically isomorphic to $X\hat{\otimes}_\epsilon(Y\hat{\otimes}_\epsilon Z)$. It is not hard to see that the same is true for the projective tensor product. However, we can conclude from Theorem 12 that what is known as the $\gamma_2^*$-tensor product is not an associative operation.

If $E$ and $F$ are Banach spaces, and $T: E \longrightarrow F$ is a bounded linear operator, following [10], we say that $T$ **factors through a Hilbert space** if there is a Hilbert space $H$, and operators $B: E \longrightarrow H$ and $A: H \longrightarrow F$ such that $T = A \circ B$. We let $\gamma_2(T) = \inf\{\| A \| \| B \|\}$, where the infimum runs over all possible factorization of $T$, and denote the space of all operators $T: E \longrightarrow F$ that factor through a Hilbert space by $\Gamma_2(E, F)$. It is not hard to check that $\gamma_2$ defines a norm on $\Gamma_2(E, F)$, making $\Gamma_2(E, F)$ a Banach space. We define the $\gamma_2^*$-norm $\| \quad \|_*$ on $E \otimes F$ (see [9] or [10]) in which the dual of $E \otimes F$ is identified with $\Gamma_2(E, F^*)$, and let $E\hat{\otimes}_{\gamma_2^*}F$ denote the completion of $(E \otimes F, \| \quad \|_*)$.

The operator $T: C(K)\hat{\otimes}_{\gamma_2^*}\ell_1 \longrightarrow \ell_2$ exhibited in Theorem 12, induces a bounded linear functional on $\left[(C(K)\hat{\otimes}_{\gamma_2^*}\ell_1)\hat{\otimes}_{\gamma_2^*}\ell_2\right]^*$. Now we see that if $C(K)\hat{\otimes}_{\gamma_2^*}(\ell_1\hat{\otimes}_{\gamma_2^*}\ell_2)$ were isometrically isomorphic to $(C(K)\hat{\otimes}_{\gamma_2^*}\ell_1)\hat{\otimes}_{\gamma_2^*}\ell_2$, then the operator $T^\#: C(K) \to \mathcal{L}(\ell_1, \ell_2)$ would induce a bounded linear functional on $\left[C(K)\hat{\otimes}_{\gamma_2^*}(\ell_1\hat{\otimes}_{\gamma_2^*}\ell_2)\right]^*$, showing that $T^\# \in \Gamma_2\left(C(K), \mathcal{L}(\ell_1, \ell_2)\right)$, implying that $T^\#$ would be 2-summing [10, p. 62]. This contradiction shows that $C(K)\hat{\otimes}_{\gamma_2^*}(\ell_1\hat{\otimes}_{\gamma_2^*}\ell_2)$ and $\left(C(K)\hat{\otimes}_{\gamma_2^*}\ell_1\right)\hat{\otimes}_{\gamma_2^*}\ell_2$ cannot be isometrically isomorphic.

Another example showing that the $\gamma_2^*$-tensor product is not associative was given by Pisier (private communication).

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