

Comparison of Sums of Independent Identically Distributed Random Variables

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ABSTRACT: Let S_k be the k -th partial sum of Banach space valued independent identically distributed random variables. In this paper, we compare the tail distribution of $\|S_k\|$ with that of $\|S_j\|$, and deduce some tail distribution maximal inequalities.

The main result of this paper was inspired by the inequality from [dP–M] that says that $\Pr(\|X_1\| > t) \leq 5 \Pr(\|X_1 + X_2\| > t/2)$ whenever X_1 and X_2 are independent identically distributed. Such results for L_p ($p \geq 1$) such as $\|X_1\|_p \leq \|X_1 + X_2\|_p$ are straightforward, at least if X_2 has zero expectation. This inequality is also obvious if either X_1 is symmetric, or X_1 is real valued positive. However, for arbitrary random variables, this result is somewhat surprising to the author. Note that the identically distributed assumption cannot be dropped, as one could take $X_1 = 1$ and $X_2 = -1$.

In this paper, we prove a generalization to sums of arbitrarily many independent identically distributed random variables. Note that all results in this paper are true for Banach space valued random variables. The author would like to thank Victor de la Peña for helpful conversations.

THEOREM 1. *There exist universal constants $c_1 = 3$ and $c_2 = 10$ such that if X_1, X_2, \dots are independent identically distributed random variables, and if we set*

$$S_k = \sum_{i=1}^k X_i,$$

then for $1 \leq j \leq k$

$$\Pr(\|S_j\| > t) \leq c_1 \Pr(\|S_k\| > t/c_2).$$

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This result cannot be asymptotically improved. Consider, for example, the case where $X_1 = 1$ with very small probability, and is zero otherwise. This shows that for the inequality to be true for all X_1 , it must be that c_2 must be larger than some universal constant for all j and k . Also, it is easy to see that c_1 must be larger than some universal constant because it is easy to select X_1 and t so that $\Pr(\|S_j\| > t)$ is close to 1.

However, Latafala [L] has been able to obtain the same theorem with $c_1 = 4$ and $c_2 = 5$, or $c_1 = 2$ and $c_2 = 7$. In the case $j = 1$ and $k = 2$, he has shown that $\Pr(\|X_1\| > t) \leq 2 \Pr(\|X_1 + X_2\| > 2t/3)$, and these constants cannot be improved.

In order to show this result, we will use the following definition. We will say that x is a t -concentration point for a random variable X if $\Pr(\|X - x\| \leq t) > 2/3$.

LEMMA 2. *If x is a t -concentration point for X , and y is a t -concentration point for Y , and z is a t -concentration point for $X + Y$, then $\|x + y - z\| \leq 3t$.*

Proof:

$$\begin{aligned} \Pr(\|x + y - z\| > 3t) &\leq \Pr(\|X - x + Y - y - (X + Y - z)\| > 3t) \\ &\leq \Pr(\|X - x\| > t) + \Pr(\|Y - y\| > t) + \Pr(\|X + Y - z\| > t) \\ &< 1. \end{aligned}$$

Hence $\Pr(\|x + y - z\| \leq 3t) > 0$. Since x , y and z are fixed vectors, the result follows. \square

COROLLARY 3. *If X_1, X_2, \dots are independent identically distributed random variables, and if the partial sums $S_j = \sum_{i=1}^j X_i$ have t -concentration points s_j for $1 \leq j \leq k$, then $\|ks_j - js_k\| \leq 3(k + j)t$.*

Proof: We prove the result by induction. It is obvious if $j = k$. Otherwise,

$$\begin{aligned} \|js_k - ks_j\| &\leq \|js_{k-j} - (k-j)s_j\| + \|js_k - js_{k-j} - js_j\| \\ &\leq 3(k-j+j)t + 3jt = 3(k+j)t. \end{aligned}$$

(The observant reader will notice that we are, in fact, following the steps of Euclid's algorithm. The same proof could show $\|ks_j - js_k\| \leq 3(j+k-2h)t$ where h is the highest common factor of j and k .) \square

Proof of Theorem 1: We consider three cases. First suppose that $\Pr(\|S_{k-j}\| > 9t/10) \leq 1/3$. Note that $S_k - S_j$ is independent of S_j , and identically distributed to S_{k-j} . Then

$$\Pr(\|S_j\| > t) \leq 3/2 \Pr(\|S_j\| > t \text{ and } \|S_k - S_j\| \leq 9t/10) \leq 3/2 \Pr(\|S_k\| > t/10).$$

For the second case, suppose that there is a $1 \leq i \leq k$ such that S_i does not have any $(t/10)$ -concentration point. Then

$$\Pr(\|S_i + X_{i+1} + \dots + X_k\| > t/10 \mid \sigma(X_{i+1}, \dots, X_k)) \geq 1/3,$$

and hence $\Pr(\|S_k\| > t/10) \geq 1/3 \geq 1/3 \Pr(\|S_i\| > t)$.

Finally, we are left with the third case where $\Pr(\|S_{k-j}\| > 9t/10) > 1/3$, and S_i has a $(t/10)$ -concentration point s_i for all $1 \leq i \leq k$. Clearly $\|s_{k-j}\| \geq 8t/10$. Also, by Corollary 3,

$$\|s_k\| \geq \frac{k}{k-j} \|s_{k-j}\| - \frac{3(2k-j)t}{10(k-j)} \geq \frac{8kt}{10(k-j)} - \frac{6kt}{10(k-j)} \geq \frac{2t}{10}.$$

Therefore, $\Pr(\|S_k\| \geq t/10) \geq \Pr(\|S_k - s_k\| \leq t/10) \geq 2/3 \geq 2/3 \Pr(\|S_j\| > t)$, and we are done. \square

One might be emboldened to conjecture the following. Suppose that X_1, X_2, \dots are independent identically distributed random variables, and that $\alpha_i > 0$. Let

$$S_k = \sum_{i=1}^k \alpha_i X_i.$$

Then one might conjecture that there is a universal constant such that for $1 \leq j \leq k$

$$\Pr(\|S_j\| > t) \leq c \Pr(\|S_k\| > t/c).$$

As it turns out, this is not the case. Let Y_1, Y_2, \dots be real valued independent identically distributed random variables such that

$$\Pr(Y_i = N - 1) = 1/N$$

$$\Pr(Y_i = -1) = (N - 1)/N.$$

Then by the central limit theorem, there exists $M \geq N^3$ such that

$$\Pr\left(\left|\frac{1}{M^{2/3}} \sum_{i=1}^M Y_i\right| > \frac{1}{N}\right) \leq \frac{1}{N}.$$

Now let $X_i = Y_i + 1/M^{1/3}$, and let

$$S_M = \frac{1}{M^{2/3}} \sum_{i=1}^M X_i.$$

Then $\Pr(|S_M| > 1/2) \geq 1 - 1/N$, whereas $\Pr(|S_M + X_{M+1}| > 3/N) \leq 2/N$.

Theorem 1 has several corollaries.

COROLLARY 4. *There is a universal constant c such that if X_1, X_2, \dots are independent identically distributed random variables, and if we set*

$$S_k = \sum_{i=1}^k X_i,$$

then

$$\Pr\left(\sup_{1 \leq j \leq k} \|S_j\| > t\right) \leq c \Pr(\|S_k\| > t/c).$$

Latalo [L] has been able to obtain this result with $c_1 = 4$ and $c_2 = 6$, or with $c_1 = 2$ and $c_2 = 8$.

Proof: This follows from Proposition 1.1.1 of [K-W], that states that if X_1, X_2, \dots are independent (not necessarily identically distributed), and if $S_k = \sum_{i=1}^k X_i$, then

$$\Pr\left(\sup_{1 \leq j \leq k} \|S_j\| > t\right) \leq 3 \sup_{1 \leq j \leq k} \Pr(\|S_j\| > t/3).$$

It is also possible to prove this result directly using the techniques of the proof of Theorem 1. The third case only requires that $\Pr(\|S_{k-j}\| > 9t/10) > 1/3$ for one $j = 1, 2, \dots, k$. Hence, for the first case we may assume that $\Pr(\|S_k - S_j\| > 9t/10) \leq 1/3$ for all $1 \leq j \leq k$. Let A_j be the event $\{\|S_i\| \leq t \text{ for all } i < j \text{ and } \|S_j\| > t\}$. Then

$$\Pr(A_j) \leq 3/2 \Pr(A_j \text{ and } \|S_k - S_j\| \leq 9t/10) \leq 3/2 \Pr(A_j \text{ and } \|S_j\| > t/10).$$

Summing over j , the result follows. □

COROLLARY 5. *There is a universal constant c such that if X_1, X_2, \dots are independent identically distributed random variables, and if $|\alpha_i| \leq 1$, then*

$$\Pr\left(\left\|\sum_{i=1}^k \alpha_i X_i\right\| > t\right) \leq c \Pr\left(\left\|\sum_{i=1}^k X_i\right\| > t/c\right).$$

Proof: The technique used in this proof is well known (see for example [KW], Proposition 1.2.1), but is included for completeness.

By taking real and imaginary parts of α_i , we may suppose that the α_i are real. Without loss of generality, $1 \geq \alpha_1 \geq \dots \geq \alpha_k \geq -1$. Then we may write $\alpha_j = -1 + \sum_{i=j}^k \sigma_i$, where

$\sigma_i \geq 0$. Thus $\sum_{i=1}^k |\sigma_i| \leq 2$, and hence

$$\begin{aligned} \left\| \sum_{j=1}^k \alpha_j X_j \right\| &= \left\| \sum_{j=1}^k \left(-1 + \sum_{i=j}^k \sigma_i \right) X_j \right\| \\ &= \left\| - \left(\sum_{i=1}^k X_i \right) + \left(\sum_{j=1}^k \sigma_j \sum_{i=1}^j X_i \right) \right\| \\ &\leq \left\| \sum_{i=1}^k X_i \right\| + \left(\sum_{j=1}^k |\sigma_j| \right) \sup_{1 \leq j \leq k} \left\| \sum_{i=1}^j X_i \right\|. \end{aligned}$$

Applying Corollary 4, the result follows. \square

COROLLARY 6. *There are universal constants c_1 and c_2 such that if X_1, X_2, \dots are independent identically distributed random variables, and if we set*

$$S_k = \sum_{i=1}^k X_i,$$

then for $1 \leq k \leq j$

$$\Pr(\|S_j\| > t) \leq c_1 j/k \Pr(\|S_k\| > kt/c_2 j).$$

Proof: Let m be the least integer such that $mk \geq j$. By Theorem 1, it follows that $\Pr(\|S_j\| > t) \leq c \Pr(\|S_{mk}\| > t/c)$. That $\Pr(\|S_{mk}\| > t) \leq m \Pr(\|S_k\| > t/m)$ is straightforward. \square

The example where X_1 is constant shows that c_2 cannot be made smaller than some universal constant. The example where $X_1 = 1$ with very small probability and is zero otherwise shows the same is true for c_1 .

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