Complemented Subspaces of Spaces Obtained by Interpolation

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ABSTRACT: If Z is a quotient of a subspace of a separable Banach space X, and V is any separable Banach space, then there is a Banach couple (A_0, A_1) such that A_0 and A_1 are isometric to $X \oplus V$, and any intermediate space obtained using the real or complex interpolation method contains a complemented subspace isomorphic to Z. Thus many properties of Banach spaces, including having non-trivial cotype, having the Radon–Nikodym property, and having the analytic unconditional martingale difference sequence property, do not pass to intermediate spaces.

There are many Banach space properties that pass to spaces obtained by the complex method of interpolation. For example, it is known that if a couple (A_0, A_1) is such that A_0 and A_1 both have the UMD (unconditional martingale difference sequence) property, and if A_θ is the space obtained using the complex interpolation method with parameter θ, then A_θ has the UMD property whenever $0 < \theta < 1$. Another example is type of Banach spaces: if A_0 has type p_0 and A_1 has type p_1, then A_θ has type p_θ, where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$.

Similar results are true for the real method of interpolation. If we denote by $A_{\theta,p}$ the space obtained using the real interpolation method from a couple (A_0, A_1) with parameters θ and p, then $A_{\theta,p}$ has the UMD property whenever A_0 and A_1 have the UMD property, $0 < \theta < 1$, and $1 < p < \infty$. Similarly, if A_0 has type p_0 and A_1 has type p_1, then $A_{\theta,p}$ has type p_θ, where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ and $p = p_\theta$ (see [**5**] 2.g.22).

However, there are other properties for which it has been hitherto unknown whether they pass to the intermediate spaces. Examples include the Radon–Nikodym property, the AUMD (analytic unconditional martingale difference sequence) property, and having non-trivial cotype.

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This paper deals with these properties, showing that they do not pass to the intermediate spaces. Indeed, we show the surprising fact that there is a couple (A_0, A_1) such that A_0 and A_1 are both isometric to l_1, but all the real or complex intermediate spaces contain a complemented subspace isomorphic to c_0. This improves a result due to Pisier, who gave an example of a couple (A_0, A_1) for which A_0 is isometric to L_1, A_1 is isometric to a dense subspace of c_0, and c_0 is finitely represented in every intermediate space A_θ obtained by the complex interpolation method (see [**3**]).

Notation

Here we outline the notation we will use about interpolation couples. The reader is referred to [**1**] or [**2**] for details.

A *Banach couple* is a pair of Banach spaces (A_0, A_1) such that A_0 and A_1 both embed into a common topological vector space, Ω, which we shall call the *ambient space*. Given such a couple, we define Banach spaces $A_0 + A_1$ (with norm $\|x\| = \inf\{\, \|x_0\|_{A_0} + \|x_1\|_{A_1} : x_0 \in A_0,\, x_1 \in A_1,\, x_0 + x_1 = x\}$) and $A_0 \cap A_1$ (with norm $\|x\| = \max\{\|x\|_{A_0}, \|x\|_{A_1}\}$). A map between two couples $T : (A_0, A_1) \to (B_0, B_1)$ is a linear map $T : A_0 + A_1 \to B_0 + B_1$ such that $T(A_0) \subseteq B_0$ and $T(A_1) \subseteq B_1$, and such that $\|T\|_{A_0 \to B_0}, \|T\|_{A_1 \to B_1} < \infty$.

An *interpolation method*, I, is a functor that takes a Banach couple (A_0, A_1) to a single Banach space A_I, such that $A_0 \cap A_1 \subseteq A_I \subseteq A_0 + A_1$ with

$$c^{-1} \|x\|_{A_0 + A_1} \le \|x\|_{A_I} \le c\, \|x\|_{A_0 \cap A_1}\,, \tag{1}$$

and so that if $T : (A_0, A_1) \to (B_0, B_1)$ is a map between couples, then $T(A_I) \subseteq B_I$ with $\|T\|_{A_I \to B_I} < \infty$.

An interpolation method is called *exponential* with *exponent* θ if $0 < \theta < 1$, and whenever $T : (A_0, A_1) \to (B_0, B_1)$ is a map between couples, then

$$\|T\|_{A_I \to B_I} \le c\, \|T\|_{A_0 \to B_0}^{1-\theta} \|T\|_{A_1 \to B_1}^{\theta}\,. \tag{2}$$

An interpolation method is called *exact exponential* if it is exponential and $c = 1$ in inequalities (1) and (2).

The most well known interpolation methods are the real interpolation method, and the complex interpolation method. They are both exponential, and the complex interpolation method is exact exponential. Another interpolation method, parameterized by $0 < \theta < 1$, is the following: if $x \in A_0 \cap A_1$, then let

$$|||x||| = \inf \left\{ \sum_{i=1}^{n} \|x_i\|_{A_0}^{1-\theta} \|x_i\|_{A_1}^{\theta} : \sum_{i=1}^{n} x_i = x \right\}.$$

Let $A_{\min,\theta}$ be the completion of $(A_0 \cap A_1, |||\cdot|||)$. This interpolation method is exact exponential of exponent θ. Furthermore, if I is another exponential method of exponent θ, then $\|x\|_{A_I} \le c\,\|x\|_{A_{\min,\theta}}$, with $c = 1$ if I is exact exponential. (In fact, this interpolation method gives a space equivalent to the space $A_{\theta,1}$ obtained by the real interpolation method.)

We denote by $X \oplus V$ the Banach space of ordered pairs (x, v) with norm $\|(x, v)\| = \|x\| + \|v\|$.

All the results given work for Banach spaces over the real or complex scalars. (Obviously the complex interpolation method will require the complex scalars.)

The Main Result

Here we present the main result of this paper.

Theorem 1. *Suppose that Z is a Banach space and that Q_i is a quotient mapping onto Z from a closed linear subspace Y_i of a separable Banach space X_i, for $i = 0, 1$, such that the following dimensional constraints hold*:

a) *the dimensions of Y_0 and Y_1 are equal*;
b) *the codimensions of Y_0 in X_0 and Y_1 in X_1 are equal*;
c) *the dimensions of the kernels of Q_0 and Q_1 are equal.*

Suppose further that $0 < \epsilon < 1$, that $0 < \theta_0 \le \theta_1 < 1$ and that V_0 and V_1 are any infinite dimensional separable Banach spaces.

Then there is a Banach couple (A_0, A_1), a subspace W of $A_0 + A_1$, and linear maps $P : W \to Z$ and $E : Z \to W$ with the following properties:

i) *A_0 is isometric to $X_0 \oplus V_0$, and A_1 is isometric to $X_1 \oplus V_1$*;
ii) *$P \circ E = \mathrm{Id}_Z$*;

if I is an exponential interpolation method of exponent θ, then

iii) *$A_I \subseteq W$ and $\|P\|_{A_I \to Z} < \infty$ (with $\|P\|_{A_I \to Z} < 1 + \epsilon/3$ if I is exact and $\theta_0 \le \theta \le \theta_1$)*;
iv) *$E(Z) \subseteq A_I$ and $\|E\|_{Z \to A_I} < \infty$ (with $\|E\|_{Z \to A_I} < 1 + \epsilon/3$ if I is exact and $\theta_0 \le \theta \le \theta_1$).*

Thus Z is isomorphic to a complemented subspace of A_I. (If I is exact and $\theta_0 \le \theta \le \theta_1$, then Z is $(1 + \epsilon)$ isomorphic to a $(1 + \epsilon)$-complemented subspace of A_I.)

This has the following corollaries.

Corollary. *Given any separable Banach space X, there is a Banach couple (A_0, A_1) such that A_0 and A_1 are isometric to l_1, and for every exponential interpolation method I, the intermediate space A_I contains a complemented subspace isomorphic to X.*

Corollary. *Given any separable Banach space X, there is a Banach couple (A_0, A_1) such that A_0 and A_1 are isometric to $C([0,1])$, and for every exponential interpolation method I, the intermediate space A_I contains a complemented subspace isomorphic to X.*

Corollary. *The following properties do not pass to intermediate spaces by any exponential interpolation methods, in the sense that there is a couple (A_0, A_1) such that A_0 and A_1 both have the property, but none of the intermediate spaces do:*

i) *the Radon–Nikodym property;*
ii) *the analytic Radon-Nikodym property;*
iii) *having non-trivial cotype;*
iv) *the AUMD property;*
v) *having a dual with non-trivial cotype;*
vi) *having a dual with the AUMD property.*

Note that parts (v) and (vi) of the above corollary follow because $C([0,1])^*$ is finitely represented in l_1.

A More Elementary Result

Before presenting the proof of Theorem 1, we first prove a more elementary result that is, in fact, a corollary of Theorem 1. The reason for this is that the proof of Theorem 1 involves many technicalities, disguising the essential idea of the proof, which is very simple.

Theorem 2. *There is a Banach couple (A_0, A_1) such that A_0 and A_1 are isometric to l_1, and for every exponential interpolation method I, the intermediate space A_I contains a complemented subspace isomorphic to c_0.*

Proof: Let e_n be the unit vectors in l_1 and c_0, and let r_n be an enumeration of the 'corners of the unit cube' in c_0, that is, all vectors of the form $(\pm 1, \pm 1, \ldots, \pm 1, 0, 0, 0, \ldots)$. Let ϵ_n be the sequence of numbers defined by

$$\epsilon_n = (2 + \|r_n\|_1)^{-n}.$$

(Any sequence of numbers tending sufficiently rapidly to zero will do.) We form an ambient space $\Omega = c_0 \oplus l_1 \oplus l_1$. We define subspaces

$$A_0 = \left\{ \left(\sum_n a_n e_n + b_n r_n, \sum_n \epsilon_n b_n e_n, \sum_n c_n e_n \right) : (a_n),\, (b_n),\, (c_n) \in l_1 \right\}$$
$$A_1 = \left\{ \left(\sum_n a_n e_n + c_n r_n, \sum_n b_n e_n, \sum_n \epsilon_n c_n e_n \right) : (a_n),\, (b_n),\, (c_n) \in l_1 \right\},$$

with norms

$$\left\|\left(\sum_n a_n e_n + b_n r_n, \sum_n \epsilon_n b_n e_n, \sum_n c_n e_n\right)\right\|_{A_0} = \sum_n \left(|a_n| + |b_n| + |c_n|\right)$$
$$\left\|\left(\sum_n a_n e_n + c_n r_n, \sum_n b_n e_n, \sum_n \epsilon_n c_n e_n\right)\right\|_{A_1} = \sum_n \left(|a_n| + |b_n| + |c_n|\right).$$

The idea is that the unit balls of A_0 and A_1 are 'slightly perturbed' versions of the unit balls of c_0, where the perturbations for A_0 and A_1 go in different directions. These perturbations cause the unit balls of A_0 and A_1 to be convex hulls of linearly independent vectors, and hence they are affine to the unit ball of l_1. The size of these perturbations is controlled by the quantities ϵ_n. The vectors that are used to perturb the unit ball of A_0 are also contained in A_1 with no control on their size. Similarly, the perturbing vectors of A_1 are contained in A_0. Thus, when we form the intermediate space A_I, the perturbations in A_0 are 'swamped out' or 'washed away' by the vectors in A_1, and similarly for the perturbations in A_1, and we are left with a complemented copy of c_0. So much for the idea—now we give the proof.

We define a projection $P : \Omega \to c_0$ by

$$P(x, y, z) = x.$$

It is easy to see that $\|P\|_{A_i \to c_0} \le 1$ for $i = 0, 1$, and hence $\|P\|_{A_I \to c_0} < \infty$ for any interpolation method I.

We define an embedding $E : c_0 \to \Omega$ by

$$E(x) = (x, 0, 0).$$

We prove that for every exponential interpolation method that $E(x) \in A_I$ $(x \in c_0)$, with $\|E\|_{c_0 \to A_I} < \infty$. To do this, it is sufficient to show that there is a constant c, depending on the exponential interpolation method I only, such that for every $n \ge 1$ we have

$$\|(r_n, 0, 0)\|_{A_I} \le c.$$

We first note the following facts.

$$\|(r_n, \epsilon_n e_n, 0)\|_{A_0} = \|(r_n, 0, \epsilon_n e_n)\|_{A_1} = 1$$

$$\|(0, 0, \epsilon_n e_n)\|_{A_0} = \|(0, \epsilon_n e_n, 0)\|_{A_1} = \epsilon_n$$

$$\|(r_n,0,0)\|_{A_0} = \|(r_n,0,0)\|_{A_1} = \|r_n\|_1 .$$

Let us suppose that I is of exponent θ. We shall give the details in the case where I is exact exponential: a similar argument works in the general case, with less control of the constants. We have the following inequalities.

$$\|(r_n,0,0)\|_{A_I} \le \|(r_n,\epsilon_n e_n,\epsilon_n e_n)\|_{A_I} + \|(0,\epsilon_n e_n,0)\|_{A_I} + \|(0,0,\epsilon_n e_n)\|_{A_I} ,$$

and

$$\begin{aligned}\|(r_n,\epsilon_n e_n,\epsilon_n e_n)\|_{A_I} \le& \|(r_n,\epsilon_n e_n,\epsilon_n e_n)\|_{A_0}^{1-\theta} \|(r_n,\epsilon_n e_n,\epsilon_n e_n)\|_{A_1}^{\theta} \\ \le& \left(\|(r_n,\epsilon_n e_n,0)\|_{A_0} + \|(0,0,\epsilon_n e_n)\|_{A_0} \right)^{1-\theta} \\ &\times \left(\|(r_n,0,\epsilon_n e_n)\|_{A_1} + \|(0,\epsilon e_n,0)\|_{A_1} \right)^{\theta} \\ \le& (1+\epsilon_n)^{1-\theta}(1+\epsilon_n)^{\theta} \\ \le& (1+\epsilon_n),\end{aligned}$$

and

$$\|(0,\epsilon_n e_n,0)\|_{A_I} \le \|(0,\epsilon_n e_n,0)\|_{A_0}^{1-\theta} \|(0,\epsilon_n e_n,0)\|_{A_1}^{\theta} .$$

But

$$\begin{aligned}\|(0,\epsilon_n e_n,0)\|_{A_0} \le& \|(r_n,\epsilon_n e_n,0)\|_{A_0} + \|(r_n,0,0)\|_{A_0} \\ \le& 1 + \|r_n\|_1 ,\end{aligned}$$

and so

$$\|(0,\epsilon_n e_n,0)\|_{A_I} \le (1+\|r_n\|_1)^{1-\theta} \epsilon_n^{\theta}.$$

Similarly

$$\|(0,0,\epsilon_n e_n)\|_{A_I} \le (1+\|r_n\|_1)^{\theta} \epsilon_n^{1-\theta}.$$

Therefore

$$\|(r_n,0,0)\|_{A_I} \le 1 + \epsilon_n + (1+\|r_n\|_1)^{1-\theta}\epsilon_n^{\theta} + (1+\|r_n\|_1)^{\theta}\epsilon_n^{1-\theta}.$$

By our choice of ϵ_n, this is bounded by some constant c that depends only upon θ.

Thus we have bounded maps

$$P : A_I \to c_0 \qquad E : c_0 \to A_I$$

such that $P \circ I = \mathrm{Id}_{c_0}$, and hence c_0 is isomorphic to a complemented subspace of A_I.

Finally, it is very clear that A_0 and A_1 are both isometric to l_1. □

The Proof of Theorem 1

In the sequel, we will make much use of biorthogonal systems. A *biorthogonal system* of a Banach space X is a sequence $(x_n;\xi_n) \in X \times X^*$ such that $\xi_n(x_m) = 0$ if $n \neq m$ and 1 if $n = m$. A biorthogonal system is called *fundamental* if $\overline{\text{span}}\{x_n\} = X$, and it is called *total* if $x = 0$ whenever $\xi_n(x) = 0$ for all n. (Of course, if X is finite dimensional, then the sequences will be finite.) A result of Markushevich shows that every separable Banach space has a total, fundamental biorthogonal system (see [**6**] or [**4**] 1.f.3).

We shall need the following proposition and its corollary.

Lemma 3. *Let Z be a quotient of a separable Banach space Y by the quotient map Q with kernel K. Suppose that $(k_p;\kappa'_p)$ is a total fundamental biorthogonal system for K, and that $(z_n;\zeta_n)$ is a total fundamental biorthogonal system for Z. Then there are sequences (y_n) in Y, and (ψ_n) and (κ_p) in Y^*, for which the following hold:*

i) $(y_n, k_p; \psi_n, \kappa_p)$ *is a total fundamental biorthogonal system for Y;*

ii) $Q(y_n) = z_n$ *and* $\zeta_n \circ Q = \psi_n$ *for all n;*

iii) $y \in K$ *if and only if* $\psi_n(y) = 0$ *for all n.*

Proof: Choose $\tilde{y}_n \in Y$ such that $Q(\tilde{y}_n) = z_n$, and let $\psi_n = \zeta_n \circ Q$. Using the Hahn–Banach Theorem, extend κ'_p to κ_p on Y so that $\kappa_p(\tilde{y}_n) = 0$ for $n < p$. Set

$$y_n = \tilde{y}_n - \sum_{m=1}^{n} \kappa_m(\tilde{y}_n) k_m.$$

It is now straightforward to verify the conclusions of the lemma. □

Lemma 4. *Let Y be a closed subspace of a separable Banach space X, and let $(y_n;\psi'_n)$ be a total fundamental biorthogonal system for Y. Then there are sequences (x_m) in X, and (ξ_m) and (ψ_n) in X^*, such that $(x_m, y_n; \xi_m, \psi_n)$ is a total fundamental biorthogonal system for X, and such that an element x in X belongs to Y if and only if $\xi_m(x) = 0$ for all m.*

Proof: By Markushevich's result, Y has a total fundamental system $(y_n;\psi'_n)$, and X/Y has a total fundamental system $(z_m;\zeta_m)$. Then the result follows by applying Lemma 3 to the quotient mapping $Q : X \to X/Y$. □

Proof of Theorem 1: We begin by defining some biorthogonal systems. First, let (z_n, ζ_n) be a total fundamental biorthogonal system for Z, with $\|\zeta_n\| = 1$ for all n. Using Lemmas 3 and 4, for $i = 0, 1$, we can find total fundamental biorthogonal systems

$$(x^i_m, y^i_n, k^i_p; \xi^i_m, \psi^i_n, \kappa^i_p)$$

for X_i with the following properties:

i) $Q_i(y^i_n) = z_n$ and $\zeta_n \circ Q_i = \psi^i_n$ for all n;
ii) $Y_i = \{ x \in X_i : \xi^i_m(x) = 0 \text{ for all } m\}$;
iii) $(y^i_n, k^i_p; \psi^i_n|_{Y_i}, \kappa^i_p|_{Y_i})$ is a total fundamental biorthogonal system for Y_i;
iv) $(k^i_p; \kappa^i_p|_{K_i})$ is a total fundamental biorthogonal system for K_i, the kernel of Q_i;
v) $\|\xi^i_m\| = \|\psi^i_n\| = \|\kappa^i_p\| = 1$ for all m, n and p.

We also choose total fundamental biorthogonal systems $(v^i_q; \phi^i_q)$ for V_i, with $\|\phi^i_q\| = 1$ for all q. If any of these systems is finite, then we extend it to an infinite system by including zero terms.

Next we define some sequences of rapidly decreasing numbers. Let ν_n and μ_n be sequences of positive numbers for which

$$\sum L_n \nu_n \le \epsilon/12,$$
$$\max\{\mu^\theta_n L_n, \mu^{1-\theta}_n L_n\} \le \nu_n \qquad \text{for } \theta_0 \le \theta \le \theta_1,$$

and

$$\sum \mu^\theta_n L_n < \infty \qquad \text{for } 0 < \theta < 1,$$

where

$$L_n = \max\{ \|x^i_n\|, \|y^i_n\|, \|k^i_p\|, \|v^i_{2n-1}\|, \|v^i_{2n}\|, 1 : i = 0, 1\}.$$

Now we define the Banach couple. We take as the ambient space $\Omega = l_\infty \oplus l_\infty \oplus l_\infty \oplus l_\infty \oplus l_\infty$. We define linear maps $M_i : X_i \oplus V_i \to \Omega$ in the following way:

$$M_0(x, v) = \Big(\big(\psi^0_n(x)\big), \big(\xi^0_m(x)\big), \big(\mu_p \kappa^0_p(x)\big), \big(\mu_q \phi^0_{2q-1}(v)\big), \big(\phi^0_{2q}(v)\big) \Big),$$
$$M_1(x, v) = \Big(\big(\psi^1_n(x)\big), \big(\mu_q \phi^1_{2q-1}(v)\big), \big(\phi^1_{2q}(v)\big), \big(\xi^1_m(x)\big), \big(\mu_p \kappa^1_p(x)\big), \Big).$$

The important feature here is the interchange in the order of the terms. We note that M_0 and M_1 are one-one maps into Ω. For $i = 0, 1$, we set $A_i = M_i(X_i \oplus V_i)$ with their norms inherited from the domains. We take W to be the linear span of all the exponential interpolation spaces A_I.

We shall now go through the details for the case when I is exact and $\theta_0 \le \theta \le \theta_1$. Otherwise the arguments are similar, with less control of the constants. Let e_n denote the nth unit vector in l_∞. Then

$$(0, \mu_n e_n, 0, 0, 0) = \mu_n M_0(x_n^0) = M_1(v_{2n-1}^1),$$

so that

$$\|(0, \mu_n e_n, 0, 0, 0)\|_{A_0} \le \mu_n L_n,$$

and

$$\|(0, \mu_n e_n, 0, 0, 0)\|_{A_1} \le L_n.$$

Hence

$$\|(0, \mu_n e_n, 0, 0, 0)\|_{A_I} \le \mu_n^{1-\theta} L_n \le \nu_n.$$

Similarly,

$$\|(0, 0, \mu_n e_n, 0, 0)\|_{A_I} \le \nu_n,$$
$$\|(0, 0, 0, \mu_n e_n, 0)\|_{A_I} \le \nu_n,$$

and

$$\|(0, 0, 0, 0, \mu_n e_n)\|_{A_I} \le \nu_n.$$

We define linear functionals on Ω as follows: if $t = (h, a, b, c, d) \in \Omega$, then let $\eta_n(t) = h_n$, $\alpha_n(t) = a_n$, $\beta_n(t) = b_n$, $\gamma_n(t) = c_n$ and $\delta_n(t) = d_n$. Then we have that

$$\|\alpha_n\|_{A_0^*} = 1$$

and

$$\|\alpha_n\|_{A_1^*} = \mu_n,$$

and so

$$\|\alpha_n\|_{A_I^*} \le \mu_n^\theta \le \nu_n.$$

Similarly, $\|\beta_n\|_{A_I^*} \le \nu_n$, $\|\gamma_n\|_{A_I^*} \le \nu_n$ and $\|\delta_n\|_{A_I^*} \le \nu_n$.

Now we define $E : Z \to \Omega$ by $E(z) = \Big(\big(\zeta_n(z)\big), 0, 0, 0, 0\Big)$. We assert that $E(z) \in A_I$, with $\|E(z)\|_{A_I} \le (1 + \epsilon/3)\,\|z\|$. To show this, we can suppose that $\|z\| < 1$. For $i = 0, 1$, choose $y_i \in Y_i$ such that $Q_i(y_i) = z$ and $\|y_i\| \le 1$. Let

$$v_0 = \sum_p \mu_p \kappa_p^1(y_1) v_{2p}^0 \quad \text{and} \quad v_1 = \sum_p \mu_p \kappa_p^0(y_0) v_{2p}^1.$$

It follows from the choice of μ_p that these sums converge in V_0 and V_1, and that $\|v_0\| \le \epsilon/12$ and $\|v_1\| \le \epsilon/12$. Now

$$\begin{aligned} M_0(y_0, v_0) &= M_1(y_1, v_1) \\ &= \Big((\zeta_n(z)), 0, (\mu_p \kappa_p^0(y_0)), 0, (\mu_p \kappa_p^1(y_1)) \Big) \\ &= w, \end{aligned}$$

say, so that $w \in A_I$ with $\|w\|_{A_I} \le 1 + \epsilon/12$. But

$$\Big\| \Big(0, 0, (\mu_p \kappa_p^0(y_0)), 0, 0\Big) \Big\|_{A_I} \le \sum \mu_p L_p \le \epsilon/12,$$

and similarly,

$$\Big\| \Big(0, 0, 0, 0, (\mu_p \kappa_p^0(y_0))\Big) \Big\|_{A_I} \le \epsilon/12.$$

Hence $E(z) \in A_I$ with $\|E(z)\| \le 1 + \epsilon/3$, as desired.

Now we turn to the problem of defining P. To do this, we will first show how we may consider Z as a subspace of what we shall call $X_0 \oplus_Q X_1$. We let

$$D = \{\, (y_0, y_1) : y_i \in Y_i, Q_0(y_0) + Q_1(y_1) = 0 \},$$

and let $X_0 \oplus_Q X_1 = X_0 \oplus X_1 / D$. We denote by Q_D the quotient map from $X_0 \oplus X_1$ to $X_0 \oplus_Q X_1$. We define $J : Z \to X_0 \oplus_Q X_1$ by setting $J\big(Q_0(y)\big) = Q_D(y, 0)$ for $y \in Y_0$. It is easy to see that J is well defined, and that $J\big(Q_1(y)\big) = Q_D(0, y)$ for $y \in Y_1$.

Suppose that $w = Q_D(x_0, x_1) \in X_0 \oplus_Q X_1$. Then it is easy to verify that $\pi_m^0(w) = \xi_m^0(x_0)$ and $\pi_m^1(w) = \xi_m^1(x_1)$ are well defined maps. It is also easy to check that Z is isometric to the space

$$J(Z) = \{\, w : \pi_m^0(w) = \pi_m^1(w) = 0 \}.$$

Now, we will define two bounded operators S and T from A_I to $X_0 \oplus_Q X_1$. First we will concentrate on S, which will, in fact, be defined on $A_0 + A_1$.

If $t = M_0(x, v) \in A_0$, we let

$$R_0(t) = \sum \mu_m \phi_{2m-1}^0(v) x_m^1.$$

By the choice of μ_m, the sum converges in X_1 with $\|R_0(t)\| \le \|v\|$. Thus if we set $S_0(t) = Q_D\big(x, R_0(t)\big)$, then S_0 is a norm decreasing map from A_0 to $X_0 \oplus_Q X_1$. We define S_1 in a similar way.

Now, if $t = M_0(x_0, v_0) = M_1(x_1, v_1) \in A_0 \cap A_1$, then

$$\xi^0_m(x_0) = \mu_m \phi^1_{2m-1}(v_1) = \xi^0_m\big(R_1(t)\big)$$

and

$$\xi^1_m(x_1) = \mu_m \phi^0_{2m-1}(v_0) = \xi^1_m\big(R_0(t)\big),$$

so that $x_0 - R_1(t) \in Y_0$ and $x_1 - R_0(t) \in Y_1$. Further,

$$\zeta_n\big(Q_0(x_0 - R_1(t))\big) = \psi^0_n(x_0 - R_1(t)) = \psi^0_n(x_0),$$

and similarly

$$\zeta_n\big(Q_1(x_1 - R_0(t))\big) = \psi^1_n(x_1).$$

Then, since $\psi^0_n(x_0) = \psi^1_n(x_1) = \eta_n(t)$, and since (ζ_n) is total, it follows that $Q_0\big(x_0 - R_1(t)\big) = Q_1\big(x_1 - R_0(t)\big)$. Thus we see that

$$S_0(t) = Q_D\big(x_0 - R_1(t)\big) = Q_D\big(x_1 - R_0(t)\big) = S_1(t),$$

and so we may define $S : A_0 + A_1 \to X_0 \oplus_Q X_1$ by setting $S|_{X_0} = S_0$ and $S|_{X_1} = S_1$. Clearly, S maps A_I into $X_0 \oplus_Q X_1$ with $\|S\|_{A_I \to X_0 \oplus_Q X_1} \le 1$.

Now we establish the effect of π^0_m and π^1_m on S. If $t = (h, a, b, c, d) = M_0(x, v) \in A_0$, then

$$\pi^0_m\big(S_0(t)\big) = \pi^0_m\big(Q_D(x - R_0(t))\big) = \xi^0_m(x) = a_m,$$

and

$$\pi^1_m\big(S_0(t)\big) = \pi^0_m\big(Q_D(x - R_0(t))\big) = \xi^0_m\big(R_0(t)\big) = \mu_m \phi_{2m-1}(v) = c_m.$$

Similarly, if $t = (h, a, b, c, d) = M_1(x, v) \in A_1$, then $\pi^0_m\big(S_1(t)\big) = a_m$ and $\pi^1_m\big(S_1(t)\big) = c_m$. Thus, if $t = (h, a, b, c, d) \in A_I$, then

$$\pi^0_m\big(S(t)\big) = a_m \quad \text{and} \quad \pi^1_m\big(S(t)\big) = c_m.$$

Next, we define $T : A_I \to X_0 \oplus_Q X_1$ by

$$T(t) = Q_D\left(\sum \alpha_m(t) x^0_m, \sum \gamma_m(t) x^1_m\right).$$

Note that the sums converge, and that

$$\|T\| \le \sum \nu_m(\|x^0_m\| + \|x^1_m\|) \le \epsilon/6.$$

We see that, if $t = (h, a, b, c, d)$, then $\pi^0_m\big(T(t)\big) = a_m$ and $\pi^1_m\big(T(t)\big) = c_m$.

Hence $(S-T)(t) \in J(Z)$. So now we set $P = J^{-1} \circ (S-T)$. We have that $\|P\|_{A_I \to Z} \leq 1 + \epsilon/3$, and

$$\begin{aligned} P \circ E(z_n) &= P(u_n, 0, 0, 0, 0) \\ &= J^{-1} \circ (S-T)(u_n, 0, 0, 0, 0) \\ &= J^{-1} \circ S(u_n, 0, 0, 0, 0) \\ &= J^{-1} \circ S_0 \circ M_0(y^0_n, 0) \\ &= J^{-1} \circ Q_D(y^0_n, 0) = z_n, \end{aligned}$$

so that $P \circ E = \mathrm{Id}_Z$. □

Further Results and Conjectures

An obvious extension to Theorem 1 would be the following.

Conjecture. *Suppose that (Z_0, Z_1) is a Banach couple where Z_i is a quotient of a subspace of X_i, for $i = 0, 1$, suppose that X_0, X_1, V_0 and V_1 are separable Banach spaces, and suppose that suitable dimension constraints are satisfied. Then there is a Banach couple (A_0, A_1) such that*

i) A_0 is isometric to $X_0 \oplus V_0$, and A_1 is isometric to $X_1 \oplus V_1$;
ii) for every exponential interpolation method I, the space A_I contains a complemented subspace isomorphic to Z_I.

The second named author has a tentative proof of a local version of this result for the real interpolation method, which he hopes to publish later.

However, we can show the following.

Theorem 5. *Suppose (Y_0, Y_1) is a Banach couple, where $Y_1 \subset Y_0$ with $\|y\|_{Y_0} \leq c\,\|y\|_{Y_1}$. Suppose further that Y_0 is a quotient space of a separable Banach space X_0, and Y_1 is a complemented subspace of a separable Banach space X_1, such that the codimensions of Y_0 in X_0 and Y_1 in X_1 are equal. Then there is a Banach couple (A_0, A_1) such that*

i) A_0 is isometric to X_0, and A_1 is isometric to X_1;
ii) given any exponential interpolation method of exponent θ, there are maps $E : Y_{\min,\theta} \to A_I$ and $P : A_I \to Y_I$, both of bounded norm, such that $P \circ E = \mathrm{Id}_{Y_{\min,\theta}}$

Corollary. *There is a Banach couple (A_0, A_1) such that A_0 is isometric to l_1, and A_1 is isometric to l_p, and for any exponential interpolation method I of exponent θ, the interme-*

diate space A_I contains a subspace V such that $l_{p/\theta,1} \subseteq V \subseteq l_{p/\theta,\infty}$ with $c^{-1}\left\|x\right\|_{p/\theta,\infty} \le \left\|x\right\|_V \le c\left\|x\right\|_{p/\theta,1}$.

Corollary. *There is a Banach couple (A_0, A_1) such that A_0 has cotype 2, and A_1 has cotype p and is K-convex, and such that for all $0 < \theta < 1$, the real interpolation space $A_{\theta,1}$ does not have cotype r for any $r < p/\theta$.*

This shows that a result of Xu [**7**] cannot be improved. He showed that, if (A_0, A_1) is a Banach couple such that A_1 is K-convex with cotype p, then for all $0 < \theta < 1$, the real interpolation space $A_{\theta,1}$ has cotype p/θ.

Proof of Theorem 5: Suppose that Y_0 is a quotient of X_0 by a subspace Z_0, and that $X_1 = Y_1 \oplus Z_1$. Denote the quotient map from X_0 to Y_0 by Q. Let $(z_n^0; \tilde{\zeta}_n^0)$ be a total fundamental biorthogonal sequence for Z_0, and let $(z_n^1; \tilde{\zeta}_n^1)$ be a total fundamental biorthogonal sequence for Z_1. Extend $\tilde{\zeta}_n^0$ to ζ_n^0 on X_0 using the Hahn–Banach Theorem, and define ζ_n^1 on X_1 by setting $\zeta_n^1(y,z) = \tilde{\zeta}_n^1(z)$. Without loss of generality, $\left\|\zeta_n^0\right\| = \left\|\zeta_n^1\right\| = 1$.

To save chasing constants, let us assume that I is exact. Let ϵ_n be a sequence of sufficiently small positive numbers, so that

$$\delta = \sum_{n=1}^{\infty} \max\{\epsilon_n, \epsilon_n^\theta\} \max\{\left\|z_n^0\right\|, \left\|z_n^1\right\|\} < \infty,$$

for all $0 < \theta < 1$. We will let $Y_0 \oplus l_\infty$ be the ambient space. Let $M_i : X_i \to Y_0 \oplus l_\infty$ be the mappings:

$$M_0(x) = \Big(Q(x), \big(\epsilon_n \zeta_n^0(x)\big)\Big)$$
$$M_1(x) = \Big(Q(x), \big(\zeta_n^1(x)\big)\Big).$$

Since $(z_n^0; \tilde{\zeta}_n^0)$ and $(z_n^1; \tilde{\zeta}_n^1)$ are both total biorthogonal systems, M_0 and M_1 are one to one. We let $A_i = M_i(X_i)$ with norms inherited from the domains of the functions.

Define $E : Y_0 \to Y_0 \oplus l_\infty$ by

$$E(y) = (y, 0).$$

We desire to show that $E(Y_{\min,\theta}) \subseteq A_I$, with $\left\|E\right\|_{Y_{\min,\theta}\to A_I} < \infty$. First, note that

$$\left\|(0, \epsilon_n e_n)\right\|_{A_0} = \left\|z_n^0\right\|,$$

and

$$\left\|(0, \epsilon_n e_n)\right\|_{A_1} = \epsilon_n \left\|z_n^1\right\|.$$

Hence,

$$\|(0,\epsilon_n)\|_{A_I} \leq \epsilon_n^\theta \max\{\left\|z_n^0\right\|, \left\|z_n^1\right\|\},$$

and so

$$\left\|\left(0, \left(\epsilon_n \zeta_n^0(x)\right)\right)\right\|_{A_I} \leq \delta \|x\|_{X_0}.$$

Let $\nu > 0$. For $y \in Y_1$, let $x \in X_0$ be such that $Q(x) = y$ with $\|x\|_{X_0} \leq (1+\nu)\|y\|_{Y_0}$. Then

$$\left\|\left(y, \left(\epsilon_n \zeta_n^0(x)\right)\right)\right\|_{A_0} = \|x\|_{X_0} \leq (1+\nu)\|y\|_{Y_0},$$
$$\|(y,0)\|_{A_1} = \|y\|_{Y_1},$$

and

$$\begin{aligned}\left\|\left(0, \left(\epsilon_n \zeta_n^0(x)\right)\right)\right\|_{A_1} &\leq \sum_n \epsilon_n \left\|z_n^1\right\| \|x\|_{X_0} \\ &\leq (1+\nu)\delta \|y\|_{Y_0} \leq (1+\nu)\delta \|y\|_{Y_1}.\end{aligned}$$

So,

$$\left\|\left(y, \left(\epsilon_n \zeta_n^0(x)\right)\right)\right\|_{A_1} \leq (1+\nu)(1+\delta)\|y\|_{Y_1}.$$

Therefore,

$$\left\|\left(y, \left(\epsilon_n \zeta_n^0(x)\right)\right)\right\|_{A_I} \leq (1+\nu)(1+\delta)\|y\|_{Y_0}^{1-\theta}\|y\|_{Y_1}^\theta.$$

Finally,

$$\begin{aligned}\|(y,0)\|_{A_I} &\leq \delta\|x\|_{X_0} + (1+\nu)(1+\delta)\|y\|_{Y_0}^{1-\theta}\|y\|_{Y_1}^\theta \\ &\leq (1+\nu)(1+2\delta)\|y\|_{Y_0}^{1-\theta}\|y\|_{Y_1}^\theta.\end{aligned}$$

This is sufficient to show that $\|E(y)\|_{A_I} \leq (1+\nu)(1+2\delta)\|y\|_{Y_{\min,\theta}}$ as desired.

Now we let $P : Y_0 \oplus l_\infty \to Y_0$ be the map

$$P(y,a) = y.$$

Then $\|P\|_{A_0 \to Y_0} = \|P\|_{A_1 \to Y_1} = 1$, and hence $\|P\|_{A_I \to Y_I} \leq 1$. Clearly $P \circ E = \mathrm{Id}_{Y_{\min,\theta}}$. ☐

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