NEGATIVE LONGITUDINAL CORRELATION FOR ISOTROPIC FLOWS

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Abstract. Examples of physically reasonable Gaussian, homogeneous, isotropic, three-dimensional, incompressible statistics with longitudinal correlation negative for some separation interval are presented. The negativity of the longitudinal correlation persists in the Galerkin approximation of the hydrodynamic equations, at least for some time. Both the outline of the mathematical arguments and the numerical implementation are included.

1. Introduction

The longitudinal correlation function of homogeneous and isotropic flows is the correlation of the velocity components parallel to the separation vector as a function of the separation distance. For incompressible flows this function determines all components of the correlation, see Batchelor (1953), p. 46. Kolmogorov (1941) examines the longitudinal correlation and its derivatives at zero separation before formulating the famous hypotheses on the statistics of locally homogeneous and isotropic turbulence.

Based on measurements, it is often accepted as a fact that this longitudinal correlation function is strictly positive for all separation lengths. For example Batchelor (1953), p. 48, for f the longitudinal correlation and r the separation distance, notes: "A demonstration that $f(r) > 0$ for all values of r has never been given, but it is a very plausible result for an incompressible fluid and is consistent with all measurements of $f(r)$." On the other hand, disagreements between what is measured and what is expected from theory have been explained as deviations from exact isotropy, see for example Comte-Bellot & Corrsin (1971), pp. 293 –294.

The theoretical possibility of negative f for exactly isotropic statistics has been noted for flows with Dirac energy spectra in Davidson (2004), p. 328, and such flows have been dismissed, quite rightly, as unrealistic. The authors are not aware of any previous theoretical investigation of the positivity of f based on clearly stated assumptions. To the extent that $f > 0$ is the basis of investigations regarding the shape of the energy spectrum at the origin, the existence of integral invariants of flows, and the energy decay laws, e.g. Gustafsson & George (2008), § 3.1, the problem merits reexamination.

While attempting to provide a rigorous proof of Batchelor's "plausible result" for realistic isotropic flows, the authors noticed that it possible to produce examples of homogeneous and isotropic Gaussian statistics with negative longitudinal correlations. These examples are presented in § 2.

The examples are random combinations of the eigenfunctions of the correlation. To avoid the impression that such statistics assign non-zero probability only to

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physically spurious configurations, it is shown that any smooth, solenoidal field which vanishes outside a bounded domain is surrounded by fields that occur with non-zero probability.

Given that "random Gaussian modes represent a perfectly legitimate initial condition" for admissible flows, (Ishida, Davidson, & Kaneda (2006), p. 457), the examples of § 2 here are used to initialize Galerkin flow approximations of Navier-Stokes statistics. And it is shown in § 3 that, on some time interval, the longitudinal correlation of these flows remains negative on whole intervals of separation.

A numerical implementation for the periodic case and the numerical simulation of the corresponding flow evolution are presented in § 4.

2. Gaussian Homogeneous and Isotropic Random fields

2.1. **The construction.** This section constructs isotropic statistics with longitudinal correlation not everywhere positive. The construction starts with choosing a reasonable candidate for the correlation $\overline{u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{h})}$ for the i and j components of the random $\mathbf{u}$, cf. Yaglom (1957), § 4.

For each $\mathbf{h}$ in 3-space, and setting $h = |\mathbf{h}|$, define

$$B_{ij}(\mathbf{h}) = \int_0^\infty \left\{ \left[\frac{3\sin(kh)}{(kh)^3} - \frac{3\cos(kh)}{(kh)^2} - \frac{\sin(kh)}{kh} \right] \frac{h_i h_j}{h^2} \right. \tag{1}$$

$$\left. + \left[\frac{\cos(kh)}{(kh)^2} - \frac{\sin(kh)}{(kh)^3} + \frac{\sin(kh)}{kh} \right] \delta_{ij} \right\} \Phi'(k)\ dk, \tag{2}$$

$i, j = 1, 2, 3$, for any $\Phi : [0, +\infty) \to [0, +\infty)$ non decreasing, differentiable on $(0, \infty)$, with $\Phi(0) = 0$, and satisfying

$$\int_0^\infty \Phi'(k) dk < \infty. \tag{3}$$

(For isotropic fields that will be constructed from B_{ij}, it can be seen that Φ' is the energy spectrum function E.) Then formulas (4.20) and (4.25) in Yaglom (1957) yield

$$B_{ij}(\mathbf{h}) = \int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot\mathbf{h}} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2} \right) \Phi'(|\mathbf{k}|)\ d\mathbf{k}. \tag{4}$$

Consider now fields $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x}))$, $\mathbf{x} = (x_1, x_2, x_3)$ on 3-space, not necessarily solenoidal, and with the property that

$$\int_{\mathbb{R}^3} \left(1 + |\mathbf{x}|^2\right)^r |\mathbf{u}(\mathbf{x})|^2\ d\mathbf{x}, \quad r < -\tfrac{3}{2}, \tag{5}$$

is finite. The restriction on r implies that the "weight" $\left(1 + |\mathbf{x}|^2\right)^r$ has finite integral over $\mathbb{R}^3$. Such a weight is necessary for non-trivial homogeneous statistics to exist, see Vishik & Fursikov (1988), p. 208. In addition to fields with spatial decay, constant and periodic fields, amongst others, satisfy (5).

For any field $\mathbf{u}$ satisfying (5) define $\mathcal{B}\mathbf{u}$ by

$$(\mathcal{B}\mathbf{u}(\mathbf{y}))_i = \sum_{j=1}^{3} \int_{\mathbb{R}^3} B_{ij}(\mathbf{x} - \mathbf{y}) u_j(\mathbf{x}) (1 + |\mathbf{x}|^2)^r\ d\mathbf{x}. \tag{6}$$

Let $\{\mathbf{e_n}\}$ be the eigenfunctions of $\mathcal{B}$ and $\{k_n\}$ the corresponding eigenvalues. (Dostoglou & Gastler (2009) explains why these exist and why $k_n \geq 0$.)

For f_n a sequence of independent, Gaussian, real-valued random variables, distributed as $N(0,1)$, consider the ensemble of fields of the form

$$\mathbf{u}(x) = \sum_{n\geq 0} \sqrt{k_n} f_n \mathbf{e_n}(\mathbf{x}). \tag{7}$$

By construction, the B_{ij}'s satisfy

$$\sum_{j=1}^{3} \partial_j B_{ij} = 0 \tag{8}$$

for all i. This yields $\mathcal{B}(\nabla\phi) = 0$ for any test function ϕ and it shows that the summation in (7) takes place only over divergence-free elements. That is, the ensemble consists purely of solenoidal fields.

This turbulence field has Gaussian statistics with zero mean and correlation $\mathcal{B}$:

$$< \mathcal{B}\mathbf{v}_1, \mathbf{v}_2 >= \overline{< \mathbf{u}, \mathbf{v}_1 >< \mathbf{u}, \mathbf{v}_2 >}, \tag{9}$$

see, for example, Bogachev (1998), p. 49. In addition, the field is homogeneous and isotropic: As a Gaussian field of mean zero, its statistics are determined by $\mathcal{B}$ which, by construction, is invariant under translations and rotations, see Dostoglou & Gastler (2009).

2.1.1. *Averages.* All overlines denote here ensemble averages. Nevertheless, the correlations constructed are decaying in space. For homogeneous statistics, this decay, regardless of rate, guarantees that ensemble averages equal (for almost all fields) the space averages used in measurements or numerical simulations, see Androulakis & Dostoglou (2004), p. 8, Theorem 4.6, and Tempelman (1992), p. 66, Corollary 2.7.

2.2. **The longitudinal correlation.** For $\mathbf{e}_L$ a unit vector, h a positive number, and $\mathbf{u}_L$ the projection of $\mathbf{u}$ on $\mathbf{e}_L$, the longitudinal correlation function

$$B_{LL}(h) = \overline{u_L(x + h\mathbf{e}_L)u_L(x)} \tag{10}$$

corresponding to (1) is

$$B_{LL}(h) = \int_0^\infty \left[\frac{2\sin(hk)}{(hk)^3} - \frac{2\cos(hk)}{(hk)^2} \right] \Phi'(k)dk, \tag{11}$$

see Yaglom (1957), p. 305. Choosing

$$\Phi(k) = \begin{cases} 0 & \text{if } 0 \leq k < 3\pi/2 \\ A(1+\cos(2k)) & \text{if } 3\pi/2 \leq k \leq 2\pi \\ 2A & \text{if } 2\pi < k, \end{cases} \tag{12}$$

where $A > 0$ is an arbitrary constant, all conditions for (1) are satisfied and (11) becomes

$$B_{LL}(h) = -4A \int_{3\pi/2}^{2\pi} \left[\frac{\sin(hk)}{(hk)^3} - \frac{\cos(hk)}{(hk)^2} \right] \sin(2k)\ dk. \tag{13}$$

As

$$B_{LL}(1) = -4A \int_{3\pi/2}^{2\pi} \left[\frac{\sin k}{k^3} - \frac{\cos k}{k^2} \right] \sin(2k)\ dk < 0, \tag{14}$$

and B_{LL} is continuous,

$$B_{LL}(h) < 0, \tag{15}$$

for all h in some interval. A slight modification exhibits a whole family $\Phi_\epsilon, \epsilon \leq \epsilon_0$, such that $\Phi_\epsilon > 0$ for all k and the longitudinal correlations B^ϵ_{LL} are arbitrarily negative on intervals.

Therefore, for this choice of Φ, the fields constructed in the previous section are Gaussian, homogeneous, isotropic, solenoidal, and have negative longitudinal correlation on intervals.

2.3. **The physical relevance of the random fields.** An important feature of the ensemble (7) is that it contains realistic fields. It makes, of course, no sense to claim that a certain field occurs with non-zero probability. What does make sense is to identify fields with the property that any open sub-ensemble containing the field, no matter how small, has non-zero probability to occur. Call a field with this property *realizable*.

All smooth, divergence free fields vanishing outside a bounded domain of $\mathbb{R}^3$ are realizable for (7). To see this, recall from Itô (1970) that, as a Gaussian ensemble, the elements of (7) that are realizable are precisely those elements that are orthogonal, with respect to (5), to all $\mathbf{v}$ satisfying

$$< \mathcal{B}\mathbf{v}, \mathbf{v} >= 0. \tag{16}$$

This, by (4), is the same as

$$\int_{\mathbb{R}^3} \left(|\widetilde{\phi}|^2(\mathbf{k}) - \frac{|\mathbf{k} \cdot \widetilde{\phi}(\mathbf{k})|^2}{|\mathbf{k}|^2} \right) \Phi'(|\mathbf{k}|) d\mathbf{k} = 0, \tag{17}$$

for $\phi(\mathbf{x}) = (1 + |\mathbf{x}|^2)^r \mathbf{v}(\mathbf{x})$ and $\tilde{\phi}$ is its Fourier transform.

For $\Phi' > 0$ everywhere, it follows that (16) is satisfied by $\mathbf{v}$'s such that $\widetilde{\phi}(\mathbf{k}) = M(\mathbf{k})\mathbf{k}$ for all $\mathbf{k}$, with M square integrable, by Dostoglou & Gastler (2009). One then obtains an m such that $\nabla m = \phi$. Therefore, realizable elements of the ensemble are orthogonal to gradients and, in particular, all solenoidal square integrable fields are realizable.

3. Galerkin approximation and time evolution

Dostoglou, Fursikov & Kahl (2006) obtains homogeneous and isotropic ensembles P of Navier-Stokes flows on any finite time interval as the limit of homogeneous and isotropic ensembles $P_{l,n}$ of Galerkin flows approximating the Navier-Stokes flow, as $l, n \to \infty$. The Galerkin flows take place on divergence-free, vector valued trigonometric polynomials of degree n and period $2l$:

$$\mathcal{M}_{l,n} = \left\{ \sum_{|k| \leq n} a_k e^{ik \cdot x} : a_k \cdot k = 0,\ a_k = \overline{a}_{-k},\ k \in \frac{\pi}{l}\mathbb{Z}^3 \right\}, \tag{18}$$

augmented with all its rotations to a space $\widehat{\mathcal{M}}_{l,n}$. The Navier-Stokes flow is approximated at each step by

$$\partial_t u - \nu \Delta u + \widehat{P}_{l,n}[(u, \nabla)u] = 0, \tag{19}$$

where $\widehat{P}_{l,n}$ projects onto $\widehat{\mathcal{M}}_{l,n}$.

The correlation of the (l, n)-Galerkin ensemble $P_{l,n}$ at time t

$$B^{(l,n)}_{ij}(t, \mathbf{h}) = \overline{u_i(t, \mathbf{x}) u_j(t, \mathbf{x} + \mathbf{h})} \tag{20}$$

is continuous in t, see Dostoglou & Gastler (2009). This relies on the fact that the only realizable elements of $P_{l,n}$ are continuous in t. The argument does not use the evolution equation for the correlations as these equations suffer from the well-known closure problem. The argument breaks down for the correlation of the ensemble P itself, as it might contain realizable elements that are not continuous in t. No rigorous proof exists at this point that the Galerkin correlations at time $t > 0$ converge to the correlation of P.

Now since isotropic correlations satisfy

$$B_{ij}(\mathbf{h}) = [B_{LL}(h) - B_{KK}(h)] \frac{h_i h_j}{h^2} + B_{KK}(h)\delta ij \tag{21}$$

for some appropriate function B_{KK}, (for example, see Yaglom (1957), formula (4.36), p. 305), then

$$B_{11}(h, 0, 0) = B_{LL}(h). \tag{22}$$

On the other hand,

$$B_{ij}^{(l,n)}(0, \mathbf{h}) \to B_{ij}(0, \mathbf{h}), \qquad l, n \to \infty, \tag{23}$$

see Dostoglou & Kahl (2009). Therefore $B_{LL}^{(l,n)}(0, h)$ is negative on h-intervals, for l and n large enough. And $B_{LL}^{(l,n)}$ is continuous in time by the continuity of $B_{ij}^{(l,n)}$ and (22). Therefore, starting with the example of § 2.2 as initial condition, for l and n large enough

$$B_{LL}^{(l,n)}(t, h) < 0, \tag{24}$$

on h-intervals and for $t \in [0, t_{l,n})$.

4. Numerical implementation

For $\mathbf{v}$ a (possibly random) field that is $2l$ periodic

$$\mathbf{v}(\mathbf{x}) = \sum_{\mathbf{k} \in \pi l^{-1} \mathbb{Z}^3} \hat{\mathbf{v}}_{\mathbf{k}} e^{i\mathbf{x}\cdot\mathbf{k}}, \tag{25}$$

define a random field by

$$\mathbf{u}(\mathbf{x}) = W \mathbf{v}(W^{-1}(\mathbf{x} - \xi)), \tag{26}$$

where W is a randomly chosen rotation matrix (using the standard probability law on the group of rotation matrices), and ξ is chosen uniformly from $[0, 2l]^3$. Thus $\mathbf{u}$ is distributed according to a homogeneous, isotropic distribution, and

$$B_{ij}(\mathbf{h}) = \overline{u_i(\cdot) u_j(\cdot - \mathbf{h})} \tag{27}$$

and Φ in equation (1) is now replaced by the (not differentiable any more)

$$\Phi(k) = \sum_{\mathbf{k} \in \pi l^{-1} \mathbb{Z}^3 : |\mathbf{k}| < k} \tfrac{1}{2} \overline{|\hat{\mathbf{v}}_{\mathbf{k}}|^2}. \tag{28}$$

Furthermore

$$B_{LL}(h) = \sum_{\mathbf{k} \in \pi l^{-1} \mathbb{Z}^3} \tfrac{1}{2} \overline{|\hat{\mathbf{v}}_{\mathbf{k}}|^2} \left(\frac{2\sin(hk)}{(hk)^3} - \frac{2\cos(hk)}{(hk)^2} \right). \tag{29}$$

For example, suppose that $\mathbf{v}$ is chosen randomly using

$$\hat{\mathbf{v}}_{\mathbf{k}} = \sigma_{|\mathbf{k}|} P_{\mathbf{k}}(\gamma_{\mathbf{k},1}, \gamma_{\mathbf{k},2}, \gamma_{\mathbf{k},3}). \tag{30}$$

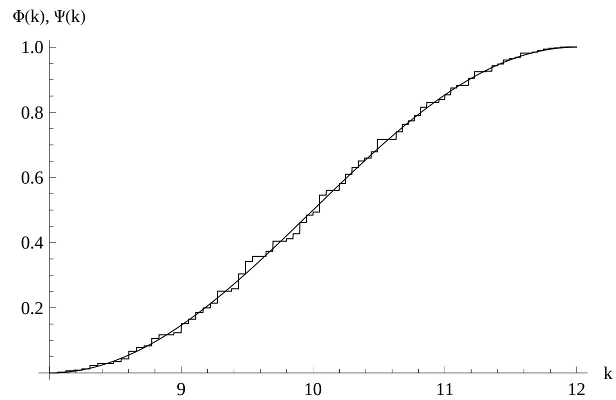

FIGURE 1. A graph showing the similarity between $\Psi(k)$ given by by equation (32), and the resulting $\Phi(k)$ derived from equation (28).

Here $P_{\mathbf{k}}\mathbf{w} = \mathbf{w} - |\mathbf{k}|^{-2}\mathbf{w}\cdot\mathbf{k}\,\mathbf{k}$ and $\gamma_{\mathbf{k},1}$, $\gamma_{\mathbf{k},2}$, $\gamma_{\mathbf{k},3}$ are independent mean zero, variance one, complex Gaussian random variables. The complete joint distribution between all the $\gamma_{\mathbf{k},i}$'s is not specified, although in all numerical simulations they are independent, except, of course, that $\hat{\mathbf{v}}_{\mathbf{k}}$ is the complex conjugate of $\hat{\mathbf{v}}_{-\mathbf{k}}$. Then

$$\frac{1}{2}\overline{|\hat{\mathbf{v}}_{\mathbf{k}}|^2} = |\sigma_{|\mathbf{k}|}|^2. \tag{31}$$

Now for the case $l = \pi$, set

$$\Psi(k) = \begin{cases} 0 & \text{if } k < 8 \\ \frac{1}{2}(1-\cos(2\pi k/8)) & \text{if } 8 \le k \le 12 \\ 1 & \text{if } k > 12 \end{cases} \tag{32}$$

and set

$$\sigma_k = \left(\frac{\Psi'(k)}{4\pi k^2}\right)^{1/2}. \tag{33}$$

Then $\Phi(k)$ and $\Psi(k)$ are related as in figure 1.

A sample $\mathbf{v}$ was chosen using the distribution equation 30, and it was assumed that $\mathbf{u}$ was built as described in equation 26. Putting this into a numerical simulation of the Navier-Stokes equation, with $n = 31$, and viscosity equal to 10^{-4}, and noting that solving the Navier-Stokes for $\mathbf{u}$ is equivalent to solving the Navier-Stokes equation for $\mathbf{v}$ and then applying equation 26 to the solution obtained, the plots of $f(h) = B_{LL}(h)/B_{LL}(0)$ for various values of t are shown in figures 2, 3, 4 and 5. It can be seen that the non-positivity of $f(h)$ seems to persist until at least $t = 2$. After $t = 2$ the precision of the numerical calculations becomes doubtful, or the possibility of negativity might be an artifact of the Gibbs phenomenon as suggested by the apparent lack of smoothness of the graphs in figure 5.

The complete software and data can be downloaded at:

`http://www.math.missouri.edu/~stephen/navier3d/navier3d-1.1-d.tar.gz`

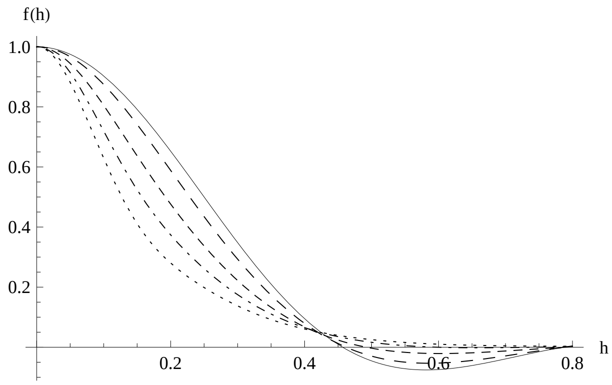

FIGURE 2. Time evolution of $f(h) = B_{LL}(h)/B_{LL}(0)$ for $0 \leq h \leq 0.8$, $0 \leq t \leq 0.4$:
——, t=0; — —, t=0.1; – – –, t=0.4; · – ·–, t=0.3; · · · · · ·, t=0.4;

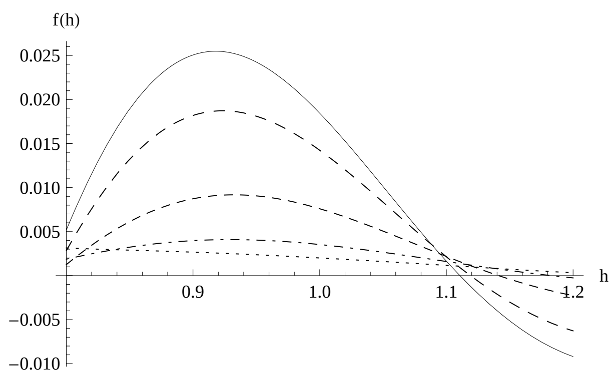

FIGURE 3. Time evolution of $f(h) = B_{LL}(h)/B_{LL}(0)$ for $0.8 \leq h \leq 1.2$, $0 \leq t \leq 0.4$:
——, t=0; — —, t=0.1; – – –, t=0.4; · – ·–, t=0.3; · · · · · ·, t=0.4;

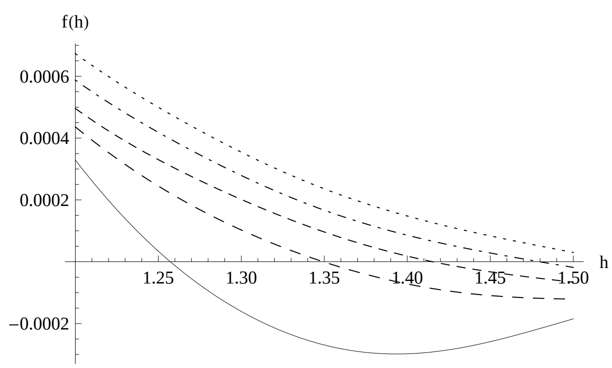

FIGURE 4. Time evolution of $f(h) = B_{LL}(h)/B_{LL}(0)$ for $1.2 \leq h \leq 1.5$, $0.4 \leq t \leq 0.8$:
——, t=0.4; — —, t=0.5; – – –, t=0.6; ·–·–, t=0.7; · · · · · ·, t=0.8;

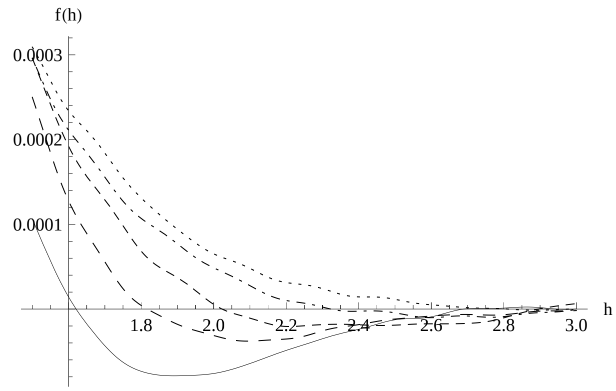

FIGURE 5. Time evolution of $f(h) = B_{LL}(h)/B_{LL}(0)$ for $1.5 \leq h \leq 3$, $1 \leq t \leq 3$:
——, t=1; — —, t=1.5; – – –, t=2; · – ·–, t=2.5; · · · · · ·, t=3;

5. Conclusion

To the extent that Gaussian initial conditions are acceptable and Galerkin approximations are widely used for numerical results, the possibility of negative correlation remains. In experiments, "fully developed turbulence" is created by a process that agitates the fluid for some length of time. Then immediately after the agitation ceases, and only then, the measurements are taken. Thus it would seem reasonable that the class of possibilities for the profile of the fluid velocity would be more restrictive than that it is merely isotropic, homogeneous and/or Gaussian. While some readers might regard the example presented here as an exotic velocity field, this paper does raise the question as to why it should be considered exotic. That is, what is the new property that should be considered, that this velocity field violates.

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