Evolutionary Semigroups and Lyapunov Theorems in Banach Spaces

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1. Introduction

Let us consider an autonomous differential equation $v' = Av$ in a Banach space E, where A is a generator of C_0-semigroup $\{e^{tA}\}_{t\geq 0}$. Denote, as usual, $s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ and $\omega(A) = \inf\{\omega : \|e^{tA}\| \leq Me^{\omega t}\}$.

A classical result of A. M. Lyapunov (see, e.g., [9]) shows that for any bounded operator $A \in B(E)$ the spectrum $\sigma(A)$ of A is responsible for the asymptotic behavior of the solution $y(t) = e^{At}y(0)$ of the above equation. For example, if $\sigma(A)$ is contained in the left half-plane, that is $s(A) < 0$, then the trivial solution is uniformly asymptotically stable, that is $\omega(A) < 0$, and $\|e^{tA}\| \to 0$ as $t \to \infty$. This fact follows from the spectral mapping theorem (see, e.g., [21]):

$$\sigma(e^{tA})\backslash\{0\} = \exp t\sigma(A), \quad t > 0, \tag{1}$$

which always holds for bounded A.

For unbounded A, equation (1) is not always true. Moreover, there are examples of generators A (see [21]) such that even $s(A) < 0$ does not guarantee $\omega(A) < 0$ and $\|e^{tA}\| \to 0$ as $t \to \infty$. Since $\sigma(A)$ does not characterize the asymptotic behavior of the solution $v(\cdot)$, we would like to find some other characterization that still does not involve solving the differential equation.

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In this paper we solve precisely this problem in the following manner. Consider the space $L_p(\mathbb{R};E)$ of E-valued functions for $1 \le p < \infty$, or the space $C_0(\mathbb{R};E)$ of continuous vanishing at $\pm\infty$ functions on $\mathbb{R}$, and the semigroup $\{e^{tB}\}_{t\ge 0}$ of evolutionary operators

$$(e^{tB}f)(x) = e^{tA}f(x-t), t \ge 0, \tag{2}$$

generated by the operator B that is the closure of $-\frac{d}{dx} + A$, $x \in \mathbb{R}$. It turns out that it is $\sigma(B)$ in $L_p(\mathbb{R};E)$ (or in $C_0(\mathbb{R};E)$) that is responsible for the asymptotic behavior of $v(\cdot)$ in E. For example, $s(B) = \omega(A)$, and $s(B) < 0$ in $L_p(\mathbb{R};E)$ or $C_0(\mathbb{R};E)$ implies $\|e^{tA}\| \to 0$ as $t \to \infty$ on E.

The order of the proofs is as follows. First, we consider the evolutionary semigroup $\{e^{tB}\}$ in the space $L_p([0,2\pi];E)$ or $C([0,2\pi];E)$ of 2π-periodic functions. We prove that $1 \notin \sigma(e^{2\pi A})$ in E is equivalent to $1 \notin \sigma(e^{2\pi B})$ or $0 \notin \sigma(B)$ in $L_p([0,2\pi];E)$ or $C([0,2\pi];E)$. The main part of the proof uses a modification of an idea due to C. Chicone and R. Swanson [6]. Next, using this result, we give a variant of the spectral mapping theorem for a semigroup $\{e^{tA}\}$ in a Banach space E. This spectral mapping theorem is a direct generalization of L. Gearhart's spectral mapping theorem for Hilbert spaces (see, e.g., [21, p. 95]), and is related to the spectral mapping theorem of G. Greiner [21, p. 94]. Finally, using a simple change of variables arguments, we prove that $\sigma(e^{tA}) \cap \mathbb{T} = \emptyset$, $t \ne 0$ in E, $\mathbb{T} = \{z : |z| = 1\}$ is equivalent to $\sigma(e^{tB}) \cap \mathbb{T} = \emptyset$, $t \ne 0$ which in turn is equivalent to $0 \notin \sigma(B)$ in $L_p(\mathbb{R};E)$ or $C_0(\mathbb{R};E)$.

We will also consider the well-posed nonautonomous equation $v' = A(x)v$ in E, and its associated evolutionary family $\{U(x,s)\}_{x\ge s}$, which can be thought as a propagator of this equation, that is $v(x) = U(x,s)v(s)$. We assume that $U(\cdot\,,\cdot)$ is strongly continuous and satisfies the usual ([31, p. 89]) algebraic properties of the propagator. Instead of the semigroup given by (2) we consider on $L_p(\mathbb{R};E)$ or $C_0(\mathbb{R};E)$ the evolutionary semigroup

$$(e^{tG}f)(x) = U(x,x-t)f(x-t), \quad x \in \mathbb{R}, t \ge 0. \tag{3}$$

We will show that $\sigma(G)$ characterizes the asymptotic behavior of $v(\cdot)$ and prove the spectral mapping theorem for the semigroup $\{e^{tG}\}$.

We will be considering not only stability but also the exponential dichotomy (hyperbolicity) for the solutions of the equation $v' = A(x)v$ or evolutionary family $\{U(x,s)\}$. We say, that an evolutionary family $\{U(x,s)\}_{x\geq s}$ is (spectrally) hyperbolic if there exists a continuous in the strong sense, bounded, projection-valued function $P : \mathbb{R} \to B(E)$ such that: a) The norm of the restrictions $U(x,s)\Big|\operatorname{Im} P(s)$ (resp. $U(x,s)\Big|\operatorname{Ker} P(s)$) exponentially decreases (resp. increases) with $x - s$, and b) $\operatorname{Im} U(x,s)\Big|\operatorname{Ker} P(s)$ is dense in $\operatorname{Ker} P(x)$. Note, that b) automatically follows from a) if the operators $U(x,s)$ are invertible and defined for all $(x,s) \in \mathbb{R}^2$. This happens, in particular, if $U(\cdot\,,\,\cdot)$ is a norm-continuous propagator for the differential equation $v' = A(x)v$ with continuous and bounded $A : \mathbb{R} \to B(E)$. For this case the hyperbolicity of $\{U(x,s)\}$ coincides with the exponential dichotomy (see, e.g., [9]) of the equation. However, generally a) does not imply b) (see [27]).

Exponential dichotomy in the theory of differential equations with bounded coefficients on E is a major tool used for proving instability theorems for nonlinear equations, and for showing existence and uniqueness of bounded solutions and Green's functions, etc. (see, e.g. [7, 9]). The spectral mapping theorem for the semigroup (3) which is given here allows one to extend these ideas to the case of unbounded coefficients.

It turns out that the spectrum $\sigma(e^{tG})$ for nonperiodic $A(\cdot)$ plays the same role in the description of exponential dichotomy as the spectrum of the monodromy operator does in the usual Floquet theory for the periodic case. That is, the condition $\sigma(e^{tG}) \cap \mathbb{T} = \emptyset$, $t \neq 0$, or equivalently $0 \notin \sigma(G)$, is equivalent to the (spectral) hyperbolicity of the evolutionary family $\{U(x,s)\}_{x\geq s}$.

Showing that the hyperbolicity $\sigma(e^G) \cap \mathbb{T} = \emptyset$ of the operator e^G implies the hyperbolicity of $\{U(x,s)\}$ is a delicate matter. It turns out that the Riesz projection $\mathcal{P}$ for e^G on $L_p(\mathbb{R}, E)$ or $C_0(\mathbb{R}; E)$, that corresponds to $\sigma(e^G) \cap \mathbb{D}$, $\mathbb{D} = \{z : |z| < 1\}$, has the form $(\mathcal{P}f)(x) = P(x)f(x)$. Here $P(\cdot)$ is a continuous in the strong sense, projection-valued function, that defines the hyperbolicity of $\{U(x,s)\}$. Note that $e^G = aR$, where a is an operator of multiplication by the function $a(x) = U(x, x-1)$, that is $(af)(x) = a(x)f(x)$, and R is a translation operator $(Rf)(x) = f(x-1)$, $x \in \mathbb{R}$. Therefore, e^G falls into the class of so-called weighted translation operators, which are well-understood in the case that E is Hilbert space and $p = 2$ (see [1, 2, 18, 27], and also [8, 23] and references therein). If $U(\cdot\,,\,\cdot)$ is norm-continuous, then $\mathcal{P}$ is an operator from a C^*-algebra generated by R and the C^*-algebra $\mathfrak{A}_{nc}$ of operators of multiplication by the norm-continuous, bounded functions from $\mathbb{R}$ to $B(E)$. The techniques from the theory of weighted translation operators (see [1, 2, 18, 27]) allows one to conclude that $\mathcal{P} \in \mathfrak{A}_{nc}$. This technique is not applicable to the case where $\{U(\cdot,\cdot)\}$ is only strongly-continuous, nor also to the case when E is not Hilbert space.

In this paper we present some new approaches, which allows one to derive the above result for any Banach space E and is new even for the Hilbert space case and when it is only known that $U(\cdot,\cdot)$ is strongly continuous. The main idea is to "discretize" the operator aR, that is to represent it by the family of operators $\pi_x(a)S$, $x \in \mathbb{R}$, acting on the "discrete" space $l_p(\mathbb{Z}; E)$. Here $S : (v_n)_{n\in\mathbb{Z}} \mapsto (v_{n-1})_{n\in\mathbb{Z}}$ is the shift operator and $\pi_x(a) : (v_n)_{n\in\mathbb{Z}} \mapsto (a(x+n)v_n)_{n\in\mathbb{Z}}$ is a diagonal operator on $l_p(\mathbb{Z}; E)$. This idea goes back to the theory of regular representations of C^*-algebras [26], and is related to works [1, 2, 13, 16, 17, 18]. As a result we prove that $\sigma(aR)\cap\mathbb{T} = \emptyset$ in $L_p(\mathbb{R}; E)$ implies $\sigma(\pi_x(a)S)\cap\mathbb{T} = \emptyset$ in $l_p(\mathbb{Z}; E)$ for each $x \in \mathbb{R}$, and derive from this fact that $\mathcal{P} \in \mathfrak{A}$, where $\mathfrak{A}$ is the set of bounded functions $a : \mathbb{R} \to B(E)$ which are continuous in strong operator topology on $B(E)$.

We point out that the investigation of evolutionary operators (2)–(3) has a long history, probably starting from [14] (see also [10, 11, 19, 22]). Recently significant progress has been made in the papers [3, 4, 25, 27]. It is these papers that essentially motivated and influenced this present work.

Finally, the results of this article can be generalized to the case of the variational equation $v'(t) = A(\varphi^t x)v(t)$ for a flow $\{\varphi^t\}$ on a compact metric space X, or to the linear skew-product flow $\hat{\varphi}^t : X \times E \to X \times E : (x, v) \mapsto (\varphi^t x, \Phi(x,t)v)$, $t \geq 0$ (see [6, 12, 18, 29, 30] and references contained therein). Here $\Phi : X \times \mathbb{R}_+ \to L(E)$ is a cocycle over φ^t, that is, $\Phi(x, t+s) = \Phi(\varphi^t x, s)\Phi(x, t)$. Let us recall (see [29, 30]) that part of the purpose of the theory of linear skew-product flows was to be able to handle the equation $v' = A(t)v$ in the case when $A(\cdot)$ is almost-periodic .

To answer the question when $\hat{\varphi}^t$ is hyperbolic (or Anosov), instead of (3) one considers the semigroup of so called weighted composition operators (see [6, 15, 18]) on $L_p(X;\mu;E)$:

$$(T^t f)(x) = \left(\frac{d\mu\circ\varphi^{-t}}{d\mu}\right)^{1/p}\Phi(\varphi^{-t}x,t)f(\varphi^{-t}x), \quad x\in X,\ t\geq 0. \tag{4}$$

Here μ is a φ^t-quasi-invariant Borel measure on X. As above, the condition $\sigma(T^t)\cap\mathbb{T} = \emptyset$ is equivalent to the spectral hyperbolicity of the linear skew-product flow $\hat{\varphi}^t$. The spectral hyperbolicity coincides with the usual hyperbolicity if $\Phi(x,t)$, $x\in X$, $t\geq 0$ are invertible or compact operators. A detailed investigation of weighted composition operators and their connections with the spectral theory of linear skew-product flows and other questions of dynamical system theory may be found in [18] (see also [27]).

We will use the following notations: $\mathbb{D} = \{z : |z| < 1\}$; $\mathbb{T} = \{z : |z| = 1\}$; "$\big|$" denotes the restriction of an operator; $\mathcal{D}(\cdot) = \mathcal{D}_F(\cdot)$ denotes the domain of an operator in a space F; $\sigma(\cdot) = \sigma(\cdot\,; F)$ denotes the spectrum; $\sigma_{ap}(\cdot) = \sigma_{ap}(\cdot\,;F)$ denotes the approximative point spectrum; $\sigma_r(\cdot) = \sigma_r(\cdot\,;F)$ denotes the residual spectrum; and $\rho(\cdot) = \rho(\cdot\,;F)$ denotes the resolvent set of an operator on F. For an operator A in E we denote by $\mathcal{A}$ the operator of multiplication by A in a space of E-valued functions: $(\mathcal{A}f)(x) = Af(x)$, $f:\mathbb{R}\to E$.

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2. Autonomous Case

Let A be a generator of a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ on a Banach space E. The semigroup is called *hyperbolic* if $\sigma(e^{tA})\cap\mathbb{T} = \emptyset$ for $t\neq 0$. In this section we will characterize the hyperbolicity of the semigroup $\{e^{tA}\}$ in terms of evolutionary semigroup $\{e^{tB}\}_{t\geq 0}$ and its generator B. This semigroup acts by the rule $(e^{tB}f)(x) = e^{tA}f(x-t)$ on functions f with values in E. In Subsection 2.1 we consider $\{e^{tB}\}$ acting on the space $L_p([0,2\pi];E)$ and $C([0,2\pi];E)$. In Subsection 2.3 $\{e^{tB}\}$ acts on $L_p(\mathbb{R};E)$ and $C_0(\mathbb{R};E)$. Subsection 2.2 is devoted to a spectral mapping theorem for $\{e^{tA}\}_{t\geq 0}$ on E which generalizes the spectral mapping theorem of L. Gearhart for Hilbert space.

2.1. Periodic Case

Let F denote one of the spaces $L_p([0,2\pi],E)$, $1\leq p<\infty$ or $C([0,2\pi],E)$ of 2π-periodic E-valued functions f, $f(0)=f(2\pi)$. Consider the evolutionary semigroup $\{e^{tB}\}_{t\geq 0}$ acting on F, defined by the rule

$$(e^{tB}f)(x)=e^{tA}f([x-t](\operatorname{mod}2\pi)),\ x\in[0,2\pi].$$

Of course, $[0,2\pi]$ here was chosen for convenience, and for a semigroup $(e^{tB}f)(x)=e^{tA}f\ ([x-t](\operatorname{mod}t_0))$ the proofs below remain the same for any $t_0>0$.

Note that e^{tB} in F is a product of two commuting semigroups $(U^tf)(x)=f([x-t](\operatorname{mod}2\pi))$ and $(e^{t\mathcal{A}}f)(x)=e^{tA}f(x)$. Hence the generator B is the closure of the operator

$$(B_0f)(x)=-\frac{d}{dx}f(x)+Af(x), \tag{5}$$

where B_0 is defined on the core $\mathcal{D}(B_0)$ of B (see [21, p. 24]). Moreover, $\mathcal{D}_F(B_0)=\mathcal{D}_F(-d/dx)\cap\mathcal{D}_F(\mathcal{A})$, where the derivative d/dx is taken in the strong sense in E, and $\mathcal{D}_F(B_0)=\{f:[0,2\pi]\to\mathcal{D}(A)\big|f\in F$ is absolutely continuous, $\frac{d}{dx}f\in F,$ and $\mathcal{A}f\in F\}$.

Since $Be^{ik\cdot}f(\cdot)=e^{ik\cdot}(B-ik)f(\cdot)$, $k\in\mathbb{Z}$, for the operator $(L_kf)(x)=e^{ikx}f(x)$ one has $BL_k=L_k(B-ik)$. Therefore, the spectrum $\sigma(B)$ in F is invariant under translations by i.

We will need the following Lemma.

Lemma 2.1. *If $1\in\sigma_{ap}(e^{2\pi A})$ in E, then $0\in\sigma_{ap}(B)$ in F.*

Proof. Fix $m\in\mathbb{N}$, $m\geq 2$. Since $1\in\sigma_{ap}(e^{2\pi A})$, we can choose $v\in E$ such that $\|v\|_E=1$ and $\|v-e^{2\pi A}v\|_E<\frac{1}{m}$. Note also that $\|e^{2\pi A}v\|_E\geq 1-\frac{1}{m}$.

Let $\alpha:[0,2\pi]\to[0,1]$ be any smooth function with bounded derivative such that $\alpha(x)=0$ for $x\in[0,\frac{2\pi}{3}]$ and $\alpha(x)=1$ for $x\in[\frac{4\pi}{3},2\pi]$. Define a function $g:[0,2\pi]\to E$ by the the formula

$$g(x)=[1-\alpha(x)]e^{(2\pi+x)A}v+\alpha(x)e^{xA}v,\ x\in[0,2\pi]. \tag{6}$$

Note that $g(0) = g(2\pi) = e^{2\pi A}v$. Obviously, $g \in F$. Also,

$$(e^{tB}g)(x) = [1 - \alpha(x-t)]e^{(2\pi+x)A}v + \alpha(x-t)e^{xA}v,$$

$g \in \mathcal{D}_F(B)$, and

$$(Bg)(x) = \alpha'(x)e^{xA}[e^{2\pi A}v - v], \quad x \in [0, 2\pi]. \tag{7}$$

Let us denote $a = \max\{|\alpha'(x)| : x \in [0, 2\pi]\}$ and $b = \max\{\|e^{xA}\| : x \in [0, 2\pi]\}$. Note, that $\|e^{2\pi A}v\| = \|e^{(2\pi - x)A}e^{xA}v\| \le b\|e^{xA}v\|$ for any $x \in [0, 2\pi]$.

First let us suppose that $F = L_p([0, 2\pi]; E)$. Then

$$\|Bg\|_{L_p([0,2\pi];E)} \le (2\pi)^{1/p}\frac{ab}{m}.$$

On the other hand,

$$\|g\|^p_{L_p([0,2\pi];E)} \ge \int_{\frac{4\pi}{3}}^{2\pi} \|e^{xA}v\|^p dx \ge \frac{2\pi}{3}b^{-p}\|e^{2\pi A}v\|^p \ge \frac{2\pi}{3}b^{-p}(1 - \frac{1}{m})^p.$$

Finally,

$$\|Bg\|_{L_p([0,2\pi];E)} \le (2\pi)^{1/p}\frac{ab}{m} \le 3^{1/p}ab^2\|g\|_{L_p([0,2\pi];E)} \cdot \frac{1}{m-1}. \tag{8}$$

Since this holds for all m, it follows that $0 \in \sigma_{ap}(B)$.

Now suppose that $F = C([0, 2\pi], E)$. Then

$$\|Bg\|_{C([0,2\pi],E)} \le \frac{ab}{m}, \ \|g\|_{C([0,2\pi],E)} \ge \|g(0)\|_E = \|e^{2\pi A}v\| \ge 1 - \frac{1}{m} \tag{9}$$

and hence

$$\|Bg\|_{C([0,2\pi];E)} \le \frac{ab}{m-1}\|g\|_{C([0,2\pi];E)}. \tag{10}$$

Since this is true for all m, it follows that $0 \in \sigma_{ap}(B)$. □

Theorem 2.2. *Let F be one of the spaces $L_p([0, 2\pi], E)$, $1 \le p < \infty$ or $C([0, 2\pi], E)$. Then the following are equivalent:*

1) $1 \in \rho(e^{2\pi A})$ in E;
2) $1 \in \rho(e^{2\pi B})$ in F;
3) $0 \in \rho(B)$ in F.

Proof. 1) ⇒ 2). Note that $(e^{2\pi B}f)(x) = e^{2\pi A}f(x)$. Hence $\sigma(e^{2\pi A}; E) = \sigma(e^{2\pi B}; F)$. Note also that $\sigma_r(e^{2\pi A}; E) = \sigma_r(e^{2\pi B}; F)$.

2) ⇒ 3) follows from the spectral inclusion theorem $e^{2\pi\sigma(B)} \subset \sigma(e^{2\pi B})$ (see [24, p. 45]).

3) $\Rightarrow$ 1). Assume $0 \in \rho(B; F)$ but $1 \in \sigma(e^{2\pi A}; E) = \sigma_{ap}(e^{2\pi A}; E) \cup \sigma_r(e^{2\pi A}; E)$. By Lemma 2.1 it follows that $1 \in \sigma_r(e^{2\pi A}; E)$, and hence $1 \in \sigma_r(e^{2\pi B}; F)$. By the spectral mapping theorem for the residual spectrum ([24, Theorem 2.5 (ii)]) it follows that $ik \in \sigma_r(B; F)$ for some $k \in \mathbb{Z}$. Since $\sigma(B; F)$ is invariant under translations by i, we have that $0 \in \sigma(B; F)$, contradicting 3). □

2.2. Spectral Mapping Theorem for Banach Spaces

As it is well-known (see, e.g., [21, p. 82–89]), that in general, the inclusion $e^{t\sigma(A)} \subset \sigma(e^{At})$, $t \neq 0$ for a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ on a Banach space E is improper. In particular, $i\mathbb{Z} \subset \rho(A)$ is implied by but does not imply $1 \in \rho(e^{2\pi A})$. For Hilbert space E, however, the following spectral mapping theorem of L. Gearhart (see [21, p. 95]) is true: $1 \in \rho(e^{2\pi A})$ *if and only if* $i\mathbb{Z} \subset \rho(A)$ *and* $\sup_{k\in\mathbb{Z}} \|(A - ik)^{-1}\| < \infty$. We will now give a direct generalization of this result to any arbitrary Banach space E. This generalization is related (but independent) to G. Greiner's spectral mapping theorem [21, p. 94] that involves Césaro summability of the series $\sum_k (A - ik)^{-1} v$, $v \in E$.

Theorem 2.3. *Let $\{e^{tA}\}$ be any C_0-semigroup on a Banach space E and let F be one of the spaces $L_p([0, 2\pi], E)$, $1 \leq p < \infty$ or $C([0, 2\pi], E)$. Then the following are equivalent:*

1) $1 \in \rho(e^{2\pi A})$;

2) $i\mathbb{Z} \subset \rho(A)$ and there is a constant $C > 0$ such that

$$\| \sum_k (A - ik)^{-1} e^{ikx} v_k \|_F \leq C \| \sum_k e^{ikx} v_k \|_F \tag{11}$$

for any finite sequence $\{v_k\} \subset E$.

Proof. Consider the evolutionary semigroup $\{e^{tB}\}_{t\geq 0}$ on F from the previous subsection. Consider a finite sequence $\{v_k\} \subset E$. Assume that $(A - ik)^{-1}$ exists for all $k \in \mathbb{Z}$. Define functions f, $g \in F$ by the rule

$$f(x) = \sum_k (A - ik)^{-1} e^{ikx} v_k, \; g(x) = \sum_k e^{ikx} v_k, \quad x \in [0, 2\pi]. \tag{12}$$

Since $(A-ik)^{-1}: E \to \mathcal{D}(A)$, one has $Bf = g$. Indeed

$$\begin{aligned}(Bf)(x) &= \frac{d}{dt} e^{tA} f([x-t](\operatorname{mod} 2\pi))\Big|_{t=0} \\ &= \sum_k [A(A-ik)^{-1} e^{ikx} v_k - ik(A-ik)^{-1} e^{ikx} v_k] = g.\end{aligned}$$

1) $\Rightarrow$ 2). If $1 \in \rho(e^{2\pi A})$, then the inclusion $i\mathbb{Z} \subset \rho(A)$ follows from the spectral inclusion theorem $e^{2\pi\sigma(A)} \subset \sigma(e^{2\pi A})$. In accordance with part 1) $\Rightarrow$ 3) of Theorem 2.2, the operator B has bounded inverse B^{-1} on F provided that $1 \in \rho(e^{2\pi A})$. Denote $C = \|B^{-1}\|$, and consider functions (12). Then $\|f\|_F = \|B^{-1}g\|_F \le C\|g\|_F$, and (11) is proved.

2) $\Rightarrow$ 1). First, we show that 2) implies $0 \notin \sigma_{ap}(B)$. Indeed, the functions of type g in (12) are dense in F. If we let $u_k = (A-ik)^{-1}v_k$, then we note that the functions of type f are also dense in F. Now (11) implies $\|Bf\|_F = \|g\|_F \ge C^{-1}\|f\|_F$, and $0 \notin \sigma_{ap}(B)$.

Assume that 2) is fulfilled, but $1 \in \sigma(e^{2\pi A}) = \sigma_r(e^{2\pi A}) \cup \sigma_{ap}(e^{2\pi A})$. If $1 \in \sigma_{ap}(e^{2\pi A})$ in E then, by Lemma 2.1, $0 \in \sigma_{ap}(B)$, in contradiction to the previous paragraph. On the other hand, $1 \in \sigma_r(e^{2\pi A})$ implies, by the spectral mapping theorem for residual spectrum, that $ik \in \sigma_r(A)$ for some $k \in \mathbb{Z}$, contradicting $i\mathbb{Z} \subset \rho(A)$. □

Remark. We note, that 1) $\Rightarrow$ 2) can be also seen directly. Indeed, assuming 1), let us denote $\phi(s) = (e^{2\pi A} - I)^{-1} e^{sA}$, $s \in [0, 2\pi]$. Then the convolution operator

$$(Kf)(x) = \int_0^{2\pi} \phi(s) f(x-s)\, ds$$

is a bounded operator on F. But

$$(A-ik)^{-1} = \int_0^{2\pi} e^{-iks}(e^{2\pi A} - I)^{-1} e^{sA}\, ds, \quad k \in \mathbb{Z},$$

are Fourier coefficients of $\phi : [0, 2\pi] \to B(E)$. Inequality (11) can be viewed as the condition of boundedness of K, which gives 2).

Let us show now that Theorem 2.3 is really a direct generalization of L. Gearhart's Theorem, mentioned above. Indeed, for Hilbert space

E and $p = 2$, Parseval's identity implies:

$$\|\sum_k (A - ik)^{-1} e^{ik\cdot} v_k\|_{L_2([0,2\pi];E)} = \Big(2\pi \sum_k \|(A - ik)^{-1} v_k\|_E^2\Big)^{1/2}$$
$$\|\sum_k e^{ik\cdot} v_k\|_{L_2([0,2\pi];E)} = \Big(2\pi \sum_k \|v_k\|_E^2\Big)^{1/2}.$$

Clearly, (11) is equivalent to the condition $\sup\{\|(A-ik)^{-1}\| : k \in \mathbb{Z}\} < \infty$.

We conclude this subsection by giving four more statements equivalent to 1) and 2) in Theorem 2.3:

3) $i\mathbb{Z} \subset \rho(A)$ and there is a constant $C > 0$ such that

$$\|Bf\|_{L_1([0,2\pi];E)} \geq C^{-1} \|f\|_{C([0,2\pi];E)}$$

for all $f \in C([0, 2\pi]; E)$ such that $Bf \in L_1([0, 2\pi]; E)$;

4) $i\mathbb{Z} \subset \rho(A)$ and there is a constant $C > 0$ such that

$$\|Bf\|_{C([0,2\pi];E)} \geq C^{-1} \|f\|_{L_1([0,2\pi];E)}$$

for all $f \in L_1([0, 2\pi]; E)$ such that $Bf \in C([0, 2\pi]; E)$;

5) $i\mathbb{Z} \subset \rho(A)$ and there is a constant $C > 0$ such that

$$\|\sum_k (A - ik)^{-1} e^{ikx} v_k\|_{C([0,2\pi];E)} \leq C\|\sum_k e^{ikx} v\|_{L_1([0,2\pi];E)}$$

for any finite sequence $\{v_k\} \subset E$;

6) $i\mathbb{Z} \subset \rho(A)$ and there is a constant $C > 0$ such that

$$\|\sum_k (A - ik)^{-1} e^{ikx} v_k\|_{L_1([0,2\pi];E)} \leq C\|\sum_k e^{ikx} v\|_{C([0,2\pi];E)}$$

for any finite sequence $\{v_k\} \subset E$.

2.3. Real Line

Consider now the evolutionary semigroup $\{e^{tB}\}_{t\geq 0}$,

$$(e^{tB} f)(x) = e^{tA} f(x - t) \tag{13}$$

acting on the space $F = L_p(\mathbb{R}; E)$, $1 \leq p < \infty$, or $F = C_0(\mathbb{R}; E)$. Formula (5) is valid and the identities

$$e^{tB} e^{i\xi\cdot} f(\cdot) = e^{i\xi\cdot} e^{-i\xi t} e^{tB} f(\cdot),$$
$$B e^{i\xi\cdot} f(\cdot) = e^{i\xi\cdot} (B - i\xi) f(\cdot), \quad \xi \in \mathbb{R},$$

show that $\sigma(e^{tB})$ in F is invariant under rotations centered at the origin, and that $\sigma(B)$, $\sigma_{ap}(B)$ and $\sigma_r(B)$ in F are invariant under translations parallel to $i\mathbb{R}$.

First we state a simple lemma. Let F_s be one of the spaces $F_s = l_p(\mathbb{Z}; E)$, $1 \le p < \infty$ or $F_s = c_0(\mathbb{Z}; E)$ (of sequences $(v_n)_{n\in\mathbb{Z}}$ such that $v_n \to 0$ as $n \to \pm\infty$). Let S be a shift operator on F_s, that is $S : (v_n) \mapsto (v_{n-1})$. For an operator a on E we will denote by D_a the diagonal operator on F_s, acting by the rule $D_a : (v_n)_{n\in\mathbb{Z}} \mapsto (av_n)_{n\in\mathbb{Z}}$.

Lemma 2.4. *The following are equivalent:*

1) $\sigma(a) \cap \mathbb{T} = \emptyset$ *in* E*;*
2) $\sigma(D_a S) \cap \mathbb{T} = \emptyset$ *in* F_s*.*

Proof. We will give the proof for the case when $F_s = l_p = l_p(\mathbb{Z}; E)$. The case $F_s = c_0(\mathbb{Z}; E)$ can be considered similarly.

1) $\Rightarrow$ 2). Since $\sigma(a) \cap \mathbb{T} = \emptyset$, there exists a Riesz projection $\hat{p}$ for a in E that corresponds to the part of the spectrum $\sigma(a) \cap \mathbb{D}$. Define $\hat{q} = I - \hat{p}$, and consider in F complimentary projections $D_{\hat{p}}$ and $D_{\hat{q}}$. Since $D_a S D_{\hat{p}} = D_a D_{\hat{p}} S$, the decomposition $F_s = \operatorname{Im} D_{\hat{p}} \oplus \operatorname{Im} D_{\hat{q}}$ is $D_a S$-invariant. For the spectral radius

$$r(\cdot) = \lim_{n\to\infty} \|(\cdot)^n\|^{\frac{1}{n}}$$

one has $r(\hat{p}a\hat{p}) < 1$ and $r([\hat{q}a\hat{q}]^{-1}) < 1$ in E. Hence, $r(D_a S D_{\hat{p}}) < 1$, $r([D_a S D_{\hat{q}}]^{-1}) < 1$, and $\sigma(D_a S) \cap \mathbb{T} = \emptyset$ in F_s.

2) $\Rightarrow$ 1). Assume that $I - D_a S$ is invertible in l_p, but for any $\epsilon > 0$ there is a vector $v \in E$ such that $\|v\|_E = 1$ and $\|v - av\|_E < \epsilon$. Fix $q > 0$ such that

$$\left|1 - e^{\pm q}\right| < \epsilon,$$

and define a sequence $(v_n) \in l_p$ by $v_n = e^{-q|n|}v$, $n \in \mathbb{Z}$. Then

$$I - D_a S : (v_n)_{n\in\mathbb{Z}} \mapsto \left(e^{-q|n|}(v - av) + (e^{-q|n|} - e^{-q|n-1|})av\right)_{n\in\mathbb{Z}}.$$

A direct calculation shows that

$$\|(I - D_a S)(v_n)\|_{l_p} \le (1 + \|a\|) \cdot \epsilon \cdot \|(v_n)\|_{l_p},$$

contradicting the invertibility of $I - D_a S$ in l_p.

Let us show now that $I - a$ has a dense range in E, provided $I - D_a S$ is an operator onto l_p. Indeed, for any $u \in E$ consider a sequence $(u_n) \in l_p$ defined by $u_0 = u$ and $u_n = 0$ for $n \ne 0$. Find a sequence

$(v_n) \in l_p$ such that $(I - D_a S)(v_n) = (u_n)$, that is $v_n - av_{n-1} = u_n$ for $n \in \mathbb{Z}$. But then for $k \in \mathbb{N}$ one has

$$\begin{aligned} u &= \sum_{n=-k}^{k} (v_n - av_{n-1}) \\ &= v_k - av_{-k-1} + (y_k - ay_k), \end{aligned}$$

where

$$y_k = \sum_{n=-k}^{k-1} v_n.$$

Therefore, $\operatorname{Im}(I - a) \ni y_k - ay_k \to u$, since $v_k \to 0$ and $av_{-k-1} \to 0$ as $k \to \infty$. ☐

Theorem 2.5. *Let F be one of the spaces $L_p(\mathbb{R}; E)$, $1 \le p < \infty$ or $C_0(\mathbb{R}; E)$, and let $t > 0$. Then the following are equivalent:*

1) $\sigma(e^{tA}) \cap \mathbb{T} = \emptyset$ in E;
2) $\sigma(e^{tB}) \cap \mathbb{T} = \emptyset$ in F;
3) $0 \in \rho(B)$ in F.

Proof. 2) $\Rightarrow$ 3) follows from the spectral inclusion theorem for $\{e^{tB}\}$.

3) $\Rightarrow$ 2) we will prove for $F = L_p(\mathbb{R}; E)$; the arguments for $F = C_0(\mathbb{R}; E)$ are similar.

Since $\sigma(e^{tB})$ is invariant under the rotations with the center at origin, it suffices to prove that 3) implies $1 \in \rho(e^{tB})$. Also, to confirm our previous notations, we will consider only the case $t = 2\pi$. The proof stays the same for any t.

The idea is to apply Theorem 2.2, to show that $1 \in \rho(e^{2\pi B})$ is implied by $0 \in \rho(B')$. Here the operator $B' = -\frac{d}{ds} - \frac{d}{dx} + A$ acts on $L_p([0, 2\pi] \times \mathbb{R}; E)$, $s \in [0, 2\pi]$, $x \in \mathbb{R}$. Indeed, by formula (5) one has $B = -\frac{d}{dx} + A$. Hence, B' on $L_p([0, 2\pi]; L_p(\mathbb{R}; E))$ is the generator of the evolutionary semigroup for the semigroup $\{e^{tB}\}$ on $L_p(\mathbb{R}; E)$. But the change of variables $u = [s - x](\operatorname{mod} 2\pi)$, $v = x$ shows that $\rho(B') = \rho(-\frac{d}{dv} + A) = \rho(B)$. Let us now make this argument more formal.

Consider the semigroups

$$(e^{tB'}h)(s, x) = e^{tA}h([s - t](\operatorname{mod} 2\pi),\ x - t),\ \ t > 0,\ \ s \in [0, 2\pi],\ \ x \in \mathbb{R},$$
$$(e^{t\mathcal{B}}h)(s, x) = e^{tA}h(s,\ x - t),\ \ t > 0,\ \ s \in [0, 2\pi],\ \ x \in \mathbb{R},$$

and an invertible isometry J,

$$(Jh)(s, x) = h([s + x](\operatorname{mod} 2\pi),\ x),$$

acting on the space

$$L_p([0,2\pi]\times\mathbb{R};\ E) = L_p([0,2\pi]; L_p(\mathbb{R};E)).$$

Since $e^{t\mathcal{B}}$ acting on $L_p([0,2\pi]; L_p(\mathbb{R};E))$ is actually the operator of multiplication by the operator e^{tB} in $L_p(\mathbb{R};E)$, one has:

$$\sigma(e^{t\mathcal{B}}) = \sigma(e^{tB}) \text{ and } \sigma(\mathcal{B}; L_p([0,2\pi]; L_p(\mathbb{R};E))) = \sigma(B; L_p(\mathbb{R};E)).$$

Also,

$$\Big(Je^{tB'}h\Big)(s,x) = e^{tA}h([s+x-t](\operatorname{mod} 2\pi), x-t) = \Big(e^{t\mathcal{B}}Jh\Big)(s,x),$$

and hence one has $Je^{tB'} = e^{t\mathcal{B}}J$ and $JB' = \mathcal{B}J$. Therefore,

$$\sigma(e^{t\mathcal{B}}) = \sigma(e^{tB'}) \text{ and } \sigma(\mathcal{B}) = \sigma(B') \text{ in } L_p([0,2\pi]; L_p(\mathbb{R};E)).$$

Thus 3) implies $0 \in \rho(\mathcal{B})$ and $0 \in \rho(B')$.

The semigroup $\{e^{tB'}\}$ acts on $L_p([0,2\pi]; L_p(\mathbb{R};E))$ by the rule

$$(e^{tB'}f)(s) = e^{tB}f([s-t](\operatorname{mod} 2\pi)),$$

where $f(s) = h(s,\cdot) \in L_p(\mathbb{R};E)$ for almost all $s \in [0,2\pi]$. Hence, $\{e^{tB'}\}$ on the space $L_p([0,2\pi]; L_p(\mathbb{R};E))$ is the evolutionary semigroup for the semigroup $\{e^{tB}\}$ on $L_p(\mathbb{R};E)$. Now one can apply the part 3) $\Rightarrow$ 1) of the Theorem 2.2 and conclude that $1 \in \rho(e^{2\pi B})$ on $L_p(\mathbb{R};E)$.

1) $\Leftrightarrow$ 2) we will prove for $F = L_p(\mathbb{R};E)$; the arguments for $F = C_0(\mathbb{R};E)$ are similar.

Let us denote, for brevity, $a = e^{2\pi\mathcal{A}}$ and $(Rf)(x) = f(x-2\pi)$ on $L_p(\mathbb{R};E)$. Thus $e^{2\pi B} = aR$. Consider the invertible isometry

$$\begin{aligned} &j : L_p(\mathbb{R};E) \to l_p(\mathbb{Z}; L_p([0,2\pi];E)) :\\ &f \mapsto (f_n),\ \ f_n(s) = f(s+2\pi n),\ \ n \in \mathbb{Z}, s \in [0,2\pi). \end{aligned}$$

Let $S : (f_n) \mapsto (f_{n-1})$ be a shift operator on $l_p(\mathbb{Z}; L_p([0,2\pi];E))$. Then $jaR = D_aSj$ and $\sigma(aR) = \sigma(D_aS)$. Therefore, 2) is equivalent to $\sigma(D_aS)\cap\mathbb{T} = \emptyset$. By Lemma 2.4 this in turn is equivalent to $\sigma(a)\cap\mathbb{T} = \emptyset$ in $L_p(\mathbb{R};E)$. □

Note, that 3) $\Rightarrow$ 1) in the above theorem can also be derived directly by constructing a function g in a similar manner as in the proof of Lemma 2.1. This proof will be given elsewhere.

3. Non-autonomous Case

Consider a non-autonomous differential equation $v'(x) = A(x)v(x)$, $x \in \mathbb{R}$ in E. We will assume that this equation is well-posed. This means that there exists an evolutionary family $\{U(x,s)\}_{x \geq s}$ (propagator) for the equation, that is $v(x) = U(x,s)v(s)$, $x \geq s$. Recall the definition of evolutionary family (see, e.g., [27, 31]).

Definition 3.1. A family $\{U(x,s)\}_{x\geq s}$ of bounded in E operators $U(x,s)$ is called an *evolutionary family* if the following conditions are fulfilled:
(i) for each $v \in E$ the function $(x,s) \mapsto U(x,s)v$ is continuous for $x \geq s$;
(ii) $U(x,s) = U(x,r)U(r,s)$, $U(x,x) = I$, $x \geq r \geq s$;
(iii) $\|U(x,s)\| \leq Ce^{\beta(x-s)}$, $x \geq s$ for some constants C, $\beta > 0$.

The evolutionary family $\{U(x,s)\}$ generates an evolutionary semigroup $\{e^{tG}\}_{t\geq 0}$ acting on the space $F = L_p(\mathbb{R}; E)$, $1 \leq p < \infty$ or $F = C_0(\mathbb{R}; E)$ by the rule

$$(e^{tG}f)(x) = U(x, x-t)f(x-t), \ x \in \mathbb{R}. \tag{14}$$

In Subsection 3.1 we will prove the spectral mapping theorem $\sigma(e^{tG})\backslash\{0\} = e^{t\sigma(G)}$, $t \neq 0$ for $\{e^{tG}\}$. We will achieve this by applying a simple change of variables argument (cf. the proof of Theorem 2.5) to deduce this from Theorem 2.5. In Subsection 3.2 we will prove that the hyperbolicity $\sigma(e^{tG}) \cap \mathbb{T} = \emptyset$ of the semigroup in F is equivalent to the so-called spectral hyperbolicity of the family $\{U(x,s)\}$. Spectral hyperbolicity is a generalization of the notion of exponential dichotomy (see, e.g., [9]) for the equation $v'(x) = A(x)v(x)$ with bounded $A : \mathbb{R} \to B(E)$.

3.1. The Spectral Mapping Theorem for Evolutionary Semigroup

Let G be the generator of the evolutionary semigroup $\{e^{tG}\}_{t\geq 0}$ acting on the space $F = L_p(\mathbb{R}; E)$, $1 \leq p < \infty$ or $F = C_0(\mathbb{R}; E)$ by equation (14).

Theorem 3.1. *The spectrum $\sigma(G)$ is invariant under translations along the imaginary axis, and the following are equivalent:*
1) $0 \in \rho(G)$ on F;
2) $\sigma(e^{tG}) \cap \mathbb{T} = \emptyset$ on F, $t > 0$.

Proof. For any $\xi \in \mathbb{R}$ it is true that $e^{tG}e^{i\xi\cdot}f(\cdot) = e^{i\xi(\cdot - t)}e^{tG}f(\cdot)$ and $Ge^{i\xi\cdot} = e^{i\xi\cdot}(G - i\xi)$. Hence $\sigma(e^{tG})$ is invariant under rotations centered at the origin, and $\sigma(G)$ is invariant under translations along the imaginary axis.

2) $\Rightarrow$ 1) follows from the spectral inclusion theorem for $\{e^{tE}\}$.

1) $\Rightarrow$ 2). We will first consider the case when $F = L_p(\mathbb{R}; E)$, $1 \leq p < \infty$.

The idea of the proof is almost identical to the proof of 3) $\Rightarrow$ 2) from Theorem 2.5. If $U(\cdot,\cdot)$ is a smooth propagator for the equation $v' = A(x)v$, then $G = -\frac{d}{dx} + A(x)$. Consider the evolutionary semigroup for $\{e^{tG}\}$, that is the semigroup with the generator $B = -\frac{d}{ds} + G$ on $L_p(\mathbb{R}; L_p(\mathbb{R}; E))$. Theorem 2.5 shows that $1 \in \rho(e^{tG})$ is implied by $0 \in \rho(B)$, where $B = -\frac{d}{ds} - \frac{d}{dx} + A(x)$, $s, x \in \mathbb{R}$ by formula (5). The change of variables $u = s - x$, $v = x$ shows that $\rho(B) = \rho(-\frac{d}{dv} + A(v)) = \rho(G)$. Let us now make this argument more formal.

Consider the semigroups $\{e^{tB}\}_{t\geq 0}$ and $\{e^{t\mathcal{G}}\}_{t\geq 0}$ acting on the space $L_p(\mathbb{R}\times\mathbb{R}; E) = L_p(\mathbb{R}; L_p(\mathbb{R}; E))$ by

$$
\begin{aligned}
&(e^{tB}h)(s,x) = U(x, x-t)h(s-t, x-t),\ (s,x) \in \mathbb{R}^2,\ t > 0,\\
&(e^{t\mathcal{G}}h)(s,x) = U(x, x-t)h(s, x-t).
\end{aligned}
$$

Note that $e^{t\mathcal{G}}$ in $L_p(\mathbb{R}; L_p(\mathbb{R}; E))$ is the operator of multiplication by e^{tG}, that is $(e^{t\mathcal{G}}f)(s) = e^{tG}f(s)$, where $f(s) = h(s,\cdot) \in L_p(\mathbb{R}; E)$. Similarly, $(\mathcal{G}f)(s) = Gf(s) = Gh(s,\cdot)$ for $f(s) = h(s,\cdot) \in \mathcal{D}_{L_p(\mathbb{R};E)}(G)$ for almost all $s \in \mathbb{R}$.

Consider an isometry J on $L_p(\mathbb{R}\times\mathbb{R}; E)$ defined by $(Jh)(s,x) = h(s+x, x)$. Then for $h \in L_p(\mathbb{R}\times\mathbb{R}; E)$ one has:

$$
(e^{t\mathcal{G}}Jh)(s,x) = U(x, x-t)h(s+x-t, x-t) = (Je^{tB}h)(s,x). \tag{15}
$$

Also (15) implies

$$
\mathcal{G}Jh = JBh,\ h \in \mathcal{D}(B) \text{ and } J^{-1}\mathcal{G}h = BJ^{-1}h,\ h \in \mathcal{D}(\mathcal{G}).
$$

Therefore, $\sigma(\mathcal{G}) = \sigma(B)$ on $L_p(\mathbb{R}\times\mathbb{R}; E)$.

Note that $\mathcal{G}$ on $L_p(\mathbb{R}\times\mathbb{R}; E)$ has bounded inverse $(\mathcal{G}^{-1}f)(s) = G^{-1}f(s)$, $s \in \mathbb{R}$, provided G has bounded inverse G^{-1} on $L_p(\mathbb{R}; E)$. Hence 1) implies $0 \in \rho(B)$.

Let us apply the part 3) $\Rightarrow$ 1) of Theorem 2.5 to the semigroups $\{e^{tG}\}$ and $\{e^{tB}\}$. To this end we note that $(e^{tB}f)(s) = e^{tG}f(s-t)$ for $f : \mathbb{R} \to L_p(\mathbb{R}; E) : s \mapsto h(s, \cdot)$. Hence $0 \in \rho(B)$ on $L_p(\mathbb{R}; L_p(\mathbb{R}; E))$ implies $\sigma(e^{tG}) \cap \mathbb{T} = \emptyset$, $t \neq 0$ on $L_p(\mathbb{R}; E)$.

The proof for the case $F = C_0(\mathbb{R}; E)$ is identical, and uses exactly the same semigroups and isometries on $C_0(\mathbb{R}; F) = C_0(\mathbb{R} \times \mathbb{R}; E)$. $\square$

3.2. Hyperbolicity

Let $\{U(x,s)\}_{x \geq s}$ be an evolutionary family on a Banach space E. In this subsection we relate the (spectral) hyperbolicity of the evolutionary family and the hyperbolicity of the evolutionary semigroup $\{e^{tG}\}$ on the space $L_p = L_p(\mathbb{R}; E)$ in the case when the Banach space E is separable. The case $F = C_0(\mathbb{R}; E)$ (without the assumption of separability) and the case of a Hilbert space E and $p = 2$ was considered in [27, 28].

Definition 3.2. An evolutionary family $\{U(x,s)\}_{x \geq s}$ is called (spectrally) hyperbolic if there exists a projection-valued, bounded function $P : \mathbb{R} \to B(E)$ such that the function $\mathbb{R} \ni x \mapsto P(x)v \in E$ is continuous for every $v \in E$ and for some constants M, $\lambda > 0$ and all $x \geq s$ the following conditions are fulfilled:

a) $P(x)U(x,s) = U(x,s)P(s)$:
b) $\|U(x,s)v\| \leq Me^{-\lambda(x-s)}\|v\|$ if $v \in \operatorname{Im} P(s)$,
$\|U(x,s)v\| \geq M^{-1}e^{\lambda(x-s)}\|v\|$ if $v \in \operatorname{Ker} P(s)$;
c) $\operatorname{Im}(U(x,s)|\operatorname{Ker} P(s))$ is dense in $\operatorname{Ker} P(x)$.

This notion generalizes the notion of exponential dichotomy (see, e.g., [9]) for the solutions of differential equation $v'(x) = A(x)v(x)$, $x \in \mathbb{R}$, with bounded and continuous $A : \mathbb{R} \to B(E)$. In this case the evolutionary family (propagator) $\{U(x,s)\}_{(x,s) \in \mathbb{R}^2}$ consists of invertible operators, the function $(x,s) \mapsto U(x,s)$ is norm-continuous, and $P(\cdot)$ from Definition 3.2 is also a bounded, norm-continuous function $P : \mathbb{R} \to B(E)$.

The second inequality in b) implies that the restriction $U(x,s)|\operatorname{Ker} P(s)$ is uniformly injective as an operator from $\operatorname{Ker} P(s)$ to $\operatorname{Ker} P(x)$ (that is $\|U(x,s)v\| \geq c\|v\|$ for some $c > 0$ and all $v \in \operatorname{Ker} P(s)$). Thus condition c) implies that $U(x,s)|\operatorname{Ker} P(s)$ is invertible as an operator from $\operatorname{Ker} P(s)$ to $\operatorname{Ker} P(x)$. Obviously, if $U(x,s)$ is invertible in

E or $\dim \operatorname{Ker} P(x) \le d < \infty$, condition c) in Definition 3.2 is redundant. The inequality $\dim \operatorname{Ker} P(x) \le d < \infty$ holds, for example, if the $U(x,s)$ are compact operators in E ([R. Rau, private communication]). Generally, of course, b) does not imply c).

¿From now on we will assume that the Banach space E is separable.

As we will see below, the spectral hyperbolicity of the evolutionary family $\{U(x,s)\}$ is equivalent to the hyperbolicity $\sigma(e^{tG}) \cap \mathbb{T} = \emptyset$, $t > 0$ of the evolutionary semigroup $\{e^{tG}\}_{t \ge 0}$ in $L_p(\mathbb{R}; E)$. Therefore, by Theorem 3.1 the spectral hyperbolicity of the evolutionary family $\{U(x,s)\}_{x \ge s}$ is also equivalent to the condition $\sigma(G) \cap i\mathbb{R} = \emptyset$. That is why we used the term *spectral* hyperbolicity in Definition 3.2. A remarkable observation by R. Rau [27] shows that generally the condition c) in Definition 3.2 cannot be dropped: there exists an evolutionary family that satisfies conditions a) and b) but $\sigma(e^{tG}) \cap \mathbb{T} \ne \emptyset$ for the associated evolutionary semigroup.

If the operator $T = e^G$ is hyperbolic in $L_p(\mathbb{R}; E)$, that is $\sigma(T) \cap \mathbb{T} = \emptyset$, we let $\mathcal{P}$ denote the Riesz projection for T, corresponding to the part $\sigma(T)$ lying inside the unit disk $\mathbb{D}$, and set $\mathcal{Q} = I - \mathcal{P}$.

Lemma 3.2. *If $\sigma(e^{tG}) \cap \mathbb{T} = \emptyset$, $t > 0$, then $\mathcal{P}$ has a form $(\mathcal{P}f)(x) = P(x)f(x)$, where $P : \mathbb{R} \to B(E)$ is a bounded projection-valued function such that the function $\mathbb{R} \ni x \mapsto P(x)v \in E$ is (strongly) measurable for each $v \in E$.*

Proof. We will show first that

$$\chi \mathcal{P} = \mathcal{P} \chi \tag{16}$$

for any scalar function $\chi \in L_\infty(\mathbb{R}; \mathbb{R})$. Then we will derive that $(\mathcal{P}f)(x) = P(x)f(x)$ from (16).

Note that the decomposition $L_p(\mathbb{R}; E) = \operatorname{Im} \mathcal{P} \oplus \operatorname{Im} \mathcal{Q}$ is T-invariant. Denote $T_P = \mathcal{P}T\mathcal{P} = T|\operatorname{Im}\mathcal{P}$, $T_Q = \mathcal{Q}T\mathcal{Q} = T|\operatorname{Im}\mathcal{Q}$. Note that $\sigma(T_P) \subset \mathbb{D}$, and T_Q is invertible with $\sigma(T_Q^{-1}) \subset \mathbb{D}$ in $\operatorname{Im}\mathcal{Q}$. Hence for some λ, $M > 0$ and all $n \in \mathbb{N}$, the following inequalities hold:

$$\|T_P^n f\|_{L_p} \le M e^{-\lambda n} \|f\|_{L_p}, \quad f \in \operatorname{Im}\mathcal{P}, \tag{17}$$

$$\|T_Q^n f\|_{L_p} \ge M^{-1} e^{\lambda n} \|f\|_{L_p}, \quad f \in \operatorname{Im}\mathcal{Q}. \tag{18}$$

We show first that $\operatorname{Im}\mathcal{P} = \{f \in L_p(\mathbb{R}; E) : T^n f \to 0 \text{ as } n \to \infty\}$. Indeed, $f \in \operatorname{Im}\mathcal{P}$ implies that $T^n f \to 0$ by (17). Conversely, if $T^n f \to$

0, then for $f = \mathcal{P}f + \mathcal{Q}f$, the inequality (18) implies

$$\|\mathcal{Q}f\| \le Me^{-\lambda n}\|T_Q^n \mathcal{Q}f\| \le Me^{-\lambda n}\{\|T^n f\| + \|T_P^n f\|\} \to 0,$$

and hence $f \in \operatorname{Im}\mathcal{P}$.

Consider on $L_p = L_p(\mathbb{R}; E)$ the operator χ of multiplication by $\chi(\cdot) \in L_\infty(\mathbb{R};\mathbb{R})$. Note that $(T^n \chi f)(x) = \chi(x-n)(T^n f)(x)$. Hence for $f \in \operatorname{Im}\mathcal{P}$

$$\|T^n \chi f\|_{L_p} \le \|\chi\|_{L_\infty}\|T^n f\|_{L_p} \to 0 \text{ as } n \to \infty,$$

and so $\chi f \in \operatorname{Im}\mathcal{P}$. Thus, to prove (16), it suffices to show that $f \in \operatorname{Im}\mathcal{Q}$ implies $\chi f \in \operatorname{Im}\mathcal{Q}$.

Fix $f \in \operatorname{Im}\mathcal{Q}$. Recall that T_Q is invertible on $\operatorname{Im}\mathcal{Q}$. Let $f_n = T_Q^{-n} f$, and define functions $g_n(x) = \chi(x+n) f_n(x)$, $n = 0, 1, \dots$. Decompose $g_n = \mathcal{P}g_n + \mathcal{Q}g_n$. Since the decomposition $L_p(\mathbb{R}; E) = \operatorname{Im}\mathcal{P} \oplus \operatorname{Im}\mathcal{Q}$ is T-invariant, one has:

$$\chi f = T^n g_n,\ \mathcal{P}\chi f = T_P^n \mathcal{P} g_n,\ \mathcal{Q}\chi f = T_Q^n \mathcal{Q} g_n,\ n = 0, 1, \dots .$$

Now (17)–(18) imply:

$$\begin{aligned} \|\mathcal{P}\chi f\| &\le Me^{-\lambda n}\|\mathcal{P}g_n\| \le Me^{-\lambda n}\{\|g_n\| + \|\mathcal{Q}g_n\|\} \\ &\le Me^{-\lambda n}\{\|\chi\|_{L_\infty}\|f_n\| + Me^{-\lambda n}\|\mathcal{Q}\chi f\|\} \\ &\le Me^{-\lambda n}\{\|\chi\|_{L_\infty} Me^{-\lambda n}\|f\| + Me^{-\lambda n}\|\mathcal{Q}\chi f\|\} \to 0, \end{aligned}$$

and hence $\chi f \in \operatorname{Im}\mathcal{Q}$. Thus (16) is proved.

In order to define $P(\cdot)$ such that $(\mathcal{P}f)(x) = P(x)f(x)$, fix $m \in \mathbb{Z}$ and let $\chi_m(x) = 1$ if $x \in [m, m+1)$, and $\chi_m(x) = 0$ otherwise. Let $\{e_n\}_{n\in\mathbb{Z}}$ be a linearly independent set with dense span E_0.

Consider the function $f \in L_p(\mathbb{R}; E)$, defined by $f(x) = \chi_m(x) e_n$. Since $\mathcal{P}$ is bounded on $L_p(\mathbb{R}; E)$, it is true that $\mathcal{P}f \in L_p(\mathbb{R}; E)$. For $x \in [m, m+1)$ define a vector $P(x)e_n \in E$ as $P(x)e_n = (\mathcal{P}f)(x)$. For $v = \sum_{n=1}^k d_n e_n \in E_0$ set $P(x)v = \sum_{n=1}^k d_n P(x) e_n$.

Let Δ be a measurable subset in $[m, m+1)$, and let χ_Δ be its characteristic function. Now (16) implies that

$$\begin{aligned} \int_\Delta \|P(x)v\|_E^p \, dx &= \int_{\mathbb{R}} \|\chi_\Delta(\mathcal{P}\chi_m v)(x)\|_E^p \, dx = \int_{\mathbb{R}} \|(\mathcal{P}\chi_\Delta v)(x)\|_E^p \, dx \\ &\le \|\mathcal{P}\|_{B(L_p(\mathbb{R};E))}^p \int_\Delta \|v\|_E^p \, dx. \end{aligned}$$

Therefore, $\|P(x)v\|_E \le \|\mathcal{P}\|_{B(L_p)}\|v\|_E$ for a.e. $x \in \mathbb{R}$ and all $v \in E_0$. Hence, $P(x)$ can be extended to a bounded operator on E, such that

$\|P(x)\| \le \|\mathcal{P}\|$ for a.e. $x \in \mathbb{R}$. That the function $x \mapsto P(x)v$ is a measurable function for all $v \in E_0$ (and, hence, for all $v \in E$) follows from the fact that the function $x \mapsto (\mathcal{P}f)(x)$ is measurable.

To show that $(\mathcal{P}f)(x) = P(x)f(x)$ we can assume that f is a simple function, $f = \sum \chi_{\Delta_k} v_k$, where $\Delta_k \subset [m_k, m_k+1)$, $m_k \in \mathbb{Z}$, and $v_k \in E_0$. Then (16) implies:

$$(\mathcal{P}f)(x) = \sum \chi_{m_k}(x)(\mathcal{P}\chi_{\Delta_k} v_k)(x) = \sum \chi_{\Delta_k}(x)P(x)v_k = P(x)f(x).$$

□

Let us stress that the function $P(\cdot)$ above is only defined on a set $\mathbb{R}_0 \subset \mathbb{R}$ such that $\mathrm{mes}(\mathbb{R} \setminus \mathbb{R}_0) = 0$. In Theorem 3.4 below we will establish that, in fact, this function $P(\cdot)$ can be extended to all of $\mathbb{R}$ as a continuous function (in the strong operator topology in $B(E)$). To prove this fact we will need a few definitions and Lemma.

Let $\mathfrak{A}$ be the set of all operators a in $L_p(\mathbb{R}; E)$ of the form $(af)(x) = a(x)f(x)$, where the function $a : \mathbb{R} \to B(E)$ is bounded and the function $\mathbb{R} \ni x \mapsto a(x)v \in E$ is continuous for each $v \in E$. For $a \in \mathfrak{A}$ and $x \in \mathbb{R}$ let us define an operator $\pi_x(a)$ on $l_p(\mathbb{Z}; E)$ by the rule

$$\pi_x(a) = \mathrm{diag}\{a(x+n)\}_{n\in\mathbb{Z}} : (v_n)_{n\in\mathbb{Z}} \mapsto (a(x+n)v_n)_{n\in\mathbb{Z}}. \tag{19}$$

Finally, let $S : (v_n)_{n\in\mathbb{Z}} \mapsto (v_{n-1})_{n\in\mathbb{Z}}$ be a shift operator on $l_p(\mathbb{Z}; E)$.

Let us denote: $T = e^G$, $a(x) = U(x, x-1)$, $(Rf)(x) = f(x-1)$. Then $T = aR$. For $\lambda \in \mathbb{T}$ set $b = \lambda I - aR$, and for $x \in \mathbb{R}$ set $\pi_x(b) = \lambda I - \pi_x(a)S$.

Lemma 3.3. *If $\sigma(T) \cap \mathbb{T} = \emptyset$ in $L_p(\mathbb{R}; E)$ then $\sigma(\pi_x(a)S) \cap \mathbb{T} = \emptyset$ in $l_p(\mathbb{Z}; E)$ for all $x \in \mathbb{R}$. Moreover, for all $\lambda \in \mathbb{T}$ the following estimate holds:*

$$\|[\pi_x(b)]^{-1}\|_{B(l_p(\mathbb{Z};E))} \le \|b^{-1}\|_{B(L_p(\mathbb{R};E))}, \quad x \in \mathbb{R}. \tag{20}$$

Proof. First, for any $\xi \in \mathbb{R}$ one has:

$$\pi_x(a)SL = e^{-i\xi} L\pi_x(a)S,$$

where L is the operator $(v_n) \mapsto (e^{i\xi n}v_n)$. Hence $\sigma(\pi_x(a)S)$ is invariant under rotations centered at the origin. Thus it suffices to prove the Lemma for the special case $\lambda = 1$, that is, to show that if $b = I - aR$ is invertible in $L_p(\mathbb{R}; E)$ then for each $x_0 \in \mathbb{R}$ that the operator $\pi_{x_0}(b) =$

$I - \pi_{x_0}(a)S$ is invertible in $l_p(\mathbb{Z}; E)$, and that estimate (22) is valid for this b and $x = x_0$.

Further, it suffices to prove the Lemma only for $x_0 = 0$. Indeed, let us denote $\hat{a}(x) = a(x + x_0)$, $x \in \mathbb{R}$ for any fixed $x_0 \in \mathbb{R}$. Obviously,

$$\pi_{x_0}(I - aR) = I - \pi_{x_0}(a)S = I - \pi_0(\hat{a})S = \pi_0(I - \hat{a}R).$$

Consider the invertible isometry J_{x_0} on $L_p(\mathbb{R}; E)$, defined as $(J_{x_0}f)(x) = f(x+x_0)$. Clearly, $I-\hat{a}R = J_{x_0}(I-aR)J_{x_0}^{-1}$. Hence, the operator $I-\hat{a}R$ is invertible if and only if the operator $b = I - aR$ is invertible, and $\|(I - \hat{a}R)^{-1}\| = \|b^{-1}\|$. Therefore, the estimate (20) for $x = x_0$ follows from the estimate (20) for $x = 0$.

Thus our purpose is to prove if $b = I - aR$ is invertible in $L_p(\mathbb{R}; E)$, then the operator $\pi_0(b) = I - \pi_0(a)S$ is invertible in $l_p(\mathbb{Z}; E)$, and

$$\|[\pi_0(b)]^{-1}\|_{B(l_p(\mathbb{Z};E))} \leq \|b^{-1}\|_{B(L_p(\mathbb{R};E))}. \tag{21}$$

We first show that for any $(v_n) \in l_p(\mathbb{Z}; E)$ the following estimate holds:

$$\|(v_n)\|_{l_p(\mathbb{Z};E)} \leq \|b^{-1}\|_{B(L_p(\mathbb{R};E))}\|\pi_0(b)(v_n)\|_{l_p(\mathbb{Z};E)}. \tag{22}$$

Let us fix a sequence $(v_n)_{n\in\mathbb{Z}} \in l_p(\mathbb{Z}; E)$, a natural number $N > 1$, and $\epsilon > 0$.

Recall that the function $\mathbb{R} \ni x \mapsto a(x)v \in E$ is continuous for each $v \in E$. Choose $\delta < 1$ such that

$$\|[a(x + n) - a(n)]v_{n-1}\|_E < \epsilon, \quad \forall x \in [0, \delta],\ n = -N, \dots , N. \tag{23}$$

Define $f \in L_p(\mathbb{R}; E)$ by $f(x) = v_n$ for $x \in [n, n + \delta]$, $|n| \leq N$, and $f(x) = 0$ otherwise. Since b is an invertible operator in $L_p(\mathbb{R}; E)$, it follows that:

$$\|b^{-1}\|^p_{B(L_p)}\|bf\|^p_{L_p} \geq \|f\|^p_{L_p} = \sum_{n=-N}^{N} \int_n^{n+\delta} \|v_n\|^p \, dx = \delta \sum_{n=-N}^{N} \|v_n\|^p. \tag{24}$$

On the other hand, using (23), one has:

$$\begin{aligned}\|bf\|_{L_p}^p &= \int_{\mathbb{R}} \|f(x) - a(x)f(x-1)\|_E^p \, dx \\ &= \sum_{n=-N+1}^{N} \int_n^{n+\delta} \|v_n - a(x)v_{n-1}\|_E^p \, dx + \int_{-N}^{-N+\delta} \|v_{-N}\|_E^p \, dx \\ &\quad + \int_{N+1}^{N+1+\delta} \|a(x)v_N\|_E^p \, dx \\ &\le \sum_{n=-N+1}^{N} \int_0^{\delta} \|v_n - a(n)v_{n-1} - [a(x+n) - a(n)]v_{n-1}\|_E^p \, dx \\ &\quad + \delta\|v_{-N}\|_E^p + \delta \max_{x\in\mathbb{R}} \|a(x)\|^p \cdot \|v_N\|_E^p \\ &\le \delta \sum_{n=-N}^{N} (\|v_n - a(n)v_{n-1}\|_E + \epsilon)^p + \delta\|v_{-N}\|_E^p + \delta \max_{x\in\mathbb{R}} \|a(x)\|^p \|v_N\|_E^p.\end{aligned}$$

Combining this inequality with (24), one has:

$$\begin{aligned}\sum_{n=-N}^{N} \|v_n\|^p \le \|b^{-1}\|_{B(L_p)}^p \Big\{ &\sum_{n=-N}^{N} (\|v_n - a(n)v_{n-1}\|_E + \epsilon)^p \\ &+ \|v_{-N}\|_E^p + \max_{x\in\mathbb{R}} \|a(x)\|^p \|v_N\|_E^p \Big\}.\end{aligned}$$

If $\epsilon \to 0$ and $N \to \infty$, then

$$\|(v_n)\|_{l_p(\mathbb{Z};E)} \le \|b^{-1}\|_{B(L_p)} \Big(\sum_{n=-\infty}^{\infty} \|v_n - a(n)v_{n-1}\|_E^p \Big)^{1/p}$$

and (22) is proved.

Note, that (22) is sufficient to show (21) provided the operator $\pi_0(b)$ is invertible, and so it only remains to show that $\pi_0(b)$ is an operator onto $l_p(\mathbb{Z}; E)$.

Fix any $(v_n) \in l_p(\mathbb{Z}; E)$ and let $f(x) = U(x, n-1)v_{n-1}$, $x \in [n - 1/2, n + 1/2)$, $n \in \mathbb{Z}$. Property (iii) from the Definition 3.1 of the evolutionary family $\{U(x, s)\}$ implies that

$$\|U(x, n-1)\| \le Ce^{\beta(x-n+1)} \le Ce^{\frac{3}{2}\beta} \text{ for } x \in [n - 1/2, n + 1/2).$$

Hence $f \in L_p(\mathbb{R}; E)$. Since the operator b, defined by $(bg)(x) = g(x) - U(x, x-1)g(x-1)$, is invertible in $L_p(\mathbb{R}; E)$, it follows that there exists a unique function $g \in L_p(\mathbb{R}; E)$ such that

$$g(x) - U(x, x-1)g(x-1) = f(x) \tag{25}$$

for almost all $x \in \mathbb{R}$. Since

$$\|g\|_{L_p}^p = \sum_{n\in\mathbb{Z}} \int_{-n-1/2}^{-n+1/2} \|g(s)\|^p \, ds = \int_{-1/2}^{1/2} \Big(\sum_{n\in\mathbb{Z}} \|g(s+n)\|^p \Big) \, ds < \infty,$$

the sequence $(g(s+n))_{n\in\mathbb{Z}}$ belongs to $l_p(\mathbb{Z}; E)$ for all $s \in \Omega$ for some subset $\Omega \subset (-1/2, 1/2)$ of full measure. For each $s \in \Omega$, let us define a function h_s by the rule:

$$h_s(x) = \begin{cases} g(x), & \text{if } n - \dfrac{1}{2} \le x \le n + s, \\ U(x, n+s) g(n+s), & \text{if } n + s \le x < n + \dfrac{1}{2}, \end{cases} \qquad n \in \mathbb{Z}.$$

Clearly, $h_s \in L_p(\mathbb{R}; E)$ for each $s \in \Omega$ because $(g(s+n))_{n\in\mathbb{Z}} \in l_p(\mathbb{Z}; E)$, and

$$\|U(x, n+s)\| \le C e^{\beta(1/2-s)} \text{ for } x \in [n+s, n+1/2).$$

We note that h_s is a solution of equation (25). Indeed, for $x \in [n-1/2, n+s]$, equation (25) implies $h_s(x) - U(x, x-1) h_s(x-1) = f(x)$. For $x \in [n+s, n+1/2)$, one has:

$$\begin{aligned} & h_s(x) - U(x, x-1) h_s(x-1) = \\ & U(x, n+s)[g(n+s) - U(n+s, n-1+s) g(n-1+s)] \\ & = U(x, n+s) f(n+s) = U(x, n+s) U(n+s, n-1) v_{n-1} = f(x). \end{aligned}$$

But equation (25) has only one solution g in $L_p(\mathbb{R}; E)$. Hence $g = h_s$ for all $s \in \Omega$.

Fix $s < 0$, $s \in \Omega$. The function $h_s(\cdot)$ is defined for $x = n$, $n \in \mathbb{Z}$. Moreover, the sequence $(h_s(n))_{n\in\mathbb{Z}}$ belongs to $l_p(\mathbb{Z}; E)$. Indeed, $h_s(n) = U(n, n+s) g(n+s)$, the sequence $(g(n+s))_{n\in\mathbb{Z}} \in l_p(\mathbb{Z}; E)$, and $\|U(n, n+s)\| \le C e^{-\beta s}$ by (iii) from Definition 3.1.

Set $u_n = h_s(n) + v_n$. Since $h_s(\cdot)$ satisfies the equation (25) for $x = n$, $n \in \mathbb{Z}$, we have:

$$\begin{aligned} & \pi_0(b)(u_n) = \\ & u_n - U(n, n-1) u_{n-1} = h_s(n) + v_n - U(n, n-1) h_s(n-1) - U(n, n-1) v_{n-1} = \\ & v_n + h_s(n) - U(n, n-1) h_s(n-1) - f(n) = (v_n). \end{aligned}$$

This identity proves that $\pi_0(b)$ is an operator onto $L_p(\mathbb{Z}; E)$. □

Theorem 3.4. *Let $\{U(x,s)\}_{x\ge s}$ be an evolutionary family in a separable Banach space E, and let $\{e^{tG}\}_{t\ge 0}$ be the evolutionary semigroup acting on $L_p(\mathbb{R};E)$, $1\le p<\infty$ by the rule $(e^{tG}f)(x)=U(x,x-t)f(x-t)$. The following conditions are equivalent:*

1) $\{U(x,s)\}_{x\ge s}$ is (spectrally) hyperbolic in E;
2) $\sigma(e^{tG})\cap\mathbb{T}=\emptyset$, $t\ne 0$, in $L_p(\mathbb{R};E)$;
3) $0\in\rho(G)$ in $L_p(\mathbb{R};E)$.

Moreover, the Riesz projection $\mathcal{P}$ that corresponds to the part $\sigma(e^G)\cap\mathbb{D}$ of the spectrum of the hyperbolic operator e^G is related to a strongly continuous, projection-valued function $P:\mathbb{R}\to B(E)$ that satisfies Definition 3.2 by the formula $(\mathcal{P}f)(x)=P(x)f(x)$, $x\in\mathbb{R}$, $f\in L_p(\mathbb{R};E)$.

Proof. 2) $\Leftrightarrow$ 3) was proved in Theorem 3.1.

1) $\Rightarrow$ 2). Without loss of generality assume $t=1$. ¿From the projection-valued function $P(\cdot)$ from Definition 3.2, let us define a projection $\mathcal{P}$ in $L_p(\mathbb{R};E)$ by the rule $(\mathcal{P}f)(x)=P(x)f(x)$. Denote $\mathcal{Q}=I-\mathcal{P}$. For $T=e^G$, condition a) of Definition 3.2 implies $\mathcal{P}T=T\mathcal{P}$. Set $T_P=\mathcal{P}T\mathcal{P}$ and $T_Q=\mathcal{Q}T\mathcal{Q}$. Then b) implies $\sigma(T_P)\subset\mathbb{D}$ in $\operatorname{Im}\mathcal{P}=\{f\in L_p(\mathbb{R};E): f(x)\in\operatorname{Im}P(x)\}$. Also b) and c) imply that the operator T_Q, which can be written as $(T_Qf)(x)=Q(x)U(x,x-1)Q(x-1)f(x-1)$, is an invertible operator, and $\sigma(T_Q^{-1})\subset\mathbb{D}$ in $\operatorname{Im}\mathcal{Q}=\operatorname{Ker}\mathcal{P}$. Hence, $\sigma(T)\cap\mathbb{T}=\emptyset$.

2) $\Rightarrow$ 1). Let $\mathfrak{B}$ be a Banach algebra with a norm $\|\cdot\|_1$ consisting of the operators d on $L_p(\mathbb{R};E)$ of the form

$$d=\sum_{k=-\infty}^{\infty}a_kR^k,\ a_k\in\mathfrak{A},\ \|d\|_1:=\sum_{k=-\infty}^{\infty}\|a_k\|_{B(L_p(\mathbb{R};E))}<\infty.$$

We first show that if $b=\lambda-T$ is invertible in $L_p(\mathbb{R};E)$ for all $\lambda\in\mathbb{T}$, then $(\lambda-T)^{-1}\in\mathfrak{B}$. This fact will be proved in several steps.

First, without loss of generality let $\lambda=1$. Since $\sigma(T)\cap\mathbb{T}=\emptyset$, by Lemma 3.2 the Riesz projection $\mathcal{P}$ has a form $(\mathcal{P}f)(x)=P(x)f(x)$, where the function $\mathbb{R}_0\ni x\mapsto P(x)v\in E$ is a bounded, measurable (in the strong sense in E) function for each $v\in E$, where $\mathbb{R}_0$ is a set of full measure in $\mathbb{R}$. Recall also that $Q(x)=I-P(x)$. Decompose $b=I-T=(\mathcal{P}-T_P)\oplus(\mathcal{Q}-T_Q)$. Since $\sigma(T_P)\subset\mathbb{D}$ and $\sigma(T_Q^{-1})\subset\mathbb{D}$, one has that $b^{-1} = (\mathcal{P} - T_P)^{-1} \oplus (\mathcal{Q} - T_Q)^{-1}$, where

$$(\mathcal{P} - T_P)^{-1} = \sum_{k=0}^{\infty} T_P^k; \quad (\mathcal{Q} - T_Q)^{-1} = [-T_Q(\mathcal{Q} - T_Q^{-1})]^{-1} = -\sum_{k=-\infty}^{-1} T_Q^k. \tag{26}$$

Notice that $T_Q^{-1} = (\mathcal{Q}aR\mathcal{Q})^{-1} = (QaQ(\cdot - 1)R)^{-1} = R^{-1}(QaQ(\cdot - 1))^{-1} = [Q(\cdot + 1)a(\cdot + 1)Q(\cdot)]^{-1}R^{-1}$, and that $T_P = aR\mathcal{P} = aP(\cdot - 1)R$. Hence both operators T_P^k and T_Q^k can be written as $a_k R^k$ for some multiplication operators a_k. The Neumann series in (26) converge absolutely. Therefore,

$$b^{-1} = \sum_{k=-\infty}^{\infty} a_k R^k, \quad \sum_{k=-\infty}^{\infty} \|a_k\|_{B(L_p(\mathbb{R};E))} < \infty, \tag{27}$$

for each $k \in \mathbb{Z}$ the function $a_k : \mathbb{R}_0 \to B(E)$ is bounded, and the function $\mathbb{R}_0 \ni x \mapsto a_k(x)v \in E$ is measurable for each $v \in E$.

Our next aim is to show that the a_k from (27) belong to $\mathfrak{A}$, that is, the function $x \mapsto a_k(x)v \in E$ extends to a continuous function from $\mathbb{R}$ for each $v \in E$. To this end let us define for a_k from (27) and all $x \in \mathbb{R}_0$ the operator $\pi_x(a_k)$ in $l_p(\mathbb{Z}; E)$ as in (19). Denote:

$$\pi_x(b^{-1}) = \sum \pi_x(a_k) S^k, \quad \pi_x(a_k) = \mathrm{diag}\{a_k(x+n)\}_{n\in\mathbb{Z}} \quad \text{for } x \in \mathbb{R}_0. \tag{28}$$

Identities $bb^{-1} = b^{-1}b = I$ in $L_p(\mathbb{R}; E)$ imply that $\pi_x(b) \cdot \pi_x(b^{-1}) = \pi_x(b^{-1}) \cdot \pi_x(b) = I$ in $l_p(\mathbb{Z}; E)$ for $x \in \mathbb{R}_0$. Since the operator b is invertible in $L_p(\mathbb{R}; E)$ by assumption, for each $x \in \mathbb{R}$ the operator $\pi_x(b)$ is invertible in $l_p(\mathbb{Z}; E)$ by Lemma 3.3. Hence

$$\pi_x(b^{-1}) = [\pi_x(b)]^{-1} \text{ for } x \in \mathbb{R}_0. \tag{29}$$

Recall that the function $\mathbb{R} \ni x \mapsto a(x)v \in E$ is continuous for each $v \in E$. Also, the function $\mathbb{R} \ni x \mapsto \|a(x)\| \in \mathbb{R}_+$ is bounded. Hence for each $(v_n) \in l_p(\mathbb{Z}; E)$ the function $\mathbb{R} \ni x \mapsto \pi_x(b)(v_n) \in l_p(\mathbb{Z}; E)$ is continuous. By Lemma 3.3 $\|[\pi_x(b)]^{-1}\|_{B(l_p)}$ are uniformly bounded for $x \in \mathbb{R}$. This implies that the function $\mathbb{R} \ni x \mapsto [\pi_x(b)]^{-1}(v_n) \in l_p(\mathbb{Z}; E)$ is continuous for each $(v_n) \in l_p(\mathbb{Z}; E)$. Indeed,

$$\left\| \left([\pi_x(b)]^{-1} - [\pi_{x_0}(b)]^{-1} \right)(v_n) \right\|_{l_p(\mathbb{Z};E)} =$$
$$\left\| [\pi_x(b)]^{-1} \cdot [\pi_x(b) - \pi_{x_0}(b)] \cdot [\pi_{x_0}(b)]^{-1}(v_n) \right\|_{l_p(\mathbb{Z};E)}$$

for any $x, x_0 \in \mathbb{R}$, and the function $\mathbb{R} \ni x \mapsto [\pi_x(b)]^{-1}(v_n) \in l_p(\mathbb{Z}; E)$ is continuous at $x = x_0$.

Fix $k_0 \in \mathbb{Z}$, $x_0 \in \mathbb{R}$, and $v \in E$. Define $(\tilde{v}_n) \in l_p(\mathbb{Z}; E)$ as $\tilde{v}_{-k_0} = v$ and $\tilde{v}_n = 0$ for $n \neq -k_0$. Consider a sequence $x_m \to x_0$, $x_m \in \mathbb{R}_0$. We will show that $\{a_{k_0}(x_m)v\}_{m\in\mathbb{N}}$ is a Cauchy sequence in E and will define $a_{k_0}(x_0)v = \lim_{m\to\infty} a_{k_0}(x_m)v$. Then the function $\mathbb{R} \ni x \mapsto a_{k_0}(x)v \in E$ becomes a continuous function, and $a_{k_0} \in \mathfrak{A}$.

Note, that the $\pi_{x_m}(b^{-1})$ are defined by the formula (28) since $x_m \in \mathbb{R}_0$. For the sequence $(\tilde{v}_n)$ one has the following estimate:

$$
\begin{aligned}
&\|[\pi_{x_{m'}}(b^{-1}) - \pi_{x_{m''}}(b^{-1})](\tilde{v}_n)\|_{l_p}^p \\
&= \sum_{n\in\mathbb{Z}} \left\| \sum_{k\in\mathbb{Z}} [a_k(x_{m'} + n) - a_k(x_{m''} + n)]\tilde{v}_{n-k} \right\|_E^p \\
&\geq \left\| \sum_k [a_k(x_{m'}) - a(x_{m''})]\tilde{v}_{-k} \right\|_E^p \\
&= \|[a_{k_0}(x_{m'}) - a_{k_0}(x_{m''})]v\|_E^p.
\end{aligned} \tag{30}
$$

Since $x_m \in \mathbb{R}_0$, formula (29) is applicable. Then the sequence

$$
\pi_{x_m}(b^{-1})(\tilde{v}_n) = [\pi_{x_m}(b)]^{-1}(\tilde{v}_n)
$$

is a Cauchy sequence in $l_p(\mathbb{Z}; E)$ since the function $\mathbb{R} \ni x \mapsto [\pi_x(b)]^{-1}(\tilde{v}_n) \in l_p(\mathbb{Z}; E)$ is continuous. In accordance with (30) the sequence $\{a_{k_0}(x_m)v\}_{m\in\mathbb{N}}$ is a Cauchy sequence in E, and $a_{k_0} \in \mathfrak{A}$.

Since the a_k from (27) are continuous, we have proved that $(\lambda I - T)^{-1} \in \mathfrak{B}$ for all $\lambda \in \mathbb{T}$.

The rest of the proof is standard (cf. [2, 18, 27]). Indeed, consider the absolutely convergent Fourier series $f : \lambda \mapsto \lambda I - aR\lambda^0$ with the coefficients from $\mathfrak{B}$. For each $\lambda \in \mathbb{T}$, the operator $f(\lambda) = b$ is invertible in $\mathfrak{B}$. Hence the function $\mathbb{T} \ni \lambda \mapsto [f(\lambda)]^{-1} \in \mathfrak{B}$ is expandable (see, e.g., [5]) into an absolutely convergent Fourier series

$$
[f(\lambda)]^{-1} = (\lambda I - aR)^{-1} = \sum_{k=-\infty}^{\infty} d_k \lambda^k, \; \sum_k \|d_k\| < \infty, \; d_k \in \mathfrak{B}.
$$

By the integral formula (see, e.g., [9, p. 20]) for the Riesz projection $\mathcal{P}$, we conclude that $\mathcal{P} = d_{-1} \in \mathfrak{B}$. Hence for some $a_k \in \mathfrak{A}$ one has that

$$
\mathcal{P} = \sum_{k=-\infty}^{\infty} a_k R^k, \text{ where } \sum_{k=-\infty}^{\infty} \|a_k\| < \infty.
$$

We will show that $a_k = 0$ for $k \neq 0$, and $\mathcal{P} = a_0 \in \mathfrak{A}$. Indeed, by (16), $\chi\mathcal{P} = \mathcal{P}\chi$ for any bounded continuous scalar function χ. Then

$$\chi\mathcal{P} - \mathcal{P}\chi = \sum_k a_k(\cdot)[\chi(\cdot) - \chi(\cdot - k)]R^k = 0.$$

Then by picking x_0 and χ such that $\chi(x_0) \neq \chi(x_0 - k)$ for $k \neq 0$, it follows that $a_k(x_0) = 0$, $k \neq 0$. $\square$

As we have mentioned above, for the space $C_0(\mathbb{R}; E)$, part 1) $\Leftrightarrow$ 2) of Theorem 3.4 was proved in [27].

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