Orlicz–Lorentz Spaces

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It is a great honor to be asked to write this article for the Proceedings of the Conference in honor of W. Orlicz. I cannot say that I knew him personally, but it is obvious from the many people I have met that knew him that he had a tremendous influence. Certainly, his ideas have found their way into much of my own research.

The purpose of this article is to summerize some recent results of the author about Orlicz-Lorentz spaces — function spaces that provide a common generalization of Orlicz spaces and Lorentz spaces.

Let us first introduce the background to these spaces. The most well known examples of Banach spaces are the L_p spaces. Their definition is very well known. We will restrict ourselves to function spaces on $[0,\infty)$ with Lebesgue measure λ. If $1 \le p \le \infty$, then for any measurable function f, the L_p-norm is defined to be

$$\|f\|_p = \left(\int |f(x)|^p \, dx \right)^{1/p}$$

for $p < \infty$, and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{0 \le x < \infty} |f(x)|$$

for $p = \infty$. The Banach space L_p is the vector space of all measurable functions f for which $\|f\|_p$ is finite.

Now these spaces can be generalized in two different ways. The first generalization is due to Orlicz **[O]** (see also **[Lu]**). If $F : [0,\infty) \to [0,\infty)$ is non-decreasing and convex with $F(0) = 0$, we define the *Luxemburg norm* of a measurable function f by

$$\|f\|_F = \inf \left\{ c : \int F\bigl(|f(x)| / c\bigr) \, dx \le 1 \right\}.$$

We define the *Orlicz space* L_F to be those measurable functions f for which $\|f\|_F$ is finite. We see that the Orlicz space L_F really is a true generalization of L_p, at least for $p < \infty$: if $F(t) = t^p$, then $L_F = L_p$ with equality of norms.

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In this article, we will not always require that the Luxemburg norm actually be a norm, that is, we will not always require the triangle inequality. For this reason, we will allow F in the above definition to be a *φ-function*, namely, that F be continuous and strictly increasing, and that

$$F(0) = 0, \qquad \lim_{n\to\infty} F(t) = \infty.$$

However, we will often desire that the function F has some control on its growth, both from above and below. For this reason we will often require that F be *dilatory*, that is, for some $c_1, c_2 > 1$ we have $F(c_1 t) \ge c_2 F(t)$ for all $0 \le t < \infty$, and that F satisfy the *Δ_2-condition*, that is, that F^{-1} is dilatory.

The second collection of examples are the *Lorentz spaces*. These were introduced by Lorentz **[Lo1]**, **[Lo2]**. If f is a measurable function, we define the *non-increasing rearrangement* of f to be

$$f^*(x) = \sup\bigl\{\, t : \lambda(|f| \ge t) \ge x \,\bigr\}.$$

If $1 \le q < \infty$, and if $w : (0,\infty) \to (0,\infty)$ is a non-increasing function, we define the *Lorentz norm* of a measurable function f to be

$$\|f\|_{w,q} = \left(\int_0^\infty w(x) f^*(x)^q \, dx \right)^{1/q}.$$

Then the *Lorentz space* $\Lambda_{w,q}$ is defined to be the space of those measurable functions f for which $\|f\|_{w,q}$ is finite. These spaces also represent a generalization of the L_p spaces: if $w(x) = 1$ for all $0 \le x < \infty$, then $\Lambda_{w,p} = L_p$ with equality of norms.

There is one, rather peculiar, choice of the function w which turns out to be rather useful. If $1 \le q \le p < \infty$, we define the spaces $L_{p,q}$ to be $\Lambda_{w,q}$ with $w(x) = \frac{q}{p} x^{q/p-1}$. We can also allow $q > p$, but at the loss of the triangle inequality. A good reference for a description of these spaces is Hunt **[H]**. By a suitable change of variables, the $L_{p,q}$ norm may also be defined in the following fashion:

$$\|f\|_{p,q} = \left(\int_0^\infty \left| f^*(x^{p/q}) \right|^q \, dx \right)^{1/q}.$$

Thus $L_{p,p} = L_p$ with equality of norms. The reason for this definition is that for any measurable set $A \in \mathcal{F}$, we have that $\|\chi_A\|_{p,q} = \|\chi_A\|_p = \lambda(A)^{1/p}$. Thus $L_{p,q}$ is a space identical to L_p for characteristic functions, but 'glued' together in a L_q fashion.

Now we come to the object of the article, the *Orlicz–Lorentz spaces*. These are a common generalization of the Orlicz spaces and the Lorentz spaces. They have been studied by Mastyło (see part 4 of [**My**]), Maligranda [**Ma**], and Kamińska [**Ka1**], [**Ka2**], [**Ka3**]. For instance, Kamińska calculated many of the isometric properties for these spaces. However, this author's work is concerned with isomorphic properties.

If G is an Orlicz function, and if $w : [0,\infty) \to [0,\infty)$ is a non-increasing function, we define the *Orlicz–Lorentz norm* of a measurable function f to be

$$\|f\|_{w,G} = \inf \left\{ c : \int_0^\infty w(x) G\big(f^*(x)/c\big)\, dx \le 1 \right\} .$$

We define the *Orlicz–Lorentz space* $\Lambda_{w,G}$ to be the vector space of measurable functions f for which $\|f\|_{w,G}$ is finite.

We shall not work with this definition of the Orlicz–Lorentz space, however, but with a different, equivalent definition that bears more resemblance to the spaces $L_{p,q}$. If F and G are φ-functions, we would like to define our spaces $L_{F,G}$ to satisfy the following properties:

i) that $\|\chi_A\|_{F,G} = \|\chi_A\|_F$ whenever A is a measurable subset;
ii) that $L_{F,G}$ be glued together in a L_G fashion.

It turns out that the required definition is the following. (In the sequel, $\tilde F(t)$ will always denote the function $1/F(1/t)$. Thus $\|\chi_A\|_F = \tilde F^{-1}\big(\lambda(A)\big)$.)

Definition: If F and G are φ-functions, then we define the *Orlicz–Lorentz functional* of a measurable function f by

$$\|f\|_{F,G} = \left\| f^* \circ \tilde F \circ \tilde G^{-1} \right\|_G .$$

We define the *Orlicz–Lorentz space*, $L_{F,G}$, to be the vector space of measurable functions f for which $\|f\|_{F,G} < \infty$, modulo functions that are zero almost everywhere.

We also have the following definition corresponding to the $L_{p,\infty}$ spaces.

Definition: If F is a φ-function, then we define the (*weak-*)*Orlicz–Lorentz functional* by

$$\|f\|_{F,\infty} = \sup_{x \ge 0} \tilde F^{-1}(x) f^*(x).$$

We define the *Orlicz–Lorentz space*, $L_{F,\infty}$, to be the vector space of measurable functions f for which $\|f\|_{F,\infty} < \infty$, modulo functions that are zero almost everywhere.

We see that $L_{F,F} = L_F$ with equality of norms, and that if $F(t) = t^p$ and $G(t) = t^q$, then $L_{F,G} = L_{p,q}$, and $L_{F,\infty} = L_{p,\infty}$, also with equality of norms. For this reason, we shall also introduce the following notation: if $F(t) = t^p$, we shall write $L_{p,G}$ for $L_{F,G}$, and $L_{G,p}$ for $L_{G,F}$.

Now let us provide some examples. We define the *modified logarithm* and the *modified exponential* functions by

$$\operatorname{lm}(t) = \begin{cases} 1+\log t & \text{if } t \ge 1 \\ 1/\bigl(1+\log(1/t)\bigr) & \text{if } 0 < t < 1 \\ 0 & \text{if } t = 0; \end{cases}$$

$$\operatorname{em}(t) = \operatorname{lm}^{-1}(t) = \begin{cases} \exp(t-1) & \text{if } t \ge 1 \\ \exp\bigl(1-(1/t)\bigr) & \text{if } 0 < t < 1 \\ 0 & \text{if } t = 0. \end{cases}$$

These functions are designed so that for large t they behave like the logarithm and the exponential functions, so that $\operatorname{lm} 1 = 1$ and $\operatorname{em} 1 = 1$, and so that $\widetilde{\operatorname{lm}} = \operatorname{lm}$ and $\widetilde{\operatorname{em}} = \operatorname{em}$. Then the functions $t^p(\operatorname{lm} t)^\alpha$ and $\operatorname{em}(t^p)$ are φ-functions whenever $0 < p < \infty$ and $-\infty < \alpha < \infty$. If the measure space is a probability space, then the Orlicz spaces created using these functions are also known as *Zygmund spaces*, and the Orlicz–Lorentz spaces $L_{t^p(\operatorname{lm} t)^\alpha,q}$ and $L_{\operatorname{em}(t^p),q}$ are known as *Lorentz–Zygmund spaces* (see, for example, [**B–S**]).

Comparison Results

A large part of my research on these spaces has asked the question: what are necessary and sufficient conditions on F_1, F_2, G_1 and G_2 so that the spaces L_{F_1,G_1} and L_{F_2,G_2} are *equivalent*, that is, that there is some constant $c < \infty$ such that

$$c^{-1} \|f\|_{F_1,G_1} \le \|f\|_{F_2,G_2} \le c\, \|f\|_{F_1,G_1} .$$

In answering these questions, it is necessary to assume that G_1 and G_2 be dilatory, and satisfy the Δ_2-condition. In all our general discussions, we shall take this as given.

First, by considering characteristic functions, it is easy to see that it must be that F_1 and F_2 are equivalent as φ-functions, that is, there is a constant $c < \infty$ such that $F_1(c^{-1}t) \le F_2(t) \le F_1(ct)$ for all $0 \le t < \infty$. In this manner, it is easy to see that without loss of generality, we may take $F_1 = F_2$. In fact, it is not hard to show that we are really asking about the equivalence of L_1 and $L_{1,H}$, where $H = G_1 \circ G_2^{-1}$ or $H = G_2 \circ G_1^{-1}$.

Results along these lines have already been obtained by G. Lorentz, and also by Y. Raynaud. I will take the liberty of translating their results into my notation. (In so doing, it may not be entirely obvious that their result as they state it, and as it is stated here, are actually the same.)

To state these results we will require some more notation. We will say that a φ-function F is an *N-function* if it is equivalent to a φ-function F_0 such that $F_0(t)/t$ is strictly increasing, $F_0(t)/t \to \infty$ as $t \to \infty$, and $F_0(t)/t \to 0$ as $t \to 0$. We will say that a φ-function F is *complementary* to a φ-function G if for some $c < \infty$ we have

$$c^{-1} t \le F^{-1}(t) \cdot G^{-1}(t) \le ct \qquad (0 \le t < \infty).$$

If F is an N-function, we will let F^* denote the (unique up to equivalence) function complementary to F.

Our definition of a complementary function differs from the usual definition. If F is an N-function that is convex, then the complementary function is usually defined by $F^*(t) = \sup_{s\geq 0}\bigl(st - F(s)\bigr)$. However, it is known that $t \leq F^{-1}(t)\cdot F^{*-1}(t) \leq 2t$ (see [**K–R**]). Thus our definition is equivalent.

Finally, we will say that an N-function H satisfies *condition* (J) if

$$\left\| 1/\tilde H^{*-1} \right\|_{H^*} < \infty.$$

(I call it condition (J) for personal reasons.)

Now we are ready to give the result of G. Lorentz [**Lo3**].

THEOREM 1. *Suppose that H is an N-function. Then the following are equivalent.*

i) *L_1 and $L_{1,H}$ are equivalent.*

ii) *H satisfies condition (J).*

What kinds of N-functions satisfy condition (J)? They are functions that satisfy growth conditions that make it 'close' to the identity function. The reader might like to verify that $t(\operatorname{lm} t)^\alpha$ satisfies this condition when $\alpha > 0$. Lorentz gave the following example:

$$H(t) = \begin{cases} t^{1+\frac{1}{1+\log(1+\log t)}} & \text{if } t \geq 1 \\ t^{1-\frac{1}{1+\log(1-\log t)}} & \text{if } t \leq 1. \end{cases}$$

In fact, we will give another characterization that shows that this example is, in some sense, on the "boundary" of satisfying condition (J).

Raynaud's result [**R**] allows one to drop the assumption that H is an N-function, but at the cost of making the implication go only one way.

THEOREM 2. *Suppose that H is a φ-function. Suppose that there exist N-functions K and L satisfying condition (J) such that $H = K \circ L^{-1}$ (or $H = K^{-1} \circ L$). Then L_1 and $L_{1,H}$ are equivalent.*

As applications, one may show that if $0 < p < \infty$ and $-\infty < \alpha < \infty$, then $L_{t^p(\operatorname{lm} t)^\alpha}$ and $L_{t^p(\operatorname{lm} t)^\alpha,p}$ are equivalent, and that if $\beta > 0$, then $L_{\operatorname{em}(t^\beta)}$ and $L_{\operatorname{em}(t^\beta),\infty}$ are equivalent. These were shown for probability spaces by Bennett and Rudnick [**B–R**] (see also [**B–S**]).

The author's contribution was to show that the converse result to Theorem 2 holds.

THEOREM 3. *Suppose that H is a φ-function such that L_1 and $L_{1,H}$ are equivalent. Then the following are true.*

i) *There exist N-functions K and L satisfying condition (J) such that $H = K \circ L^{-1}$*

ii) There exist N-functions K and L satisfying condition (J) such that $H = K^{-1} \circ L$.

In fact there are many more equivalent conditions, and we will give some more later. We will not prove any results here — the interested reader should consult **[Mo1]**. However, we will explain some of the ideas behind them.

First we will describe the simple comparison principles for Orlicz–Lorentz spaces. If the reader has studied Lorentz spaces, he will already know that $\|f\|_{p,q_1} \le \|f\|_{p,q_2}$ whenever $q_1 \ge q_2$ (see **[H]**). In our more general setting, we have the following result: if F is equivalent to a convex function, then $\|f\|_F \le c\,\|f\|_{F,1}$ for all measurable f. In fact, it is quite easy to show that if $\|\cdot\|$ is any *norm* (the triangle inequality is essential here) such that

$$|f| \le \chi A \Rightarrow \|f\| \le c_1 \tilde{F}^{-1}\big(\lambda(A)\big),$$

then $\|f\| \le c_2\,\|f\|_{F,1}$.

From this, we can deduce the following result. Let us say that G_1 is *equivalently less convex than* G_2 (in symbols $G_1 \prec G_2$) if $G_2 \circ G_1^{-1}$ is equivalent to a convex function. Then

$$G_1 \prec G_2 \Rightarrow \|f\|_{1,G_1} \ge c^{-1}\,\|f\|_{1,G_2}\,.$$

However, we can see from Theorems 1 and 2 that this is not the whole story. If we desire a converse to this implication, we will have to soften the notion of 'less convex than' to 'almost less convex than.' It turns out that we can precisely characterize this notion of 'almost convexity.'

Before doing this, let us discuss what it means for a φ-function to be equivalent to a convex function. Suppose we are given a fixed number $a > 1$. It is quite easy to see that a φ-function G is completely determined, up to equivalence, by the values $G(a^n)$ for $n \in \mathbb{Z}$. In this way, it can be easily shown that a φ-function G is equivalent to a convex function if and only there exists numbers $a > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, we have that

$$G(a^{n+m}) \ge a^{m-N} G(a^n)$$

for all $n \in \mathbb{Z}$. (Here $\mathbb{N} = \{1, 2, 3, \ldots\}$.)

It turns out that the correct definition for 'almost convex' is the following.

Definition: Let G be a φ-function. We say that G is *almost convex* if there are numbers $a > 1$, $b > 1$ and $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, the cardinality of the set of $n \in \mathbb{Z}$ such that we do **not** have

$$G(a^{n+m}) \ge a^{m-N} G(a^n)$$

is less than b^m.

In the same way, one can get notions of *almost concave*, *almost linear*, etc. It turns out that an N-function H satisfies condition (J) if and only if H^{-1} is almost convex.

Using these ideas, it is then possible to prove Theorem 3, and indeed to get the following result, that gives the desired necessary and sufficient conditions for Orlicz–Lorentz spaces to be equivalent.

THEOREM 4. *Suppose that F_1, F_2, G_1 and G_2 are φ-functions such that at least one of G_1 and G_2 are dilatory, and at least one of G_1 and G_2 satisfy the Δ_2-condition. Then the following are equivalent statements.*

i) L_{F_1,G_1} *and* L_{F_2,G_2} *are equivalent.*
ii) F_1 *and* F_2 *are equivalent, and there exist N-functions* H *and* K *that satisfy condition (J) such that* $G_1 \circ G_2^{-1} = H \circ K^{-1}$.
iii) F_1 *and* F_2 *are equivalent, and there exist N-functions* H *and* K *that satisfy condition (J) such that* $G_1 \circ G_2^{-1} = K^{-1} \circ H$.
iv) F_1 *and* F_2 *are equivalent, and* $G_1 \circ G_2^{-1}$ *is almost convex, and* $G_2 \circ G_1^{-1}$ *is almost convex.*

IS EVERY R.I. SPACE EQUIVALENT TO AN ORLICZ–LORENTZ SPACE?

Or more precisely, does there exist a rearrangement invariant space X such that the $\|\cdot\|_X$ is not equivalent to any Orlicz–Lorentz norm on the space of simple functions? It turns out that we can find an example to show that this can happen. To do this, we use the following result, which is a corollary of the proof of Theorem 4. As before, we refer the reader to **[Mo1]** for details.

THEOREM 5. *Let F_1, F_2, G_1 and G_2 be φ-functions. Suppose that one of G_1 or G_2 is dilatory, and that one of G_1 or G_2 satisfies the Δ_2-condition. Then the following are equivalent.*

i) L_{F_1,G_1} *and* L_{F_2,G_2} *are equivalent.*
ii) For some $c < \infty$ *we have that* $c^{-1}\|f\|_{F_1,G_1} \le \|f\|_{F_2,G_2} \le c\,\|f\|_{F_1,G_1}$ *whenever* f *is of the following form: there exist* $0 = a_0 < a_1 < a_2 < \ldots < a_n$ *such that*

$$F \circ f^*(x) = \begin{cases} 1/a_i & \text{if } a_{i-1} \le x < a_i \text{ and } 1 \le i \le n \\ 0 & \text{otherwise.} \end{cases}$$

Thus to compare two Orlicz–Lorentz spaces, we need only compare their norms on a certain class of test functions. Now it is easy to prove the desired result.

THEOREM 6. *There is a rearrangement invariant Banach space X such that for every Orlicz–Lorentz space $L_{F,G}$, the norms $\|\cdot\|_X$ and $\|\cdot\|_{F,G}$ are inequivalent on the vector space of simple functions.*

Proof: We define the following norm for measurable functions f:

$$\|f\|_X = \sup \|fg\|_1 \,/\, \|g\|_2\,,$$

where the supremum is over all g of the following form: there exist $0 = a_0 < a_1 < a_2 < \ldots < a_n$ such that

$$g^*(x) = \begin{cases} 1/\sqrt{a_i} & \text{if } a_{i-1} \le x < a_i \text{ and } 1 \le i \le n \\ 0 & \text{otherwise.} \end{cases}$$

Then it is easy to see from Theorem 5 that if X is equivalent to an Orlicz–Lorentz space, then it must be equivalent to L_2. That this is not the case is easily shown by the following example:

$$f(x) = \frac{1}{\sqrt{x \log x}} \qquad \text{for } x \ge 2.$$

Then $\|f\|_X < \infty$, whereas $\|f\|_2 = \infty$. □

Boyd Indices of Orlicz–Lorentz Spaces

In studying a particular rearrangement invariant space, it is very important to know its Boyd indices. Even very obvious questions, like whether it is equivalent to a normed space, or whether it is p-convex/q-concave, cannot be answered except with a knowledge of these indices.

As their name suggests, they were first studied by Boyd [**Bo**]. We will take our definition from [**L–T**]. Their definition differs from that usually used: most references reverse the words 'upper' and 'lower', and use the recipricols of the indices used here.

Essentially, they describe the norms of the following operators: for each $a > 0$ we let $d_a f(x) = f(ax)$. The *lower Boyd index* is defined to be

$$p(X) = \sup \left\{ p : \text{for some } c < \infty \text{ we have } \|d_a\|_{X \to X} \le c a^{-1/p} \text{ for } a < 1 \right\},$$

and the *upper Boyd index* is

$$q(X) = \inf \left\{ q : \text{for some } c < \infty \text{ we have } \|d_a\|_{X \to X} \le c a^{-1/q} \text{ for } a > 1 \right\}.$$

The reader should appreciate that $p(L_{p,q}) = q(L_{p,q}) = p$.

The hope is that it should be possible to calculate the Boyd indices of $L_{F,G}$ simply from knowledge of some appropriate index of F. In fact, this was the question posed by Maligranda [**Ma**]. What are the appropriate indices? For a φ-function F, we define the *lower Matuszewska–Orlicz index* to be

$$p_m(F) = \sup \{ p : \text{for some } c > 0 \text{ we have } F(at) \ge c\, a^p F(t) \text{ for } 0 \le t < \infty \text{ and } a > 1 \},$$

and the *upper Matuszewska–Orlicz index* to be

$$q_m(F) = \inf \{ q : \text{for some } c < \infty \text{ we have } F(at) \le c\, a^q F(t) \text{ for } 0 \le t < \infty \text{ and } a > 1 \}.$$

Thus, for example, $p_m(T^p) = q_m(T^p) = p$. Maligranda's conjecture is the following: is $p(L_{F,G}) = p_m(F)$ and $q(L_{F,G}) = q_m(F)$?

Without going into details, I was able to show that this is not the case. Briefly, the example is $L_{1,G}$, where G is a φ-function that spends some of the time behaving like T^p, and some of the time behaving like T^q. We refer the reader to [**Mo2**] for more details. (The first example of a rearrangement invariant space where this sort of thing happened is due to Shimogaki [**Sh**].)

However, it is possible to obtain the following result without undue stress.

PROPOSITION 7. *Let F and G be φ-functions. Then*

i) $p_m(F) \ge p(L_{F,G}) \ge p_m(F \circ G^{-1}) p_m(G) \ge p_m(F) p_m(G)/q_m(G)$;

ii) $q_m(F) \le q(L_{F,G}) \le q_m(F \circ G^{-1}) q_m(G) \le q_m(F) q_m(G)/p_m(G)$.

We are then left with the following question. Given F and G, how exactly is one to calculate the Boyd indices of $L_{F,G}$? The author does have some idea for how to approach this problem, at least for giving necessary and sufficient conditions for the indices of $L_{1,G}$ to be 1. The idea is simple: we see that if $0 < a < \infty$, then $a \left\| d_a f \right\|_{1,G} = \left\| f \right\|_{1,G_a}$, where $G_a(t) = G(at)$. Then the problem of determining the Boyd indices becomes a problem of comparing two Orlicz–Lorentz spaces, and the methods from the above section should apply. One day, the author will get around to checking these ideas out. But if anyone else would like to do this, they can, and the author won't mind. Then *they* will have the problem of finding a journal that will accept results from this tiny corner of mathematics.

Finally, I would like to mention some very recent work of Bastero and Ruiz [**Ba–R**]. They prove some results about the Hardy transform on Orlicz–Lorentz spaces. If one looks hard enough at what they did, and then twists the way they state the results, one can obtain fairly sharp estimates for Boyd indices in the following manner. Given φ-functions F and G, we define the *modular lower and upper Boyd indices* of $L_{F,G}$ as follows:

$$p_{\text{mod}}(L_{F,G}) = \sup \Big\{ p : \text{for some } c < \infty \text{ we have} \\ \int G\big(f^*(a\tilde F \circ \tilde G^{-1}(x)\big)\,dx \le \int G\big(ca^{-1/p} f^*(\tilde F \circ \tilde G^{-1}(x)\big) \text{ for } a < 1 \Big\},$$

$$q_{\text{mod}}(L_{F,G}) = \inf \Big\{ q : \text{for some } c < \infty \text{ we have} \\ \int G\big(f^*(a\tilde F \circ \tilde G^{-1}(x)\big)\,dx \le \int G\big(ca^{-1/q} f^*(\tilde F \circ \tilde G^{-1}(x)\big) \text{ for } a > 1 \Big\}.$$

Then we have the following result.

THEOREM 8. *Let F and G be φ-functions. Then*

i) $p_{\text{ mod }}(L_{F,G}) = p_m(F \circ G^{-1}) p_m(G)$;
ii) $q_{\text{ mod }}(L_{F,G}) = q_m(F \circ G^{-1}) q_m(G)$.

The Definition of Torchinsky and Raynaud

Finally, we mention that there is another possible definition for Orlicz–Lorentz spaces, first given by Torchinsky [**T**], and investigated in detail by Raynaud [**R**]. We define

$$\|f\|^T_{F,G} = \left\| \tilde{F}^{-1}(e^x) f^*(e^x) \right\|_G ,$$

and call the corresponding space $L^T_{F,G}$ (my notation). Raynaud showed that if F is dilatory and satisfies the Δ_2-condition, and if G is dilatory, then

$$\|\chi_A\|^T_{F,G} \approx \tilde{F}^{-1}\big(\lambda(A)\big).$$

Thus these spaces are really quite a good contender for a possible alternative definition. Also, the problems that I considered are very easy to solve for these spaces. Raynaud showed that if F_1 and F_2 are dilatory and satisfy the Δ_2-condition, and if G_1 and G_2 are dilatory, then $L^T_{F_1,G_1}$ and $L^T_{F_2,G_2}$ are equivalent if F_1 and F_2 are equivalent, and the *sequence* spaces l_{G_1} and l_{G_2} are equivalent. The converse result is also easy to show.

Also, the Boyd indices of these spaces are much easier to compute. If F is dilatory and satisfies the Δ_2-condition, and if G is dilatory, then $p(L_{F,G}) = p_m(F)$, and $q(L_{F,G}) = q_m(G)$.

The only problem with these spaces is that we do not always have that $L^T_{F,F}$ is equivalent to the Orlicz space L_F.

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