

The Hardy Operator and Boyd Indices

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ABSTRACT

We give necessary and sufficient conditions for the Hardy operator to be bounded on a rearrangement invariant quasi-Banach space in terms of its Boyd indices.

MAIN RESULTS

A *rearrangement invariant space* X on \mathbf{R} is a set of measurable functions (modulo functions equal almost everywhere) with a complete quasi-norm $\|\cdot\|_X$ such that the following holds:

- i) if $g^* \leq f^*$ and $f \in X$, then $g \in X$ with $\|g\|_X \leq \|f\|_X$;
 - ii) if f is simple with finite support then $f \in X$;
 - iii) either $f_n \searrow 0$ implies $\|f_n\| \searrow 0$
 - iii') or $0 \leq f_n \nearrow f$ and $\sup_n \|f_n\|_X < \infty$ imply $f \in X$ with $\|f\|_X = \sup_n \|f_n\|_X$.
- Here f^* denotes the decreasing rearrangement of $|f|$, that is, $f^*(s) = \sup\{t : \text{measure}\{|f| > t\} > s\}$.

The properties of rearrangement invariant spaces that we will use will be the *Boyd indices* defined as follows. Given a number $0 < a < \infty$, we define the operator $D_a f(t) = f(at)$. Then the lower Boyd index of X is defined by

$$p_X = \sup\{p : \exists c \forall a < 1 \|D_a f\|_X \leq c a^{-1/p} \|f\|_X\}$$

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and the upper Boyd index of X is defined by

$$q_X = \inf\{q : \exists c \forall a > 1 \|D_a f\|_X \leq c a^{-1/q} \|f\|_X\}.$$

Thus we see that $1 \leq p_X \leq q_X \leq \infty$. Also, if X is the Lorentz space $L_{p,q}$, then $p_X = q_X = p$.

We also define the *Hardy operators* as follows.

$$\begin{aligned} H^{(p,r)} f(t) &= \frac{1}{t^{1/p}} \left(\int_0^t (f^*(s))^r ds^{r/p} \right)^{1/r}, \\ H_{(q,r)} f(t) &= \frac{1}{t^{1/q}} \left(\int_t^\infty (f^*(s))^r ds^{r/q} \right)^{1/r}, \\ H^{(p,\infty)} f(t) &= \sup_{0 < s < t} (s/t)^{1/p} f^*(s), \\ H_{(q,\infty)} f(t) &= \sup_{t < s} (s/t)^{1/q} f^*(s), \\ H_{(\infty,r)} f(t) &= \left(\int_t^\infty \frac{f^*(s)^r}{s} ds \right)^{1/r}, \\ H_{(\infty,\infty)} f(t) &= f^*(t). \end{aligned}$$

The Hardy operators play a very important role in interpolation theory. The reason for this is the following result, essentially due to Holmstedt (1970).

THEOREM 1 If $0 < p < q \leq \infty$, and $0 < r, s \leq \infty$, then

$$\inf\{t^{-1/p} \|f'\|_{p,r} + t^{-1/q} \|f''\|_{q,s} : f' + f'' = f\} \approx H^{(p,r)} f(t) + H_{(q,s)} f(t).$$

Proof: In the case that $p = r$ and $q = s$, this is the result in Holmstedt (1970). Otherwise, this result follows from Theorem 7 below.

Boyd indices also play an important role in interpolation theory, because the Boyd indices are strongly connected with the Hardy operators. In fact, the purpose of this paper is to make this connection firm. In this paper, we show the following result. The implications from left to right complement known results which would yield the following in the case that X satisfied the triangle inequality (Maligranda, 1980, 1983). The following result also generalizes a result from Ariño and Muckenhoupt (1990), where they give necessary and sufficient conditions for $H^{(1)}$ to be bounded on a Lorentz space.

THEOREM 2 If X is a quasi-Banach r.i. space then we have the following.

- i) for $0 < p < \infty$ and $0 < r < \infty$ the operator $H^{(p,r)}$ is bounded from X to X if and only if $p_X > p$.
- ii) For $0 < q \leq \infty$ and $0 < r < \infty$ the operator $H_{(q,r)}$ is bounded from X to X if and only if $q_X < q$.
- iii) for $0 < p < \infty$ the operator $H^{(p,\infty)}$ is bounded from X to X if $p_X > p$.
- iv) For $0 < q < \infty$ the operator $H_{(q,\infty)}$ is bounded from X to X if $q_X < q$.

Note that the reverse implications are not true in parts (iii) and (iv). For example, the operators $H^{(p,\infty)}$ and $H_{(p,\infty)}$ are both bounded on the space $L_{p,\infty}$.

From this we can immediately generalize a result of Boyd (1967, 1969) to the following.

THEOREM 3 If $0 < p < q \leq \infty$ and $0 < r_1, r_2, s_1, s_2 \leq \infty$, and if $T : L_{p,r_1} \cap L_{q,s_1} \rightarrow L_{p,r_2} \cap L_{q,s_2}$ is a quasi-linear operator such that $\|Tf\|_{p,r_1} \leq c \|f\|_{p,r_2}$ and $\|Tf\|_{q,s_1} \leq c \|f\|_{q,s_2}$

for all $f \in L_{p,r_1} \cap L_{q,s_1}$, and if X is a quasi-Banach r.i. space with Boyd indices strictly between p and q , then $\|Tf\|_X \leq c\|f\|_X$ for all $f \in L_{p,r_1} \cap L_{q,s_1}$.

Proof: From Theorem 1, we see that

$$H^{(p,r_1)}(Tf)(t) + H_{(q,s_1)}(Tf)(t) \leq c(H^{(p,r_2)}f(t) + H_{(q,s_2)}f(t)).$$

Now the result follows easily from Theorem 2 which implies that for $i = 1, 2$

$$\|H^{(p,r_i)} + H_{(q,s_i)}\|_X \approx \|f\|_X.$$

Thus, as applications, we may obtain the following generalization of a result of Fehér, Gaspar and Johnen (1973).

THEOREM 4 The Hilbert transform is bounded on a quasi-Banach r.i. space X if and only if $p_X > 1$ and $q_X < \infty$.

Proof: The implication from right to left follows immediately from Theorem 3. As for the other way, this follows from the easy estimate:

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{f^*(y-x)}{y} dy \geq \frac{1}{2}(H^{(1)}f(x) + H^*f(x)) \quad x > 0.$$

We also obtain a result in the spirit of Ariño and Muckenhoupt (1990).

THEOREM 5 The Hardy–Littlewood maximal function is bounded from $X(\mathbf{R}^n)$ to $X(\mathbf{R}^n)$ if and only if $p_X > 1$.

Proof: Combine the argument given in Ariño and Muckenhoupt (1990) with Theorem 2 above.

Before proceeding with the proof of Theorem 2, we will require the following lemma.

LEMMA 6 Suppose that X is a quasi-Banach r.i. space. Then given any $p > 0$, there is a number $0 < u \leq p$ such that for any $f_1, f_2, \dots, f_n \in X$ we have

$$\left\| \left(\sum_{i=1}^n |f_i|^p \right)^{1/p} \right\| \leq c \left(\sum_{i=1}^n \|f_i\|^u \right)^{1/u}.$$

Proof: Let $X^{(p)}$ be the p -convexification of X , that is, $X^{(p)} = \{f : |f|^{1/p} \in X\}$ and $\|f\|_{X^{(p)}} = \left\| |f|^{1/p} \right\|^p$. Clearly $X^{(p)}$ is also a quasi-Banach space. Thus without loss of generality it is sufficient to show the above result when $p = 1$. But this follows immediately from Kalton, Peck and Roberts (1984), Lemma 1.1.

Proof of Theorem 2: Let's consider the case for the lower Boyd indices. The proof for the other cases are almost identical.

Let us start by proving the implication from left to right. Suppose that $p_X > p$. Let us also suppose that $r < \infty$. The case when $r = \infty$ then follows since $H^{(p,\infty)}f(t) \leq H^{(p,r)}f(t)$ for any $0 < r < \infty$. Note that

$$H^{(p,r)}f(t) = \frac{1}{t^{1/p}} \left(\int_0^t (f^*(s))^r ds^{r/p} \right)^{1/r}$$

$$\begin{aligned}
&= \left(\int_0^1 (D_a f^*(t))^r da^{r/p} \right)^{1/r} \\
&\leq \left(\sum_{n=-\infty}^0 2^{rn/p} (D_{2^n} f^*(t))^r \right)^{1/r}.
\end{aligned}$$

Pick $0 < u \leq p$ as given by Lemma 6. Also, there is a number $p' \in (p, p_X)$ such that

$$\|D_a f\|_X \leq ca^{-1/p'} \quad (0 < a \leq 1).$$

Therefore,

$$\begin{aligned}
\|H^{(p,r)} f\|_X &\leq c \left(\sum_{n=-\infty}^0 2^{un/p} \|D_{2^n} f\|_X^u \right)^{1/u} \\
&\leq c \left(\sum_{n=-\infty}^0 2^{un/p} 2^{-un/p'} \right)^{1/u} \|f\|_X \\
&\leq c' \|f\|_X,
\end{aligned}$$

as desired.

Now let us prove the opposite implication. Suppose that $\|H^{(p,r)} f\| \leq C \|f\|$. We are going to show that X has lower Boyd index greater than or equal to $p/(1 - 1/C^r)$. In order to do this, it is sufficient to show that there is a number $0 < k < 1$ such that for all numbers $a = k^n$ for integers $n \geq 1$, we have that

$$\|D_a f\| \leq ca^{-(1-1/C^r)/p} \|f\|.$$

Let us proceed. By induction and a straightforward use of Fubini, we obtain the following formula for the iteration:

$$(H^{(p,r)})^{n+1} f(t) = \frac{1}{t^{1/p}} \left(\int_0^t \frac{(\log(\frac{t}{s})^{r/p})^n}{n!} f^*(s)^r ds^{r/p} \right)^{1/r},$$

that is,

$$(H^{(p)})^{n+1} f = \left(\int_0^1 \frac{(\log \frac{1}{a^{r/p}})^n}{n!} (D_a f^*)^p da^{r/p} \right)^{1/r}.$$

Note that $\frac{(\log \frac{1}{a^{r/p}})^n}{n!} f^*(a)^r$ is a decreasing function in a , and hence for any $0 < a < 1$ we have that

$$(H^{(p)})^{n+1} f \geq \left(a^{r/p} \frac{(\log \frac{1}{a^{r/p}})^n}{n!} \right)^{1/r} D_a f^*.$$

Hence

$$\|D_a f\| \leq C^{m+1} \left(\frac{n!}{a^{r/p} (\log \frac{1}{a^{r/p}})^n} \right)^{1/r} \|f\|.$$

Now let $k = \exp(-\frac{p}{r} C^r)$. Then using the estimate $n! \leq ce^{-n} n^n$, we see that if $a = k^n$, then

$$\|D_a f\| \leq Cc^{1/r} a^{-(1-1/C^r)/p} \|f\|.$$

Note that the proof actually gives a quite precise result. For instance, it is known that $\|H^{(1)} f\|_p \leq \frac{p}{p-1} \|f\|_p$. The above proof would show that the lower Boyd index is greater than or equal to p , which is of course correct.

APPENDIX

The result of this section is an extension of results already known in interpolation theory. Let $\|f\|_{X+Y} = \inf\{\|f\|'_X + \|f\|''_Y : f' + f'' = f\}$, and let

$$\|f\|_{H(X,Y)} = \|f^*|_{[0,1]}\|_X + \|f^*|_{[1,\infty)}\|_Y.$$

THEOREM 7 Let X and Y be r.i. spaces such that the following hold.

- i) If f has support in $[0, 1]$, then $\|f\|_Y \leq c_1\|f\|_X$.
- ii) If f is constant on intervals of the form $[n, n+1)$, then $\|f\|_X \leq c_1\|f\|_Y$.
- iii) $\|D_{1/4}f\|_X \leq c_2\|f\|_X$ and $\|D_{1/4}f\|_Y \leq c_2\|f\|_Y$.
- iv) The quasi-triangle inequality constant for X and Y is less than c_3 .

Then

$$\|f\|_{X+Y} \leq \|f\|_{H(X,Y)} \leq 2c_1c_2c_3\|f\|_{X+Y}.$$

Proof: Clearly $\|f\|_{X+Y} \leq \|f\|_{H(X,Y)}$. To show the opposite inequality, let

$$Ef(x) = \begin{cases} f(x) & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x \geq 1, \end{cases}$$

$$Ff(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ \int_n^{n+1} f(s) ds & \text{if } n \leq x < n+1 \text{ and } n \text{ is a positive integer.} \end{cases}$$

Then we can see that $f^*(2x) \leq (E+F)f^*(x) \leq f^*(x/2)$. Now suppose that $f^* = f_1 + f_2$. Then

$$\|D_{1/2}f_1\|_X \geq \|(E+F)f_1\|_X \geq \frac{1}{2}(\|Ef_1\|_X + \|Ff_1\|_X) \geq \frac{1}{2c_1}(\|Ef_1\|_X + \|Ff_1\|_Y),$$

and

$$\|D_{1/2}f_2\|_Y \geq \|(E+F)f_2\|_Y \geq \frac{1}{2}(\|Ef_2\|_Y + \|Ff_2\|_Y) \geq \frac{1}{2c_1}(\|Ef_2\|_X + \|Ff_2\|_Y).$$

Hence

$$\|D_{1/2}f_1\|_X + \|D_{1/2}f_2\|_Y \geq \frac{1}{2c_1c_3}(\|Ef^*\|_X + \|Ff^*\|_Y),$$

and so $\|(E+F)f^*\|_{H(X,Y)} \leq 2c_1c_3\|D_{1/2}f^*\|_{X+Y}$. Therefore,

$$\|f\|_{H(X,Y)} \leq \|(E+F)D_{1/2}f^*\|_{H(X,Y)} \leq 2c_1c_3\|D_{1/4}f\|_{X+Y} \leq 2c_1c_2c_3\|f\|_{X+Y}.$$

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