Boyd Indices of Orlicz–Lorentz Spaces

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ABSTRACT

Orlicz–Lorentz spaces provide a common generalization of Orlicz spaces and Lorentz spaces. In this paper, we investigate their Boyd indices. Bounds on the Boyd indices in terms of the Matuszewska–Orlicz indices of the defining functions are given. Also, we give an example to show that the Boyd indices and Zippin indices of an Orlicz–Lorentz space need not be equal, answering a question of Maligranda. Finally, we show how the Boyd indices are related to whether an Orlicz–Lorentz space is p-convex or q-concave.

1 INTRODUCTION

The Boyd indices of a rearrangement invariant space are of fundamental importance. They were originally introduced by Boyd (1969) for the purpose of showing certain interpolation results. Since then, they have played a major role in the theory of rearrangement invariant spaces (see, for example, Bennett and Sharpley (1988), Lindenstrauss and Tzafriri (1979) or Maligranda (1984)).

Orlicz–Lorentz spaces provide a common generalization of Orlicz spaces (see Orlicz(1932) or Luxembourg (1955)) and Lorentz spaces (see Lorentz (1950) or Hunt (1966)), and have been studied by many authors, including, for example, Maligranda (1984), Mastyło (1986)

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and Kamińska (1990a, 1990b, 1991). In particular, Maligranda posed a question about the Boyd indices of these spaces.

In this paper, we first give some fairly elementary estimates for the Boyd indices of Orlicz–Lorentz spaces. Then we give an example that show that these estimates cannot be improved, thus answering Maligranda's question. Finally we show how knowledge of the Boyd indices gives information about the p-convexity or q-concavity of the Orlicz–Lorentz space.

2 DEFINITIONS

In discussing Orlicz–Lorentz spaces, it will be convenient to talk about them in the more general framework of rearrangement invariant spaces. Unfortunately, the definitions in the literature usually require that the spaces be quasi-normed, which is not always the case with the Orlicz–Lorentz spaces. For this reason we introduce the following definition of rearrangement invariant spaces.

DEFINITION If $(\Omega, \mathcal{F}, \mu)$ is a measure space, we denote the measurable functions, modulo functions equal to zero almost everywhere, by $L_0(\mu)$. We say that a *Köthe functional* is a function $\|\cdot\| : L_0(\mu) \to [0,\infty]$ satisfying

i) if $f \in L_0(\mu)$, then $\|f\| = 0 \Leftrightarrow f = 0$;
ii) if $f \in L_0(\mu)$ and $\alpha \in \mathbf{C}$, then $\|\alpha f\| = |\alpha| \, \|f\|$;
iii) if $f, g \in L_0(\mu)$, then $|f| \le |g| \Rightarrow \|f\| \le \|g\|$;
iv) if $f_n, f \in L_0(\mu)$, then $|f_n| \nearrow |f| \Rightarrow \|f_n\| \to \|f\|$;
v) if $f_n \in L_0(\mu)$, then $\|f_n\| \to 0 \Rightarrow f_n \to 0$ in the measure topology.

A *Köthe space* is a pair $(X, \|\cdot\|)$, where $\|\cdot\|$ is a Köthe functional, and $X = \{f \in L_0(\mu) : \|f\| < \infty\}$. Usually, we will denote a space by a single letter, X, and denote its functional by $\|\cdot\|_X$.

DEFINITION If $f : \Omega \to \mathbf{C}$ is a measurable function, we define the *non-increasing rearrangement* of f to be

$$f^*(x) = \sup\{\, t : \mu(|f| \ge t) \ge x \,\}.$$

A *rearrangement invariant space* is a Köthe space such that if $f, g \in L_0(\mu)$, and $f^* \le g^*$, then $\|f\| \le \|g\|$.

Now we define the Orlicz–Lorentz spaces. We refer the reader to Montgomery-Smith (1992) for a motivation of the following definitions.

DEFINITION A *φ-function* is a function $F : [0,\infty) \to [0,\infty)$ such that

i) $F(0) = 0$;
ii) $\lim_{t\to\infty} F(t) = \infty$;
iii) F is strictly increasing;
iv) F is continuous;

We will say that a φ-function F is *dilatory* if for some $1 < c_1, c_2 < \infty$ we have $F(c_1 t) \ge c_2 F(t)$ for all $0 \le t < \infty$. We will say that F satisfies the *Δ_2-condition* if F^{-1} is dilatory.

If F is a φ-function, we will define the function $\tilde{F}(t)$ to be $1/F(1/t)$ if $t > 0$, and 0 if $t = 0$.

We say that two φ-functions F and G are *equivalent* (in symbols $F \asymp G$) if for some number $c < \infty$ we have that $F(c^{-1}t) \le G(t) \le F(ct)$ for all $0 \le t < \infty$.

We will denote the φ-function $F(t) = t^p$ by T^p.

DEFINITION (See Orlicz (1932) or Luxembourg (1955).) If $(\Omega, \mathcal{F}, \mu)$ is a measure space, and F is a φ-function, then we define the *Luxemburg functional* of a measurable function f by

$$\|f\|_F = \inf \left\{ c : \int_\Omega F(|f(\omega)|/c) \, d\mu(\omega) \le 1 \right\},$$

The *Orlicz space* is the associated Köthe space, and is denoted by $L_F(\Omega, \mathcal{F}, \mu)$ (or $L_F(\mu)$, $L_F(\Omega)$ or L_F for short).

DEFINITION If $(\Omega, \mathcal{F}, \mu)$ is a measure space, and F and G are φ-functions, then we define the *Orlicz–Lorentz functional* of a measurable function f by

$$\|f\|_{F,G} = \left\| f^* \circ \tilde F \circ \tilde G^{-1} \right\|_G .$$

The *Orlicz–Lorentz space* is the associated Köthe space, and is denoted by $L_{F,G}(\Omega, \mathcal{F}, \mu)$ (or $L_{F,G}(\mu)$, $L_{F,G}(\Omega)$ or $L_{F,G}$ for short).

We will write $L_{F,p}$, $L_{p,G}$ and $L_{p,q}$ for L_{F,T^p}, $L_{T^p,G}$ and L_{T^p,T^q} respectively.

It is an elementary matter to show that the Orlicz and Orlicz–Lorentz spaces are rearrangement invariant spaces. We note that $\|\cdot\|_{F,F} = \|\cdot\|_F$, and that $\|\chi_A\|_{F,G} = \tilde F^{-1}(\mu(A))$.

Now we define the various indices that we use throughout this paper. Obviously, the most important of these are the Boyd indices. These were first introduced in Boyd (1969). We will follow Maligranda (1984) for the names of the other indices, but will modify the definitions so as to be consistent with the notation used in Lindenstrauss and Tzafriri (1979). Thus other references to these indices will often reverse the words 'upper' and 'lower', and use the reciprocals of the indices used here. The Zippin indices were introduced in Zippin (1971), and the Matuszewska–Orlicz indices in Matuszewska and Orlicz (1960 and 1965). The Zippin indices are sometimes called *fundamental indices.*

DEFINITION For a rearrangement invariant space X, we let the *dilation operators* $d_a : X \to X$ be $d_a f(x) = f(ax)$ for $0 < a < \infty$. We define the *lower Boyd index* to be

$$p(X) = \sup \left\{ p : \text{for some } c < \infty \text{ we have } \|d_a\|_{X \to X} \le c a^{-1/p} \text{ for } a < 1 \right\}.$$

We define the *upper Boyd index* to be

$$q(X) = \inf \left\{ q : \text{for some } c < \infty \text{ we have } \|d_a\|_{X \to X} \le c a^{-1/q} \text{ for } a > 1 \right\}.$$

We define the *lower Zippin index* to be

$$p_z(X) = \sup \left\{ p : \begin{array}{c} \text{for some } c < \infty \text{ we have } \|d_a \chi_A\|_X \le c a^{-1/p} \|\chi_A\|_X \\ \text{for all } a < 1 \text{ and measurable } A \end{array} \right\}.$$

We define the *upper Zippin index* to be

$$q_z(X) = \inf \left\{ q : \begin{array}{c} \text{for some } c < \infty \text{ we have } \|d_a \chi_A\|_X \le c a^{-1/q} \|\chi_A\|_X \\ \text{for all } a > 1 \text{ and measurable } A \end{array} \right\}.$$

DEFINITION For a φ-function F, we define the *lower Matuszewska–Orlicz index* to be

$$p_m(F) = \sup\{\, p : \text{for some } c > 0 \text{ we have } F(at) \ge c\, a^p F(t) \text{ for } 0 \le t < \infty \text{ and } a > 1 \,\}.$$

We define the *upper Matuszewska–Orlicz index* to be

$$q_m(F) = \inf\{\, q : \text{for some } c < \infty \text{ we have } F(at) \le c\, a^q F(t) \text{ for } 0 \le t < \infty \text{ and } a > 1 \,\}.$$

Thus, for example,

$$p(L_{p,q}) = q(L_{p,q}) = p_z(L_{p,q}) = q_z(L_{p,q}) = p_m(T^p) = q_m(T^p) = p.$$

We also note the following elementary proposition about the Matuszewska–Orlicz indices.

PROPOSITION 2.1 Let F be a φ-function.
i) F is dilatory if and only if $p_m(F) > 0$.
ii) F satisfies the Δ_2-condition if and only if $q_m(F) < \infty$.

It was conjectured, at one time, that the Boyd and Zippin indices coincide. This is a natural conjecture in view of the fact that these indices do coincide for almost all 'natural' rearrangement spaces, for example, the Orlicz spaces and the Lorentz spaces. However Shimogaki (1970) gave an example of a rearrangement invariant Banach space where these indices differ.

Maligranda (1984) posed a conjecture (Problem 6.1) that would imply that the Boyd indices and Zippin indices coincide for the Orlicz–Lorentz spaces. One of the main purposes of this paper is to show that this is not the case.

In the sequel, we will always suppose that the measure space is $[0,\infty)$ with the Lebsgue measure λ.

3 ESTIMATES FOR THE BOYD INDICES OF THE ORLICZ LORENTZ SPACES

The first results that we present give estimates for the Boyd indices. These estimates are not very sophisticated. However, as we will show in Section 4, they cannot be improved, at least in the form in which they are given. It would be nice to give better estimates at some point in the future, which would make use of more detailed structure information of the defining functions of the Orlicz–Lorentz space.

THEOREM 3.1 Let F and G be φ-functions. Then
i) $p_m(F) \ge p(L_{F,G}) \ge p_m(F \circ G^{-1}) p_m(G) \ge p_m(F) p_m(G)/q_m(G)$;
ii) $q_m(F) \le q(L_{F,G}) \le q_m(F \circ G^{-1}) q_m(G) \le q_m(F) q_m(G)/p_m(G)$.

This will follow from the following propositions.

PROPOSITION 3.2 Let X be a rearrangement invariant space, and let F and G be φ-functions.
i) $p(X) \le p_z(X)$ and $q(X) \ge q_z(X)$.
ii) $p(L_F) = p_z(L_F)$ and $q(L_F) = q_z(L_F)$.

iii) $p_z(L_{F,G}) = p_m(F)$ and $q_z(L_{F,G}) = q_m(F)$.

Proof: See Maligranda (1984) for part (i), and see Lindenstrauss and Tzafriri (1979) for part (ii). Part (iii) is clear.

PROPOSITION 3.3 Let F_1, F_2 and G be φ-functions.

i) $p(L_{F_1,G}) \ge p_m(F_1 \circ F_2^{-1}) p(L_{F_2,G})$.

ii) $q(L_{F_1,G}) \le q_m(F_1 \circ F_2^{-1}) q(L_{F_2,G})$.

Proof: We will show (i). The proof of (ii) is similar.

We note that if $p_1 < p_m(F_1 \circ F_2^{-1})$, and if $p_2 < p(L_{F_2,G})$, then there is a constant $c_1 < \infty$ such that for any $t \ge 0$ and $0 < a < 1$ we have

$$a\tilde{F}_1 \circ \tilde{F}_2^{-1}(t) \le \tilde{F}_1 \circ \tilde{F}_2^{-1}(c_1\, a^{1/p_1} t),$$

and there is a constant $c_2 < \infty$ such that for any $f \in L_0$ and $0 < b < 1$ we have

$$\|d_{c_1 b} f\|_{F_2,G} \le c_2 b^{-1/p_2} \|f\|_{F_2,G} .$$

Therefore,

$$\begin{aligned} \|d_a f\|_{F_1,G} &= \left\| x \mapsto f^* \left(a\, \tilde{F}_1 \circ \tilde{G}^{-1}(x) \right) \right\|_G \\ &\le \left\| x \mapsto f^* \circ \tilde{F}_1 \circ \tilde{F}_2^{-1} \left(c_1\, a^{1/p_1} \tilde{F}_2 \circ \tilde{G}^{-1}(x) \right) \right\|_G \\ &\le c_2 a^{-1/p_1 p_2} \|f\|_{F_1,G} . \end{aligned}$$

Therefore $p(L_{F_1,G}) \ge p_1 p_2$, and the result follows.

Proof of Theorem 3.1: The first inequality follows from Proposition 3.2. The second inequality follows from Propositions (3.2) and (3.3). The third inequality follows because

$$p_m(F \circ G^{-1}) \ge p_m(F) p_m(G^{-1}) = p_m(F)/q_m(G).$$

4 BOYD INDICES CAN DIFFER FROM ZIPPIN INDICES

Now we show that Theorem 3.1 cannot be improved. In so doing, we answer Problem 6.1 posed by Maligranda (1984), by showing that the Boyd indices and Zippin indices do not necessarily coincide for the Orlicz–Lorentz spaces.

THEOREM 4.1 Given $0 < p < q < \infty$, there is a φ-function G such that $p_m(G) = p$, $q_m(G) = q$, $p(L_{1,G}) = p/q$, and $q(L_{1,G}) = q/p$.

We also have the following interesting example, that shows that an Orlicz–Lorentz space need not be quasi normed just because its defining functions are dilatory.

THEOREM 4.2 There is a dilatory φ-function G such that $L_{1,G}$ is not a quasi-Banach space.

At the heart of these results is the following lemma.

LEMMA 4.3 Suppose that $0 < p, q < \infty$, $a > 1$ and $n_0, n_1 \in \mathbf{N}$ are such that

$$(n_1 - n_0)a^{-p}\left(1 - a^{-(p+q)}\right) + a^{-2p-q} = 1.$$

Suppose that G is a φ-function such that for some $L, M > 0$ we have that

$$\begin{aligned} \tilde{G}(Ma^{2n}t) &= La^{(p+q)n}t^p \\ \tilde{G}(Ma^{2n+1}t) &= La^{(p+q)n+p}t^q \end{aligned}$$

for $1 \leq t \leq a$ and $n_0 \leq n \leq n_1 + 1$. Then for all $0 \leq \theta \leq \inf\{q/p, 1\}$, there are functions f and g such that we have

$$\|d_{a^{-\theta}} f\|_{1,G} = a^{(q/p)\theta}\|f\|_{1,G} \quad \text{and} \quad \|d_{a^{\theta}} g\|_{1,G} = a^{-(q/p)\theta}\|g\|_{1,G}.$$

Proof: We define the functions f and g by

$$f(Mx) = \begin{cases} M^{-1}a^{-2n_0-3} & \text{if } 0 \leq x < a^{2n_0} \\ M^{-1}a^{-2n-3} & \text{if } a^{2n} \leq x < a^{2n+2} \text{ and } n_0 \leq n \leq n_1 \\ 0 & \text{if } a^{2n_1+2} \leq x \end{cases}$$

$$g(Mx) = \begin{cases} M^{-1}a^{-2n_0-3-(p/q)\theta} & \text{if } 0 \leq x < a^{2n_0+\theta} \\ M^{-1}a^{-2n-3-(p/q)\theta} & \text{if } a^{2n+\theta} \leq x < a^{2n+2+\theta} \text{ and } n_0 \leq n \leq n_1 \\ 0 & \text{if } a^{2n_1+2+\theta} \leq x \end{cases}$$

so that $g = a^{-(p/q)\theta}d_{a^{-\theta}}f$. Then it is sufficient to show that $\|f\|_{1,G} = \|g\|_{1,G} = 1$. We will only show that $\|g\|_{1,G} = 1$, as setting $\theta = 0$ gives the other equality.

First, we note that if

$$La^{(p+q)n+p\theta} \leq x < La^{(p+q)(n+1)+p\theta}$$

then

$$Ma^{2n+\theta} \leq \tilde{G}^{-1}(x) < Ma^{2n+2+\theta}$$

and so

$$g^* \circ \tilde{G}^{-1}(x) = M^{-1}a^{-2n-3-(p/q)\theta}$$

implying that

$$G \circ g^* \circ \tilde{G}^{-1}(x) = 1/\tilde{G}(Ma^{2n+3+(p/q)\theta}) = 1/\left(La^{(p+q)n+2p+q+p\theta}\right) = L^{-1}a^{-(p+q)n-2p-q-p\theta}.$$

Similarly, if $0 \leq x < La^{(p+q)n_0+p\theta}$, then $G \circ g^* \circ \tilde{G}^{-1}(x) = L^{-1}a^{-(p+q)n_0-2p-q-p\theta}$. Hence

$$\begin{aligned} &\int_0^\infty G \circ g^* \circ \tilde{G}^{-1}(x)\,dx \\ &= \sum_{n=n_0}^{n_1} \int_{La^{(p+q)n+p\theta}}^{La^{(p+q)(n+1)+p\theta}} G \circ g^* \circ \tilde{G}^{-1}(x)\,dx + \int_0^{La^{(p+q)n_0+p\theta}} G \circ g^* \circ \tilde{G}^{-1}(x)\,dx \\ &= \sum_{n=n_0}^{n_1} \left(La^{(p+q)(n+1)+p\theta} - La^{(p+q)n+p\theta}\right) L^{-1}a^{-n(p+q)-2p-q-p\theta} \\ &\qquad + La^{(p+q)n_0+p\theta}L^{-1}a^{-n_0(p+q)-2p-q-p\theta} \\ &= (n_1 - n_0)a^{-p}\left(1 - a^{-(p+q)}\right) + a^{-2p-q} \\ &= 1, \end{aligned}$$

as required.

Proof of Theorem 4.1: Construct sequences of numbers a_k, b_k, M_k and N_k $(k \geq 0)$ such that M_k and N_k are integers, $a_k, b_k > 0$,

$$\begin{aligned} M_k a_k^{-p}\left(1 - a_k^{-(p+q)}\right) + a_k^{-2p-q} &= 1, \\ N_k b_k^{-q}\left(1 - b_k^{-(p+q)}\right) + b_k^{-p-2q} &= 1, \end{aligned}$$

$a_k \to \infty$, and $b_k \to \infty$. Define sequences A_k and B_k inductively as follows: $A_0 = B_0 = 1$, $B_k = A_k a_k^{2M_k+2}$, and $A_{k+1} = B_k b_k^{2N_k+2}$ for $k \geq 0$. Define G by

$$\begin{aligned} G(1) &= 1, \\ G(A_k a_k^{2n} t) &= G(A_k) a_k^{(p+q)n} t^p \\ G(A_k a_k^{2n+1} t) &= G(A_k) a_k^{(p+q)n+p} t^q \end{aligned}$$

for $0 \leq n \leq M_k$ and $1 \leq t \leq a_k$,

$$\begin{aligned} G(B_k b_k^{2n} t) &= G(B_k) b_k^{(p+q)n} t^q \\ G(B_k b_k^{2n+1} t) &= G(B_k) b_k^{(p+q)n+q} t^p \end{aligned}$$

for $0 \leq n \leq N_k$ and $1 \leq t \leq b_k$, and

$$G(t) = \tilde{G}(t)$$

for $t < 1$. Clearly $p_m(G) = p$ and $q_m(G) = q$. From Lemma 4.3, we have that $p(L_{1,G}) = p/q$ and $q(L_{1,G}) = q/p$.

Proof of Theorem 4.2: Let $q = 1$, and construct sequences of numbers p_k, a_k and N_k $(k \geq 0)$ such that N_k is an integer, $a_k > 0$,

$$N_k a_k^{-p_k}\left(1 - a_k^{-(p_k+q)}\right) + a_k^{-2p_k-q} = 1,$$

$p_k \to \infty$, and $a_k^{q/p_k} \to \infty$. Define a sequence A_k inductively as follows: $A_0 = 1$, and $A_{k+1} = A_k a_k^{2N_k+2}$ for $k \geq 0$. Define G by

$$\begin{aligned} G(1) &= 1, \\ G(A_k a_k^{2n} t) &= G(A_k) a_k^{(p_k+q)n} t_k^p \\ G(A_k a_k^{2n+1} t) &= G(A_k) a_k^{(p_k+q)n+p_k} t^q \end{aligned}$$

for $0 \leq n \leq M_k$ and $1 \leq t \leq a_k$, and

$$G(t) = \tilde{G}(t)$$

for $t > 1$. Then $p_m(G) = 1$. From Lemma 4.3, we have that $p(L_{1,G}) = 0$, and so by Theorem 5.3(ii) below, $L_{1,G}$ cannot be a quasi-Banach space.

5 CONVEXITY AND CONCAVITY OF ORLICZ–LORENTZ SPACES

An important property that one might like to know about Köthe spaces is whether it is p-convex or q-concave for some prescribed p or q. These questions have already been settled for Orlicz spaces and Lorentz spaces.

For Lorentz spaces, it is almost immediate from their definition (Bennett and Sharpley (1988) or Hunt (1966)) that $L_{p,q}$ is q-convex if $p \geq q$, and p-concave if $p \leq q$. However, outside of these ranges, it is more difficult. In general, it is only the case that $L_{p,q}$ is $q \wedge (p-\epsilon)$-convex and $p \vee (q+\epsilon)$-concave. These results are shown in many places, for example, in Bennett and Sharpley (1988) or Hunt (1966). For Orlicz–Lorentz spaces, the same methods of proof work, and we present these results here.

First we define the notions of p-convexity and q-concavity. These notions may also be found in, for example, Lindenstrauss and Tzafriri (1979).

DEFINITION If X is a Köthe space, we say that X is *p-convex*, respectively *q-concave*, if for some $C < \infty$ we have

$$\left\| \left(\sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_X \leq C \left(\sum_{i=1}^n \|f_i\|_X^p \right)^{1/p},$$

respectively

$$\left\| \left(\sum_{i=1}^n |f_i|^q \right)^{1/q} \right\|_X \geq C^{-1} \left(\sum_{i=1}^n \|f_i\|_X^q \right)^{1/q},$$

for any $f_1, f_2, \ldots, f_n \in X$.

The most elementary result about p-concavity and q-convexity is the following. This corresponds to the result that $L_{p,q}$ is q-convex if $p \geq q$, and p-concave if $p \leq q$.

THEOREM 5.1 Let F and G be φ-functions.

i) If $G \circ T^{1/p}$ is equivalent to a convex function and $\tilde{G} \circ \tilde{F}^{-1}$ is concave, then $L_{F,G}$ is p-convex.

ii) If $G \circ T^{1/q}$ is equivalent to a concave function and $\tilde{G} \circ \tilde{F}^{-1}$ is convex, then $L_{F,G}$ is q-concave.

Proof: We will only prove (i), as the proof of (ii) is similar. We first use the identity

$$\|f\|_{F \circ T^p, G \circ T^p} = \||f|^p\|_{F,G}^{1/p}$$

to notice that without loss of generality we may take $p = 1$.

From Hardy, Littlewood and Pólya (1952), Chapter X, it follows that

$$\|f\|_{F,G} = \sup \left\| f \circ \sigma \circ \tilde{F} \circ \tilde{G}^{-1} \right\|_G,$$

where the supremum is over all measure preserving maps $\sigma : [0,\infty) \to [0,\infty)$. Since G is convex, it follows from Krasnosel'skiĭ and Rutickiĭ (1961) that $\|\cdot\|_G$ is 1-convex. Now the result follows easily.

However, if we take the Boyd indices into account, we can also obtain the following results. These correspond to the result that says that $L_{p,q}$ is $q \wedge (p-\epsilon)$-convex and $p \vee (q+\epsilon)$-concave.

To state and prove these results, it is first necessary to recall notation and results from Montgomery-Smith (1992).

DEFINITION If F and G are φ-functions, then say that F is *equivalently less convex than* G (in symbols $F \prec G$) if $G \circ F^{-1}$ is equivalent to a convex function. We say that F is *equivalently more convex than* G (in symbols $F \succ G$) if G is equivalently less convex than F.

A φ-function F is said to be an *N-function* if it is equivalent to a φ-function F_0 such that $F_0(t)/t$ is strictly increasing, $F_0(t)/t \to \infty$ as $t \to \infty$, and $F_0(t)/t \to 0$ as $t \to 0$.

A φ-function F is said to be *complementary* to a φ-function G if for some $c < \infty$ we have

$$c^{-1}t \leq F^{-1}(t) \cdot G^{-1}(t) \leq ct \qquad (0 \leq t < \infty).$$

If F is an N-function, we will let F^* denote a function complementary to F.

An N-function H is said to satisfy *condition* (J) if

$$\left\| 1/\tilde{H}^{*-1} \right\|_{H^*} < \infty.$$

To give some intuitive feeling for N-functions that satisfy condition (J), we point out that these are functions that equivalent to slowly rising convex functions, for example,

$$F(t) = \begin{cases} t^{1+1/\log(1+t)} & \text{if } t \geq 1 \\ t^{1-1/\log(1+1/t)} & \text{if } t \leq 1. \end{cases}$$

THEOREM 5.2 (Montgomery-Smith, 1992) Let F, G_1 and G_2 be φ-functions such that one of G_1 and G_2 is dilatory, and one of G_1 or G_2 satisfies the Δ_2-condition. Then the following are equivalent.

i) For some $c < \infty$, we have that $\|f\|_{F,G_1} \leq c \, \|f\|_{F,G_2}$ for all measurable f.

ii) There is an N-function H satisfying condition (J) such that $G_1 \circ G_2^{-1} \succ H^{-1}$.

Now, we are ready to state the main results of this section.

THEOREM 5.3 Let F and G be φ-functions, and $0 < p < \infty$.

i) If the lower Boyd index $p(L_{F,G}) > p$, and if $G \succ H^{-1} \circ T^p$ for some N-function satisfying condition (J), then $L_{F,G}$ is p-convex.

ii) If $L_{F,G}$ is p-convex, then the lower Boyd index $p(L_{F,G}) \geq p$, and $G \succ H^{-1} \circ T^p$ for some N-function satisfying condition (J).

Note that in part (i), it is not sufficient to take $p(L_{F,G}) = 1$. This is shown by the example $L_{1,q}$ for $1 < q < \infty$, which is known to be not 1-convex (Hunt, 1966).

THEOREM 5.4 Let F and G be φ-functions such that G is dilatory and $p(L_{F,G}) > 0$, and let $0 < q < \infty$.

i) If the lower Boyd index $q(L_{F,G}) < q$, and if $T^q \circ G^{-1} \succ H^{-1}$ for some N-function satisfying condition (J), then $L_{F,G}$ is q-concave.

ii) If $L_{F,G}$ is q-convex, then the lower Boyd index $q(L_{F,G}) \leq p$, and $T^q \circ G^{-1} \succ H^{-1}$ for some N-function satisfying condition (J).

Proof of Theorem 5.3: As in the beginning of the proof of Theorem 5.1, we may suppose without loss of generality that $p = 1$.

The proof of (i) uses fairly standard techniques (Bennett and Sharpley, 1988). First, by Theorem 5.2, we may assume that G is equivalent to a convex function. Next, for any measurable function f, we define

$$f^{**}(x) = \frac{1}{x} \int_0^x f^*(\xi) \, d\xi = \int_0^1 d_a f^*(x) \, da.$$

Then we have the Hardy inequality holding, that is, for some $c < \infty$ we have that $\|f\|_{F,G} \leq \|f^{**}\|_{F,G} \leq c \, \|f\|_{F,G}$. The left hand inequality is obvious. For the right hand inequality, since

$p(L_{F,G}) > 1$, we know that for some $p > 1$ and some $c_1 < \infty$ we have that $\|d_a\|_{L_{F,G}\to L_{F,G}} \le c_1 a^{-1/p}$ for all $a < 1$. Hence

$$\begin{aligned}
\|f^{**}\|_{F,G} &= \left\| \int_0^1 d_a f^* \, da \right\|_{F,G} \\
&= \left\| \int_0^1 d_a f^* \circ \tilde{F} \circ \tilde{G}^{-1} \, da \right\|_G \\
&\le c_2 \int_0^1 \left\| d_a f^* \circ \tilde{F} \circ \tilde{G}^{-1} \right\|_G \, da
\end{aligned}$$

(as G is equivalent to a convex function)

$$\begin{aligned}
&= c_2 \int_0^1 \|d_a f^*\|_{F,G} \, da \\
&\le c_2 \int_0^1 c_1 a^{-1/p} \, da \, \|f^*\|_{F,G} \\
&\le c_1 c_2 \frac{p}{p-1} \|f\|_{F,G} \, .
\end{aligned}$$

But, the functional that takes f to $\|f^{**}\|_{F,G}$ is 1-convex. This is because for any $x_0 > 0$, we have that

$$f^{**}(x_0) = \sup_{\lambda(A)=x_0} \int_A |f(x)| \, dx.$$

(See Hardy, Littlewood and Pólya (1952), Chapter X, or Lindenstrauss and Tzafriri (1979).) Hence, $(f+g)^{**} \le f^{**} + g^{**}$. Also, by Krasnosel'skiĭ and Rutickiĭ (1961), it follows that $\|\cdot\|_G$ is 1-convex. Therefore,

$$\begin{aligned}
\left\| \sum_{i=1}^n |f_i| \right\|_{F,G} &\le \left\| \left(\sum_{i=1}^n |f_i| \right)^{**} \right\|_{F,G} \\
&\le \left\| \sum_{i=1}^n f_i^{**} \right\|_{F,G} \\
&= \left\| \sum_{i=1}^n f_i^{**} \circ \tilde{F} \circ \tilde{G}^{-1} \right\|_G \\
&\le c_2 \sum_{i=1}^n \left\| f_i^{**} \circ \tilde{F} \circ \tilde{G}^{-1} \right\|_G \\
&= c_2 \sum_{i=1}^n \|f_i^{**}\|_{F,G} \\
&\le c\, c_2 \sum_{i=1}^n \|f_i\|_{F,G} \, ,
\end{aligned}$$

as desired.

To show (ii), we note that if a is the reciprocal of an integer, then there are functions $g_1, g_2, \ldots, g_{a^{-1}}$, with disjoint supports, and each with the same distribution as f, so that $g_1 + g_2 + \ldots + g_{a^{-1}}$ has the same distribution as $d_a f$. Hence

$$\|d_a f\|_{F,G} \le c \left(\|g_1\|_{F,G} + \|g_2\|_{F,G} + \ldots + \|g_{a^{-1}}\|_{F,G} \right) = c\, a^{-1} \|f\|_{F,G} \, .$$

Hence $p(L_{F,G}) \ge 1$.

To show that $G \succ H^{-1}$ for some N-function satisfying condition (J), we note the following inequalities.

$$\begin{aligned}
\|f\|_{F,G} &= \left\| \int_0^\infty \chi_{|f|\ge t}\, dt \right\|_{F,G} \\
&\le c \int_0^\infty \left\| \chi_{|f|\ge t} \right\|_{F,G} dt \\
&= c \int_0^\infty \tilde F^{-1}(\mu\{|f|\ge t\})\, dt \\
&= c\, \|f\|_{F,1} .
\end{aligned}$$

Now the result follows immediately from Theorem 5.2.

Proof of Theorem 5.4: As in the proof of Theorem 5.1, we may assume that $q = 1$. To prove (i) we first note, by Theorem 5.2, we may assume that G^{-1} is equivalent to a convex function. Since G is dilatory, it follows that G is equivalent to a concave function (see Montgomery-Smith (1992), Lemma 5.5.2).

Next, for any measurable function f, we define

$$f_{**}(x) = f^*(x) + \frac{1}{x}\int_x^\infty f^*(\xi)\, d\xi = f^*(x) + \int_1^\infty d_a f^*(x)\, da.$$

Then, for some $c < \infty$ we have that $\|f\|_{F,G} \le \|f_{**}\|_{F,G} \le c\, \|f\|_{F,G}$. The left hand inequality is obvious.

For the right hand inequality, we argue as follows. Since $q(L_{F,G}) < 1$, we know that for some $q < 1$ and some $c_1 < \infty$ we have that $\|d_a\|_{L_{F,G}\to L_{F,G}} \le c_1 a^{-1/q}$ for all $a > 1$. Since G is dilatory, it is easy to see that there is there some $p > 0$ such that $G \circ T^{1/p}$ is equivalent to a convex function. Let $q < r < 1$. Then there is a constant $c_2 < \infty$, depending upon r only, such that

$$\begin{aligned}
f_{**}(x) &\le c_2 \left((f^*(x))^p + \frac{1}{x^{p/r}} \int_x^\infty \xi^{p/r-1} (f^*(\xi))^p\, d\xi \right)^{1/p} \\
&= c_2 \left((f^*(x))^p + \int_1^\infty a^{p/r-1} (d_a f^*(x))^p\, da \right)^{1/p} .
\end{aligned}$$

For if the right hand side is less than or equal to 1, then it is easily seen that

$$f^*(\xi) \le 1 \wedge \left(\frac{x}{\xi - x} \right)^{1/r} \qquad (\xi > x),$$

and hence

$$f_{**}(x) \le \int_1^\infty 1 \wedge (\theta - 1)^{-1/r}\, d\theta.$$

Thus we have the following inequalities.

$$\begin{aligned}
\|f_{**}\|_{F,G} &\le c_2 \left\| \left((f^*)^p + \int_1^\infty a^{p/r-1} (d_a f^*)^p\, da \right)^{1/p} \right\|_{F,G} \\
&= c_2 \left\| \left((f^* \circ \tilde F \circ \tilde G^{-1})^p + \int_1^\infty a^{p/r-1} (d_a f^* \circ \tilde F \circ \tilde G^{-1})^p\, da \right)^{1/p} \right\|_G \\
&\le c_3 \left(\left\| f^* \circ \tilde F \circ \tilde G^{-1} \right\|_G^p + \int_1^\infty a^{p/r-1} \left\| d_a f^* \circ \tilde F \circ \tilde G^{-1} \right\|_G^p da \right)^{1/p}
\end{aligned}$$

(as $G \circ T^{1/p}$ is equivalent to a convex function)

$$
\begin{aligned}
&= c_3 \left(\|f^*\|_{F,G}^p + \int_1^\infty a^{p/r-1} \|d_a f^*\|_{F,G}^p \, da \right)^{1/p} \\
&\leq c_1 c_3 \left(1 + \int_1^\infty a^{p/r-p/q-1} \, da \right)^{1/p} \|f\|_{F,G} .
\end{aligned}
$$

But, the functional that takes f to $\|f_{**}\|_{F,G}$ is 1-concave. This is because for any $x_0 > 0$, we have that

$$
f_{**}(x_0) = \frac{1}{x_0} \int_0^\infty f(\xi) \, d\xi - f^{**}(x_0).
$$

Hence, $(f+g)_{**} \geq f_{**} + g_{**}$. Also, by an argument similar to that given in M.A. Krasnosel'skiĭ and Rutickiĭ (1961), it follows that $\|\cdot\|_G$ is 1-concave. Therefore,

$$
\begin{aligned}
\sum_{i=1}^n \|f_i\|_{F,G} &\leq \sum_{i=1}^n \|f_{i**}\|_{F,G} \\
&= \sum_{i=1}^n \left\| f_{i**} \circ \tilde{F} \circ \tilde{G}^{-1} \right\|_G \\
&\leq c_3 \left\| \sum_{i=1}^n f_{i**} \circ \tilde{F} \circ \tilde{G}^{-1} \right\|_G \\
&= c_3 \left\| \sum_{i=1}^n f_{i**} \right\|_{F,G} \\
&\leq c_3 \left\| \left(\sum_{i=1}^n |f_i| \right)^{**} \right\|_{F,G} \\
&\leq c\, c_1 \left\| \sum_{i=1}^n |f_i| \right\|_{F,G} ,
\end{aligned}
$$

as desired.

To show (ii), we note that if a is an integer, then there are functions g_1, $g_2, \ldots,$ g_a, with disjoint supports, and each with the same distribution as $d_a f$, so that $g_1 + g_2 + \ldots + g_a$ has the same distribution as f. Hence

$$
\|f\|_{F,G} \geq c^{-1} \left(\|g_1\|_{F,G} + \|g_2\|_{F,G} + \ldots + \|g_a\|_{F,G} \right) = c^{-1} a \, \|d_a f\|_{F,G} .
$$

Hence $q(L_{F,G}) \leq 1$.

To show that $G \prec H$ for some N-function satisfying condition (J), we note the following inequalities.

$$
\begin{aligned}
\|f\|_{F,G} &= \left\| \int_0^\infty \chi_{|f| \geq t} \, dt \right\|_{F,G} \\
&\geq c^{-1} \int_0^\infty \left\| \chi_{|f| \geq t} \right\|_{F,G} dt \\
&= c^{-1} \int_0^\infty \tilde{F}^{-1}(\mu\{|f| \geq t\}) \, dt \\
&= c^{-1} \|f\|_{F,1} .
\end{aligned}
$$

Now the result follows immediately from Theorem 5.2.

6 ADDITIONAL COMMENTS

First, we remark that there is another definition of Orlicz–Lorentz spaces given by Torchinsky (1976) (see also Raynaud (1990)). If F and G are φ-functions, then we define

$$\|f\|_{F,G}^T = \inf\left\{ c : \int_0^\infty G(\tilde F^{-1}(x) f^*(x)/c)\,\frac{dx}{x} \le 1 \right\},$$

If F is dilatory and satisfy the Δ_2-condition, and if G is dilatory, then it is very easy to calculate the Boyd indices of these spaces — they are precisely the same as their corresponding Matuszewska–Orlicz indices. This follows from the fact that under these conditions, $\left\|\chi_{[0,t]}\right\|_{F,G}^T \approx \tilde F^{-1}(t)$ (See Raynaud (1990) for more details).

We also pose some questions.

i) What is the dual of an Orlicz–Lorentz space (when the space itself is 1-convex)? Is it another Orlicz–Lorentz space?

ii) Is it possible to find more precise estimates for the Boyd indices of Orlicz–Lorentz spaces?

An approach to the last problem (at least for giving necessary and sufficient conditions for $p(L_{1,G}) = q(L_{1,G}) = 1$ is suggested in Montgomery-Smith (1991).

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REFERENCES

1. C. Bennett and R. Sharpley, *Interpolation of Operators,* Academic Press 1988.
2. D.W. Boyd, Indices of function spaces and their relationship to interpolation, *Canad. J. Math.* **21** *(1969), 1245–1254.*
3. D.W. Boyd, Indices for the Orlicz spaces, *Pacific J. Math.* **38** *(1971), 315–323.*
4. G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities,* Cambridge University Press, 1952.
5. R.A. Hunt, On $L(p,q)$ spaces, *L'Enseignement Math. (2)* **12** *(1966), 249–275.*
6. A. Kamińska, Some remarks on Orlicz–Lorentz spaces, *Math. Nachr.* **147**, *(1990), 29–38.*
7. A. Kamińska, Extreme points in Orlicz–Lorentz spaces, *Arch. Math.* **55**, *(1990), 173–180.*
8. A. Kamińska, Uniform convexity of generalized Lorentz spaces, *Arch. Math.* **56**, *(1991), 181–188.*
9. M.A. Krasnosel'skiĭ and Ya.B. Rutickiĭ, *Convex Functions and Orlicz Spaces,* P. Noordhoof Ltd., 1961.

10. G.G. Lorentz, Some new function spaces, *Ann. Math.* **51** *(1950), 37–55.*
11. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I—Sequence Spaces,* Springer-Verlag 1977.
12. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II—Function Spaces,* Springer-Verlag 1979.
13. W.A.J. Luxemburg, *Banach Function Spaces,* Thesis, Delft Technical Univ. 1955.
14. L. Maligranda, Indices and interpolation, *Dissert. Math.* **234** *(1984), 1–49.*
15. M. Mastyło, Interpolation of linear operators in Calderon–Lozanovskii spaces, *Comment. Math.* **26,2** *(1986), 247–256.*
16. W. Matuszewska and W. Orlicz, On certain properties of φ-functions, *Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys.* **8** *(1960), 439–443.*
17. W. Matuszewska and W. Orlicz, On some classes of functions with regard to their orders of growth, *Studia Math.* **26** *(1965), 11–24.*
18. S.J. Montgomery-Smith, *The Cotype of Operators from $C(K)$,* Ph.D. thesis, Cambridge, August 1988.
19. S.J. Montgomery-Smith, Orlicz–Lorentz Spaces, *Proceedings of the Orlicz Memorial Conference*, (Ed. P. Kranz and I. Labuda), Oxford, Mississippi (1991).
20. S.J. Montgomery-Smith, Comparison of Orlicz–Lorentz spaces, *Studia Math.* **103**, *(1992), 161–189.*
21. W. Orlicz, Über eine gewisse Klasse von Räumen vom Typus B, *Bull. Intern. Acad. Pol.* **8** *(1932), 207–220.*
22. Y. Raynaud, On Lorentz–Sharpley spaces, *Proceedings of the Workshop "Interpolation Spaces and Related Topics", Haifa, June 1990.*
23. T. Shimogaki, A note on norms of compression operators on function spaces, *Proc. Japan Acad.* **46** *(1970), 239–242.*
24. A. Torchinsky, Interpolation of operators and Orlicz classes, *Studia Math.* **59** *(1976), 177–207.*
25. M. Zippin, Interpolation of operators of weak type between rearrangement invariant spaces, *J. Functional Analysis* **7** *(1971), 267–284.*

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